Communications in Mathematics and Applications
Volume 4 (2013), Number 1, pp. 61–73

http://www.rgnpublications.com

Morphisms of Vector Groupoids

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Abstract The main purpose is to investigate the morphisms of vector groupoids and several properties of them are established.

1. Introduction

The notion of groupoid was introduced by H. Brandt [*Math. Ann.*, **96**(1926), MR 1512323] and it is developed by P.J. Higgins in [3]. The algebraic structure of groupoid is similar to a group, with the exception that products of elements cannot are always be defined.

The concept of vector groupoid has been defined by V. Popuţa and Gh. Ivan [6]. This is an algebraic structure which combines the concepts of the groupoid and vector space such that these are compatible. In [7] the same authors introduce the algebraic structure of generalized vector groupoid. The groupoids and vector groupoids have applications in several areas of science, see [1], [10], [9], [5], [2], [8].

The paper is organized as follows. In Section 2, we present some basic concepts and results about vector groupoids. In Section 3, we introduce the notion of morphism of vector groupoids and its useful properties are discussed. In Section 4, the correspondence theorem for vector subgroupoids by a vector groupoid homomorphism is proved.

2. Vector Groupoids

We recall some necessary backgrounds on vector groupoids for our purposes (see [3], [4], [6] and references therein for more details).

A *groupoid G over G_0* ([1]) is a pair (G, G_0) of nonempty sets such that $G_0 \subseteq G$ endowed with two surjective maps $\alpha, \beta : G \to G_0$ (*source* and *target*), a partially binary operation (*multiplication*) $m : G_{(2)} \to G$, $(x, y) \mapsto m(x, y) := x \cdot y$, where

2010 *Mathematics Subject Classification.* 20L13, 20L99.
Key words and phrases. Vector groupoid; Morphism of vector groupoids.

$G_{(2)} := \{(x,y) \in G \times G \mid \beta(x) = \alpha(y)\}$ is the *set of composable pairs* and a map $i : G \to G$, $x \mapsto i(x) := x^{-1}$ (*inversion*), which verifies the following conditions:

(G1) (*associativity*): $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in the sense that if either of $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ is defined, so is the other and they are equal;

(G2) (*units*): For each $x \in G$ it follows that $(\alpha(x), x)$, $(x, \beta(x)) \in G_{(2)}$ and we have $\alpha(x) \cdot x = x \cdot \beta(x) = x$;

(G3) (*inverses*): For each $x \in G$ it follows that (x, x^{-1}), $(x^{-1}, x) \in G_{(2)}$ and we have $x^{-1} \cdot x = \beta(x)$, $x \cdot x^{-1} = \alpha(x)$.

A groupoid G over G_0 (called also G_0-*groupoid*) with the *structure functions* α, β, m, i is denoted by $(G, \alpha, \beta, m, i, G_0)$. G_0 is the *set of units* of G. For any $u \in G_0$, the set $G(u) := \{x \in G \mid \alpha(x) = \beta(x) = u\}$ has a structure of group under the restriction of m to $G(u)$, called the *isotropy group at* u of G. The map $(\alpha, \beta) : G \to G_0 \times G_0$ defined by $(\alpha, \beta)(x) := (\alpha(x), \beta(x))$, for all $x \in G$ is called the *anchor map* of G. A groupoid is said to be *transitive*, if its anchor map is surjective.

Definition 2.1 ([6]). By *vector groupoid*, we mean a V_0-groupoid $(V, \alpha, \beta, i, m, V_0)$ which verifies the following conditions:

(2.1.1) V is a vector space over a field K, and V_0 is a vector subspace of V;

(2.1.2) $\alpha, \beta : V \to V_0$ are linear maps;

(2.1.3) $i : V \to V$ is a linear map such that $x + i(x) = \alpha(x) + \beta(x)$, for all $x \in V$;

(2.1.4) The multiplication $m : V_{(2)} = \{(x,y) \in V \times V \mid \alpha(y) = \beta(x)\} \to V$, $(x,y) \rightarrowtail m(x,y) := xy$, satisfies the following conditions:

- (a) $x(y + z - \beta(x)) = xy + xz - x$, for all $x, y, z \in V$ such that $\alpha(y) = \beta(x) = \alpha(z)$;
- (b) $x(ky + (1-k)\beta(x)) = k(xy) + (1-k)x$, for all $x, y \in V$ such that $\alpha(y) = \beta(x)$;
- (c) $(y + z - \alpha(x))x = yx + zx - x$, for all $x, y, z \in V$ such that $\alpha(x) = \beta(y) = \beta(z)$;
- (d) $(ky + (1-k)\alpha(x))x = k(yx) + (1-k)x$, for all $x, y \in V$ such that $\alpha(y) = \beta(x)$.

In the following proposition we summarize the most important rules of algebraic calculation in a vector groupoid obtained directly from definitions.

Proposition 2.1 ([4], [6]). *If* $(V, \alpha, \beta, m, i, V_0)$ *is a vector groupoid, then:*

(i) $\alpha(u) = \beta(u) = u$, $u \cdot u = u$ *and* $i(u) = u$, *for all* $u \in V_0$.

(ii) $\alpha(xy) = \alpha(x)$ *and* $\beta(xy) = \beta(y)$, *for all* $(x,y) \in V_{(2)}$.

(iii) $\alpha(x^{-1}) = \beta(x)$ *and* $\beta(x^{-1}) = \alpha(x)$, *for all* $x \in V$.

(iv) α, β *are linear epimorphisms and* i *is a linear automorphism.*

(v) $0 \cdot x = x$ *and* $y \cdot 0 = y$, *for all* $x \in \alpha^{-1}(0), y \in \beta^{-1}(0)$ (0 *is null vector*).

(vi) $\alpha^{-1}(0)$, $\beta^{-1}(0)$ *and* $V(0) := \alpha^{-1}(0) \cap \beta^{-1}(0)$ *are vector subspaces in* V.

Example 2.1 ([6]). (i) Let V be a vector space. We define $\alpha_0, \beta_0 : V \to \{0\}$, $i_0 : V \to V$ and $m_0 : V \times V \to V$ by setting $\alpha_0(x) = \beta_0(x) = 0$, $i_0(x) = -x$, for all $x \in V$ and $m_0(x, y) = x + y$, for all $x, y \in V$. Then $(V, \alpha_0, \beta_0, m_0, i_0, V_0 = \{0\})$ is a vector groupoid called *vector groupoid with a single unit*.

(ii) Let (V, α, β, V_0) be a vector groupoid. $\mathrm{Is}(V) := \{x \in G \mid \alpha(x) = \beta(x)\}$ is a vector groupoid, called the *isotropy vector bundle* associated to V.

Example 2.2. Let V be a vector space.

(i) ([6]) The Cartesian product $\mathscr{V} := V \times V$ has a structure of groupoid, by taking the structure functions as follows:

$$\widetilde{\alpha}(x,y) := (x,x), \quad \widetilde{\beta}(x,y) := (y,y), \quad (x,y)\cdot(y,z) := (x,z)$$

and

$$(x,y)^{-1} := (y,x).$$

By a direct computation we prove that the conditions from Definition 2.1 are satisfied. Hence $(V \times V, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{m}, \widetilde{i}, \Delta_V)$, where $\Delta_V = \{(x,x) \in V \times V \mid x \in V\}$, is a vector groupoid called the *pair vector groupoid associated to* V.

(ii) Let $p, q, p_1, q_1 \in GL(V)$ such that $pq = p_1q_1 = 1$. The Cartesian product $V^3 := V \times V \times V$ may be endowed with a structure of vector groupoid, denoted with $V^3(p,q)$. The structure functions $\alpha, \beta, i : V^3 \to V^3$ and the multiplication are defined by:

$$\alpha(x) := (x_1, px_1, 0),$$
$$\beta(x) := (qx_2, x_2, 0),$$
$$i(x) := (qx_2, px_1, -x_3),$$
$$(x_1,x_2,x_3)\cdot(qx_2,y_2,y_3) := (x_1,y_2,x_3+y_3), \text{ for all } x = (x_1,x_2,x_3),$$
$$y = (y_1,y_2,y_3) \in V^3.$$

This is called the *vector groupoid of type* (p,q) *over* V^3.

3. **Vector Groupoid Morphisms**

Definition 3.1. Let $(V_1, \alpha_1, \beta_1, m_1, i_1, V_{1,0})$ and $(V_2, \alpha_2, \beta_2, m_2, i_2, V_{2,0})$ be two vector groupoids. A map $f : V_1 \to V_2$ is called *morphism of vector groupoids*, if

(i) f is a linear map and

(ii) $f : V_1 \to V_2$ is a groupoid morphism between the groupoids $(V_1, V_{1,0})$ and $(V_2, V_{2,0})$, i.e. the following conditions are verified:

(a) for all $(x,y) \in V_{1,(2)} \to (f(x), f(y)) \in V_{2,(2)}$;

(b) $f(m_1(x,y)) = m_2(f(x), f(y))$, for all $(x,y) \in V_{1,(2)}$.

Proposition 3.1. *Let* $f : V_1 \to V_2$ *be a vector groupoid morphfism between the vector groupoids* $(V_1, \alpha_1, \beta_1, m_1, i_1, V_{1,0})$ *and* $(V_2, \alpha_2, \beta_2, m_2, i_2, V_{2,0})$*. Then:*

(i) $f(u) \in V_{2,0}$, *for all* $u \in V_{1,0}$;
(ii) $f(x^{-1}) = (f(x))^{-1}$, *for all* $x \in V_1$;
(iii) $f_0 : V_{1,0} \to V_{2,0}$ *defined by* $f_0(u) := f(u)$, *(for all)* $u \in V_{1,0}$, *i.e. the restriction of* f *to* $V_{1,0}$, *is a linear map.*

Proof. (i) Let $u \in V_{1,0}$. Then $\alpha_1(u) = \beta_1(u) = u$. From $(\alpha_1(u), u) \in V_{1(2)}$ it follows $(f(\alpha_1(u)), f(u)) \in V_{2(2)}$ and $f(\alpha_1(u)) \cdot f(u) = f(\alpha_1(u) \cdot u) = f(u)$, since f is vector groupoid morphism. But, $\alpha_2(f(u)) \cdot f(u) = f(u)$. From $\alpha_2(f(u)) \cdot f(u) = f(u)$ and $f(\alpha_1(u)) \cdot f(u) = f(u)$ it follows $\alpha_2(f(u)) = f(\alpha_1(u))$. Hence $\alpha_2(f(u)) = f(u)$. Similarly, $\beta_2(f(u)) = f(u)$. Therefore, $f(u) \in V_{2,0}$ since $\alpha_2(f(u)) = \beta_2(f(u)) = f(u)$.

(ii) Applying the properties of structure functions for the groupoids V_1 and V_2, we can prove that $f(i_1(x)) = i_2(f(x))$ for all $x \in V_1$.

(iii) Using (i), we have $f_0(u) = \alpha_2(f(u))$ for all $u \in V_{1,0}$. The map f_0 is linear, because it is a composition of linear maps. □

Using Proposition 3.1, we say that $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ is a vector groupoid morphism. If $V_{1,0} = V_{2,0}$ and $f_0 = Id_{V_{1,0}}$, we say that $f : V_1 \to V_2$ is a $V_{1,0}$*-morphism of vector groupoids*.

A vector groupoid morphism $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ is said to be *isomorphism of vector groupoids*, if f (and hence f_0) is a linear isomorphism.

Proposition 3.2. *The pair* $(f, f_0) : (V_1, \alpha_1, \beta_1, m_1, i_1, V_{1,0}) \to (V_2, \alpha_2, \beta_2, m_2, i_2, V_{2,0})$ *where* $f : V_1 \to V_2$ *and* $f_0 : V_{1,0} \to V_{2.0}$ *is a vector groupoid morphism if and only if the following conditions are verified*:

(i) f *and* f_0 *are linear maps*;
(ii) $\alpha_2 \circ f = f_0 \circ \alpha_1$ *and* $\beta_2 \circ f = f_0 \circ \beta_1$;
(iii) $f(m_1(x, y)) = m_2(f(x), f(y))$, *for all* $(x, y) \in V_{1,(2)}$.

Proof. We suppose that (f, f_0) is a vector groupoid morphism. Then, the conditions (i) and (iii) are clearly satisfied. We show that $\beta_2 \circ f = f_0 \circ \beta_1$. For all $x \in V_1$ we have $(x, \beta_1(x)) \in V_{1,(2)}$. Then $(f(x), f(\beta_1(x))) \in V_{2,(2)}$ and $f(x) \cdot f(\beta_1(x)) = f(x \cdot \beta_1(x)) = f(x)$, since f is vector groupoid morphism. Since $f(x) \cdot \beta_2(f(x)) = f(x) \Rightarrow f(x) \cdot f(\beta_1(x)) = f(x) \cdot \beta_2(f(x))$ and so $f(\beta_1(x)) = \beta_2(f(x))$. But $f(\beta_1(x)) = f_0(\beta_1(x))$, since $\beta_1(x) \in V_{1,0}$. Hence $\beta_2(f(x)) = f_0(\beta_1(x))$ for all $x \in V_1$, i.e. $\beta_2 \circ f = f_0 \circ \beta_1$. Similarly, $\alpha_2 \circ f = f_0 \circ \alpha_1$. Therefore, the condition (ii) holds.

Conversely, we suppose that (f, f_0) verify (i), (ii) and (iii). The conditions (i) and (ii)(b) of Definition 3.1 are verified. It remains to prove that the condition (ii)(a) of Definition 3.1 holds. For this, let $(x, y) \in V_{1,(2)}$. Then $\alpha_1(y) = \beta_1(x)$. We have $f_0(\alpha_1(y)) = f_0(\beta_1(x))$, i.e. $(f_0 \circ \alpha_1(y)) = (f_0 \circ \beta_1)(x)$. Applying the hypothesis (ii), it follows that $(\alpha_2 \circ f)(y) = (\beta_2 \circ f)(x)$. Hence $\alpha_2(f(y)) =$

$\beta_2(f(x))$ and so $(f(x), f(y)) \in V_{2,(2)}$. Therefore, the pair (f, f_0) is a vector groupoid morphism. □

Proposition 3.3. *A vector groupoid morphism (f, f_0) is linked with the structure functions by the following commutative diagrams:*

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ V_{1,0} & \xrightarrow[f_0]{} & V_{2,0} \end{array} \qquad \begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \beta_1}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ V_{1,0} & \xrightarrow[f_0]{} & V_{2,0} \end{array} \qquad \begin{array}{ccc} V_{1,(2)} & \xrightarrow{f\times f} & V_{2,(2)} \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m_2} \\ V_1 & \xrightarrow[f]{} & V_2 \end{array} \qquad \begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ V_1 & \xrightarrow[f]{} & V_2 \end{array}$$

where $(f \times f)(x, y) := (f(x), f(y))$, for all $(x, y) \in V_1 \times V_1$. More precisely, the following relations hold:

$$\alpha_2 \circ f = f_0 \circ \alpha_1, \quad \beta_2 \circ f = f_0 \circ \beta_1, \quad m_2 \circ (f \times f) = f \circ m_1, \quad i_2 \circ f = f \circ i_1. \tag{3.1}$$

Proof. We apply the Propositions 3.1 and 3.2. □

Remark 3.1. A morphism of vector groupoids $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ is an isomorphism of vector groupoids if and only if f (and hence f_0) is bijective.

Proposition 3.4. *Let $(f, f_0) : (V_1, \alpha_1, \beta_1, V_{1,0}) \to (V_2, \alpha_2, \beta_2, V_{2,0})$ be a vector groupoid morphism. Then the following assertions hold:*

(i) *$f(\mathrm{Is}(V_1)) \subseteq \mathrm{Is}(V_2)$;*

(ii) *f $f : V_1 \to V_2$ is surjective and $f_0 : V_{1,0} \to V_{2,0}$ is injective (in particular every surjective $V_{1,0}$-morphism of vector groupoids), then (f, f_0) preserves the isotropy vector group bundles i.e. $f(\mathrm{Is}(V_1)) = \mathrm{Is}(V_2)$.*

Proof. (i) Let $x_2 \in f(\mathrm{Is}(V_1))$. Then $x_2 = f(x_1)$ with $x_1 \in \mathrm{Is}(V_1)$ and we have $\alpha_2(x_2) = \alpha_2(f(x_1)) = f_0(\alpha_1(x_1)) = f_0(\beta_1(x_1)) = \beta_2(f(x_1)) = \beta_2(x_2)$, since $\alpha_1(x_1) = \beta_1(x_1)$. Hence, $x_2 \in \mathrm{Is}(V_2)$ and $f(\mathrm{Is}(V_1)) \subseteq \mathrm{Is}(V_2)$.

(ii) Using (i), it suffices to prove that $\mathrm{Is}(V_2) \subseteq f(\mathrm{Is}(V_1))$. For this reason, we take $x_2 \in \mathrm{Is}(V_2)$. Then $\alpha_2(x_2) = \beta_2(x_2)$. Since f is surjective, for $x_2 \in V_2$ there exists $x_1 \in V_1$ such that $x_2 = f(x_1)$. Then, $\alpha_2(f(x_1)) = \beta_2(f(x_1))$ and hence $f_0(\alpha_1(x_1)) = f_0(\beta_1(x_1))$, because f is a vector groupoid morphism. Further, it follows $\alpha_1(x_1) = \beta_1(x_1)$, since f_0 is injective. Hence, $x_1 \in \mathrm{Is}(V_1)$ and $x_2 \in f(\mathrm{Is}(V_1))$. Consequently, it follows that $\mathrm{Is}(V_2) \subseteq f(\mathrm{Is}(V_1))$. □

Definition 3.2. By *homomorphism of vector groupoids*, we mean a morphism of vector groupoids $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ satisfying the following condition:

$$\text{for all } (x, y) \in V_1 \times V_1 \text{ such that } (f(x), f(y)) \in V_{2,(2)} \Rightarrow (x, y) \in V_{1,(2)}.$$

Example 3.1. (i) Let $(V, \alpha, \beta, m, i, V_0)$ be a vector groupoid. We consider the pair vector groupoid $(V_0 \times V_0, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{m}, \widetilde{i}, \Delta_{V_0})$. Then the anchor map $(\alpha, \beta) : V \to V_0 \times V_0$ is a homomorphism of vector groupoids.

(ii) Let V be a vector space. The map $f: V^2 \to V, f(x,y) = x - y$ is a morphism of vector groupoids between pair vector groupoid V^2 and vector groupoid V.

Theorem 3.1. *Let $(V,,V_0)$ be a vector groupoid and W be a nonempty set. For a bijection $f: V \to W$, we define the operations $\boxplus : W \times W \to W$, $\boxtimes : K \times W \to W$, the maps $\overline{\alpha}_f, \overline{\beta}_f : W \to W_0 := f(V_0)$, $\overline{i}_f : W \to W$ and the multiplication $\boxdot : W_{(2)} = \{(x,y) \in W \times W \mid \overline{\alpha}_f(y) = \overline{\beta}_f(x)\} \to W$, given by*

$$x \boxplus y := f(f^{-1}(x) + f^{-1}(y)), \quad k \boxtimes x := f(k \cdot f^{-1}(x)), \quad \text{for all } x,y \in W, k \in K,$$

$$\overline{\alpha}_f := f \circ \alpha \circ f^{-1}, \quad \overline{\beta}_f := f \circ \beta \circ f^{-1}, \quad \overline{i}_f := f \circ i \circ f^{-1},$$

$$x \boxdot y := f(f^{-1}(x) \cdot f^{-1}(y)), \quad \text{for all } x,y \in W_{(2)}.$$

Then $(W, \boxplus, \boxtimes, \overline{\alpha}_f, \overline{\beta}_f, \boxdot, \overline{i}_f, W_0)$ is a vector groupoid isomorphic to $(V, \alpha, \beta, m, i, V_0)$.

Proof. It is easy to verify that W is a vector space over the field K with respect to $\boxplus$, $\boxtimes$. Further, the other conditions of Definition 2.1 are satisfied. Also, the conditions from the definition of a vector groupoid morphism hold. Then $f: V \to W$ is an isomorphism of vector groupoids.

In fact, the structure of vector groupoid on W is obtained by transportation of the structure of vector groupoid from V by the bijection f. □

Remark 3.2. It is well-known that two vector spaces which have the same finite dimension, are isomorphic; but, in the case of vector groupoids, this is not true.

By a direct computation we prove the following proposition.

Proposition 3.5. *Let $(V_j, \alpha_j, \beta_j, m_j, i_j, V_{j,0})$, $j = 1,2,3$ be three vector groupoids. If $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ and $(g, g_0) : (V_2, V_{2,0}) \to (V_3, V_{3,0})$ are vector groupoid morphisms (resp., vector groupoid homomorphisms), then, their composition $(g \circ f, g_0 \circ f_0) : (V_1, V_{1,0}) \to (V_3, V_{3,0})$ is also a vector groupoid morphism (resp., vector groupoid homomorphism).*

Remark 3.3. The set of vector groupoids form a category denoted by *VectGroid*; its morphisms are the morphisms of vector groupoids and its composition law is the composition of vector groupoid morphisms.

4. The Correspondence Theorem for Vector Subgroupoids

Definition 4.1. Let $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ be a morphism of vector groupoids. The sets

$$\mathrm{Ker}(f) := \{x \in V_1 \mid f(x) = 0\} \quad \text{and} \quad \mathrm{Ker}_{vgr}(f) := \{x \in V_1 \mid f(x) \in V_{2,0}\}$$

are called the *kernel* and *groupoid kernel of* f, respectively.

Proposition 4.1. *If $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ is a morphism of vector groupoids, then*

(i) $\mathrm{Ker}(f) \subseteq V_{1,0} \subseteq \mathrm{Ker}_{vgr}(f)$;

(ii) *if f is injective, then* $\mathrm{Ker}_{vgr}(f) = V_{1,0}$.

Proof. (i) This sequence of inclusions follows immediately from definitions.

(ii) If $x \in \mathrm{Ker}_{vgr}(f)$, then $f(x) \in V_{2,0}$ and $\alpha_2(f(x)) = f(x) = f(\alpha_1(x))$. But f being injective, it follows $\alpha_1(x) = x$, and, therefore, $x \in V_{1,0}$. From (i) one obtains the equality desired. □

Remark 4.1. If $\mathrm{Ker}_{vgr}(f) = V_{1,0}$, it does not result that f is injective.

Proposition 4.2. *Let $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ be a vector groupoid homomorphism. For all $x \in V_1$, we have:*

$$f^{-1}(f(x)) = x \cdot \mathrm{Ker}_{vgr}(f)(\beta_1(x)),$$

where $\mathrm{Ker}_{vgr}(f)(\beta_1(x))$ is the isotropy group of $\mathrm{Ker}_{vgr}(f)$ at $\beta_1(x)$.

Proof. If $y \in f^{-1}(f(x))$, then $f(y) = f(x)$. One obtains $[f(x)]^{-1} f(y) = [f(x)]^{-1} f(x)$ or, equivalently, $f(x^{-1}y) = \beta_2(f(x)) = f(\beta_1(x)) \in V_{2,0}$. It follows $x^{-1}y \in \mathrm{Ker}_{vgr}(f)$, i.e. $\exists\ h \in \mathrm{Ker}_{vgr}(f)$ such that $y = xh$. From $y = xh$ it follows $\alpha_1(y) = \alpha_1(x), \beta_1(h) = \beta_1(y)$ and $\alpha_1(h) = \beta_1(x)$. Similarly, from $f(y) = f(x)$ one obtains $y = tx$, with $t \in \mathrm{Ker}_{vgr}(f)$ and $\beta_1(y) = \beta_1(x)$. Therefore $\alpha_1(h) = \beta_1(h) = \beta_1(x)$. So $y \in x \cdot \mathrm{Ker}_{vgr}(f)(\beta_1(x))$. We conclude that $f^{-1}(f(x)) \subset x \cdot \mathrm{Ker}_{vgr}(f)(\beta_1(x))$. In a similar way one proves the inversion inclusion. □

Corollary 4.1. *Let $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ be a vector groupoid homomorphism. For any subset $H \subseteq V_1$, we have*

$$f^{-1}(f(H)) = H \cdot \mathrm{Ker}_{vgr}(f)(\beta_1(H)),$$

where $\mathrm{Ker}_{vgr}(f)(\beta_1(H)) = \bigcup_{h \in H} \mathrm{Ker}_{vgr}(f)(\beta_1(h))$.

Definition 4.2. Let $(V, \alpha, \beta, m, i, V_0)$ be a vector groupoid. A nonempty set $S \subset V$ is called *vector subgroupoid* of V, if:

(i) S is a vector subspace of V;

(ii) S is a subgroupoid of V ([4]), i.e. the following conditions hold:

  (a) for all $x \in S \Rightarrow \alpha(x),\ \beta(x) \in S$ and $x^{-1} \in S$;

  (b) for all $x, y \in S$ such that $\alpha(y) = \beta(x) \Rightarrow xy \in S$.

The units set of S is $\alpha(S) = \beta(S) =: S_0$. It is easy to see that S itself is a vector groupoid under the restrictions of α, β, i and m to S. A vector subgroupoid S of V is called *wide*, if $S_0 = V_0$.

Proposition 4.3. *Let $(V, \alpha, \beta, m, i, V_0)$ be a vector groupoid. Then isotropy group $V(0) = \{x \in V \mid \alpha(x) = \beta(x) = 0\}$ is a vector subgroupoid of V with a single unit.*

Proof. We have $V(0)=\mathrm{Ker}(\alpha)\cap\mathrm{Ker}(\beta)$ and $V(0)$ is a vector subspace of V, since α and β are linear. Also $\alpha(V(0))=\beta(V(0))=\{0\}\subset V(0)$. For any $x\in V(0)$ we have $\alpha(x^{-1})=\beta(x)=0, \beta(x^{-1})=\alpha(x)=0$ and so $x^{-1}\in V(0)$. Let $x,y\in V(0)$ such that $\alpha(y)=\beta(x)$. Then, $\alpha(x\cdot y)=\alpha(x)=0$, $\beta(x\cdot y)=\beta(y)=0$ and so $x\cdot y\in V(0)$. Therefore, the conditions of Definition 4.2 are verified. Hence, $V(0)$ is a vector subgroupoid, called the *isotropy vector subgroupoid at null vector*. □

Definition 4.3. Let (V,α,β,m,i,V_0) be a vector groupoid. A wide vector subgroupoid S of V is called *normal vector subgroupoid* (denoted by $S\trianglelefteq V$), if for any $x\in V$ and any $h\in S$ such that $\alpha(h)=\beta(h)=\beta(x)$, we have $x\cdot h\cdot x^{-1}\in S$.

Example 4.1. Let (V,α,β,m,i,V_0) be a vector groupoid and $u\in V_0$. Then $V(0),V_0$ and V are normal vector subgroupoids in V.

Proposition 4.4. *Let $(f,f_0):(V_1,\alpha_1,\beta_1,V_{1,0})\to(V_2,\alpha_2,\beta_2,V_{2,0})$ be a vector groupoid morphism. Then*

(i) $\mathrm{Ker}(f)$ *is a vector subgroupoid of* V_1.
(ii) $\mathrm{Ker}_{vgr}(f)$ *is a normal vector subgroupoid of* V_1.

Proof. (i) $\mathrm{Ker}(f)$ is a vector subspace, since f is linear. For $x\in\mathrm{Ker}(f)$ we have $f_0(\alpha_1(x))=\alpha_2(f(x))=\alpha_2(0)=0$, i.e. $f(\alpha_1(x))=0$ and so $\alpha_1(x)\in\mathrm{Ker}(f)$. Similarly, $\beta_1(x)\in\mathrm{Ker}(f)$. We have $x+i_1(x)=\alpha_1(x)+\beta_1(x)$, since V_1 is a vector groupoid. Then $i_1(x)=\alpha_1(x)+\beta_1(x)-x\in\mathrm{Ker}(f)$, because $\mathrm{Ker}(f)$ is a vector subspace. Hence, the requirements (ii)(a) of Definition 4.2 are verified.

Let now $x,y\in\mathrm{Ker}(f)$ such that $\alpha_1(y)=\beta_1(x)$. One obtains that $f(x\cdot y)=f(x)\cdot f(y)=0\cdot 0=0$, and so $x\cdot y\in\mathrm{Ker}(f)$. Consequently, the condition (ii)(b) of Definition 4.2 is verified. Hence, $\mathrm{Ker}(f)$ is a vector subgroupoid of V.

(ii)(a) We denote $W:=\mathrm{Ker}_{vgr}(f)$. Firstly we prove that W is a vector subgroupoid. Indeed, for $x,y\in W$ and $a,b\in K$, we have $f(x),f(y)\in V_{2,0}$ and it follows $af(x)+bf(y)\in V_{2,0}$, since $V_{2,0}$ is a vector subspace. Because f is linear, we have $f(ax+by)=af(x)+bf(y)\in V_{2,0}$ and so $ax+by\in W$. Consequently, W is a vector subspace. For any $x\in W$ we have $f(\alpha_1(x))=f_0(\alpha_1(x))=\alpha_2(f(x))=f(x)\in V_{2,0}$, and so $\alpha_1(x)\in W$. Similarly, $\beta_1(x)\in W$. Because f is a vector groupoid morphism and using the equality $x+i_1(x)=\alpha_1(x)+\beta_1(x)$ with $x\in W$, we have

$$\begin{aligned}
& f(x+i_1(x))=f(\alpha_1(x)+\beta_1(x))\\
\Rightarrow\quad & f(x)+f(i_1(x))=f(\alpha_1(x))+f(\beta_1(x))\\
\Rightarrow\quad & f(x)+f(i_1(x))=f_0(\alpha_1(x))+f_0(\beta_1(x))\\
\Rightarrow\quad & f(i_1(x))=\alpha_2(f(x))+\beta_2(f(x))-f(x)\\
\Rightarrow\quad & f(i_1(x))=f(x)\\
\Rightarrow\quad & i_1(x)\in V_{2,0}.
\end{aligned}$$

Let now $x, y \in W$ such that $\alpha_1(y) = \beta_1(x)$. We have $f(\alpha_1(y)) = f(\beta_1(x))$ and it follows that $\alpha_2(f(y)) = \beta_2(f(x))$, i.e. $f(x) = f(y)$ since $f(x), f(y) \in V_{2,0}$. One obtains that $f(x \cdot y) = f(x) \cdot f(x) = f(x) \in V_{2,0}$ and hence $x \cdot y \in W$. Therefore, the requirements from Definition 4.2 are verified. Hence, W is a vector subgroupoid.

(b) We prove that W is a normal vector subgroupoid of V_1. Applying Proposition 4.1(i), we have $V_{1,0} \subseteq \mathrm{Ker}_{vgr}(f)$ and so $V_{1,0} \subseteq W \subseteq V_1$. Then, $\alpha_1(V_{1,0}) \subseteq \alpha_1(W) \subseteq \alpha_1(V_1)$, i.e. $V_{1,0} \subseteq \alpha_1(W) \subseteq V_{1,0}$. Therefore, $\alpha_1(W) = V_{1,0}$. Similarly, $\beta_1(W) = V_{1,0}$. Hence, W is a wide vector subgroupoid. Let $x \in V_1$, $h \in W$ with $\alpha_1(h) = \beta_1(x) = \beta_1(h)$. Then $f(\beta_1(x)) = f(\beta_1(h))$ and it follows $\beta_2(f(x)) = \beta_2(f(h))$. Consequently, $f(h) = \beta_2(f(x))$, since $f(h) \in V_{2,0}$. One obtains $f(x \cdot h \cdot x^{-1}) = f(x) \cdot f(h) \cdot f(x^{-1}) = f(x) \cdot \beta_2(f(x)) \cdot f(x^{-1}) = f(x) \cdot [f(x)]^{-1} = \alpha_2(f(x)) \in V_{2,0}$ and so $x \cdot h \cdot x^{-1} \in W$. Therefore, the conditions of Definition 4.3 are verified. Hence, $W \trianglelefteq V_1$. □

Proposition 4.5. *Let $(V, \alpha, \beta, m, i, V_0)$ be a vector groupoid. Then*:

(i) $\mathrm{Is}(V)$ *is a normal vector subgroupoid of* V.

(ii) $\mathrm{Is}(V) = V_0 \oplus V(0)$, *i.e.* $\mathrm{Is}(V)$ *is the direct sum of* V_0 *and* $V(0)$.

Proof. (i) We have $\mathrm{Is}(V) = \mathrm{Ker}_{vgr}(\alpha, \beta)$, where $(\alpha, \beta) : V \to V_0 \times V_0$ is the anchor map of V. Applying Proposition 4.4(ii), it follows that $\mathrm{Is}(V)$ is a normal vector subgroupoid, since (α, β) is a homomorphism of vector groupoids, see Example 3.1.

(ii) For $x \in \mathrm{Is}(V)$, let $t = x - \alpha(x)$. We have $\alpha(t) = \beta(t) = 0$, since $\alpha(x) \in V_0$ and α, β are linear maps. One obtains $x = \alpha(x) + t$, with $\alpha(x) \in V_0$ and $t \in V(0)$. So $\mathrm{Is}(V) \subset V_0 + V(0)$. Similarly, the inverse inclusion holds. Hence $\mathrm{Is}(V) = V_0 + V(0)$. For $x \in V_0 \cap V(0)$, it follows $\alpha(x) = x = 0$, and so the desired relation is proved. □

Proposition 4.6. *Let $(f, f_0) : (V_1, \alpha_1, \beta_1, V_{1,0}) \to (V_2, \alpha_2, \beta_2, V_{2,0})$ be a vector groupoid homomorphism and $u \in V_{1,0}$. Then:*

(i) *$f(V_1(u))$ is a subgroup of the group $V_2(f_0(u))$.*

(ii) *if f is surjective and f_0 is injective, then $f(V_1(u)) = V_2(f_0(u))$.*

Proof. (i) Let $y \in f(V_1(u))$. Then there is $x \in V_1(u)$ such that $f(x) = y$ and we have $\alpha_1(x) = \beta_1(x) = u$. We obtain that $\alpha_2(y) = \alpha_2(f(x)) = f_0(\alpha_1(x)) = f_0(u)$. Similarly, we prove that $\beta_2(y) = f_0(u)$. Consequently, $y \in V_2(f_0(u))$ and $f(V_1(u)) \subseteq V_2(f_0(u))$.

Let $y_1, y_2 \in f(V_1(u))$ such that the product $y_1 \cdot y_2$ is defined. Then, there are $x_1, x_2 \in V_1(u)$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. But the product $x_1 \cdot x_2$ is defined, because f is a homomorphism of vector groupoids. Also, $x_1 \cdot x_2 \in V_1(u)$, since $\alpha_1(x_1 \cdot x_2) = \alpha_1(x_1) = u$ and $\beta_1(x_1 \cdot x_2) = \beta_1(x_2) = u$.

Therefore, we have $y_1 \cdot y_2 = f(x_1) \cdot f(x_2) = f(x_1 \cdot x_2) \in f(V_1(u))$. Let $y \in f(V_1(u))$ and $y = f(x)$ with $x \in V_1(u)$. We have $x^{-1} = i_1(x) \in V_1(u)$, since $V_1(u)$ is a group. Then, $i_2(y) = i_2(f(x)) = f(i_1(x)) \in f(V_1(u))$. Hence, $f(V_1(u))$ is a subgroup of $V_2(f_0(u))$.

(ii) Using (i), it remains to prove that $V_2(f_0(u)) \subseteq f(V_1(u))$. Indeed, let $y \in V_2(f_0(u))$. Then, $\alpha_2(y) = \beta_2(y) = f_0(u)$. Since f is surjective, it follows $y = f(x)$ for some $x \in V_1$. We have $f_0(\alpha_1(x)) = \alpha_2(f(x)) = \alpha_2(y) = f_0(u)$. From $f_0(\alpha_1(x)) = f_0(u)$ we obtain $\alpha_1(x) = u$, since f_0 is injective. Similarly, $\beta_1(x) = u$. Therefore, $x \in V_1(u)$ and so $y \in f(V_1(u))$. □

We denote by $\mathscr{VS}(V, V_0)$ denotes the set of all vector subgroupoids of (V, V_0).

For a vector groupoid morphism $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ we denote by $\mathscr{VS}_f(V_1, V_{1,0})$, the set of all vector subgroupoids of V_1 which contains $\mathrm{Ker}_{vgr}(f)$, i.e.

$$\mathscr{VS}_f(V_1, V_{1,0}) := \{S \in \mathscr{VS}(V_1, V_{1,0}) \mid \mathrm{Ker}_{vgr}(f) \subseteq S\}.$$

Theorem 4.1 (Correspondence theorem for vector subgroupoids). *Let there be a vector groupoid morphism $(f, f_0) : (V_1, V_{1,0}) \to (V_2, V_{2,0})$ with $N = \mathrm{Ker}_{vgr}(f)$.*

(i) *Then, the following assertions hold:*

  (a) *if S_2 is a vector subgroupoid of V_2, then $f^{-1}(S_2)$ is a vector subgroupoid of V_1. If f is surjective, then $f(f^{-1}(S_2)) = S_2$.*

  (b) *if $W_2 \trianglelefteq S_2$, then $f^{-1}(W_2) \trianglelefteq V_1$.*

(ii) *If (f, f_0) is a vector groupoid homomorphism, then:*

  (a) *if $S_1 \subseteq V_1$ is a vector subgroupoid, then $f(S_1) \subseteq f(V_1)$ is a vector subgroupoid.*

  (b) *if $W_1 \trianglelefteq V_1$, then $f(W_1) \trianglelefteq f(V_1)$.*

(iii) *If (f, f_0) is a surjective vector groupoid homomorphism, then:*

  (a) *$\varphi : \mathscr{VS}_f(V_1, V_{1,0}) \to \mathscr{VS}(V_2, V_{2,0})$ given by $\varphi(S_1) := f(S_1)$ for all $S_1 \in \mathscr{VS}_f(V_1, V_{1,0})$, is a bijective map satisfying the property:*

$$S_1 \subseteq S_1' \Leftrightarrow \varphi(S_1) \subseteq \varphi(S_1'). \tag{4.1}$$

  (b) *φ preserves the normal vector subgroupoids, i.e.*

$$W_1 \trianglelefteq V_1 \Leftrightarrow \varphi(W_1) \trianglelefteq V_2. \tag{4.2}$$

Proof. (i)(a) The set $f^{-1}(S_2)$ is a vector subspace in V_1, because S_2 is a vector subspace and f is linear. Let any $x \in f^{-1}(S_2)$. Then $f(x) \in S_2$. It follows $\alpha_2(f(x)), \beta_2(f(x)) \in S_2$ and $i_2(f(x)) \in S_2$, since S_2 is a subgroupoid. Since f is a groupoid morphism, we have $f(\alpha_1(x)) = f_0(\alpha_1(x)) = \alpha_2(f(x)) \in S_2, f(\beta_1(x)) = f_0(\beta_1(x)) = \beta_2(f(x)) \in S_2$ and $f(i_1(x)) = i_2(f(x)) \in S_2$. Then $\alpha_1(x), \beta_1(x), i_1(x) \in f^{-1}(S_2)$. Hence, the conditions (ii)(a) of Definition 4.2 are verified.

Let $x, y \in f^{-1}(S_2)$ such that $x \cdot y$ is defined in V_1. Then, $f(x), f(y) \in S_2$ and we have $f(x) \cdot f(y) \in S_2$, since S_2 is a subgroupoid. It follows that $f(x \cdot y) = f(x) \cdot f(y) \in S_2$, since f is groupoid morphism. Therefore, $x \cdot y \in f^{-1}(S_2)$, and the condition (ii)(b) of Definition 4.2 holds. Hence $f^{-1}(S_2)$ is a vector subgroupoid. Because f is surjective, one can prove that $f(f^{-1}(S_2)) = S_2$.

(i)(b) Let $W_2 \trianglelefteq V_2$. Applying (i)(b), we have that $f^{-1}(W_2)$ is a vector subgroupoid in V_1, since W_2 is a vector subgroupoid in V_2. We have that $\alpha_2(W_2) = \beta_2(W_2) = V_{2,0}$, since W_2 is a wide subgroupoid. Using the hypothesis, we can verify $\alpha_1(f^{-1}(W_2)) = \beta_1(f^{-1}(W_2)) = V_{1,0}$, and so $f^{-1}(W_2)$ is a wide subgroupoid.

Let now $a \in V_1$ and $x \in f^{-1}(W_2)$ such that the product $a \cdot x \cdot a^{-1}$ is defined in V_1. Then, $\alpha_1(x) = \beta_1(x) = \beta_1(a)$. It follows that $f_0(\alpha_1(x)) = f_0(\beta_1(x)) = f_0(\beta_1(a))$ and hence $\alpha_2(f(x)) = \beta_2(f(x)) = \beta_2(f(a))$. Therefore, the product $f(a) \cdot f(x) \cdot (f(a))^{-1}$ is defined in V_2. Hence, for $f(a) \in V_2$ and $f(x) \in W_2$ we have $f(a) \cdot f(x) \cdot (f(a))^{-1} \in W_2$, since $W_2 \trianglelefteq V_2$. Also, we have $f(a \cdot x \cdot a^{-1}) = f(a) \cdot f(x) \cdot (f(a))^{-1} \in W_2$, since f is groupoid morphism. It follows that $a \cdot x \cdot a^{-1} \in f^{-1}(W_2)$. Hence, the conditions from Definition 4.3 are satisfied. Therefore $f^{-1}(W_2) \trianglelefteq V_1$.

(ii) $\mathrm{Im}(f) = f(V_1)$ is a subgroupoid in V_2 because (f, f_0) is a vector groupoid homomorphism and it is a vector subspace. Then, $f(V_1) \subseteq V_2$ is a vector subgroupoid.

(a) Let S_1 be a vector subgroupoid of V_1. Clearly, $f(S_1)$ is a vector subspace in $f(V_1)$, since S_1 is a vector subspace. For $y \in f(S_1), \exists x \in S_1$ such that $y = f(x)$. Then, $\alpha_2(y) = \alpha_2(f(x)) = f_0(\alpha_1(x)) = f(\alpha_1(x)) \in f(S_1)$, since $\alpha_1(x) \in S_1$. Similarly, we can verify that $\beta_2(y), i_2(y) \in f(S_1)$.

Let $y_1, y_2 \in f(S_1)$ such that $y_1 \cdot y_2$ is defined in V_2, i.e. $\beta_2(y_1) = \alpha_2(y_2)$. Then, $\exists x_1, x_2 \in S_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. From $\beta_2(y_1) = \alpha_2(y_2)$, it follows that $\beta_2(f(x_1) = \alpha_2(f(x_2)$ and so $(f(x_1), f(x_2)) \in V_{2(2)}$. We have $(x_1, x_2) \in V_{1(2)}$, since f is groupoid homomorphism. Also $x_1 \cdot x_2 \in S_1$, because S_1 is a subgroupoid. Then, $y_1 \cdot y_2 = f(x_1) \cdot f(x_2) = f(x_1 \cdot x_2) \in f(S_1)$. Hence $f(S_1)$ is a vector subgroupoid.

(b) Let $W_1 \trianglelefteq V_1$. Applying (ii)(a), we have that $f(W_1)$ is a vector subgroupoid in $f(V_1)$, since W_1 is a vector subgroupoid in V_1. Also, $\alpha_1(W_1) = \beta_1(W_1) = V_{1,0}$, since W_1 is a wide subgroupoid. Using the hypothesis, we can verify that $\alpha_2(f(W_1)) = \beta_1(f(W_1)) = \widetilde{W}_{2,0}$,where $\widetilde{W}_{2,0}$ is the unit set of the groupoid $f(V_1)$. Consequently, $f(W_1)$ is a wide subgroupoid. Let $b \in f(V_1)$ and $y \in f(W_1)$ with $\alpha_2(y) = \beta_2(y) = \beta_2(b)$. There are $a \in V_1$ and $x \in W_1$ such that $f(a) = b$ and $f(x) = y$. Then, $\alpha_2(f(x)) = \beta_2(yf(x)) = \beta_2(f(a))$ and it follows that $f(a) \cdot f(x) \cdot (f(a))^{-1}$ is defined. One obtains that $a \cdot x \cdot a^{-1}$ is defined in $V_{1,0}$, since

f is a groupoid homomorphism. It follows that $a\cdot x\cdot a^{-1}\in W_1$, since $W_1\trianglelefteq V_1$. Then $b\cdot y\cdot b^{-1}=f(a)\cdot f(x)\cdot (f(a))^{-1}=f(a\cdot x\cdot a^{-1})\in f(W_1)$. Hence $f(W_1)\trianglelefteq f(V_1)$.

(iii) Suppose that (f,f_0) is a surjective vector groupoid homomorphism.

(iii)(a) By (ii)(a), for any $S_1\in\mathcal{VS}_f(V_1,V_{1,0})$ we have $f(S_1)\in\mathcal{VS}(V_2,V_{2,0})$. Consider now a vector subgroupoid $S_2\in\mathcal{VS}(V_2,V_{2,0})$. By (i)(a) we have that $f^{-1}(S_2)$ is a vector subgroupoid in V_1. It is easy to verify that $N\subseteq f^{-1}(S_2)$. Hence, $f^{-1}(S_2)\in\mathcal{VS}_f(V_1,V_{1,0})$. Therefore, we can define the map $\psi:\mathcal{VS}(V_2,V_{2,0})\to\mathcal{VS}_f(V_1,V_{1,0})$ given by $\psi(S_2):=f^{-1}(S_2)$ for all $S_2\in\mathcal{VS}(V_2,V_{2,0})$.

Then, $(\psi\circ\varphi)(S_1)=\psi(\varphi(S_1))=f^{-1}(f(S_1))$, for all $S_1\in\mathcal{VS}_f(V_1,V_{1,0})$. We have

$$\begin{aligned} x\in f^{-1}(f(S_1)) &\Leftrightarrow f(x)\in f(S_1)\\ &\Leftrightarrow \exists\; a\in S_1,\ f(x)=f(a)\\ &\Leftrightarrow \exists\; a\in S_1,\ f(a^{-1}x)\in V_{2,0}\\ &\Leftrightarrow \exists a\in S_1, a^{-1}x\in N\\ &\Leftrightarrow x\in S_1N. \end{aligned}$$

Therefore, $f^{-1}(f(S_1))=S_1N=NS_1$, since $N\trianglelefteq V_1$. From the hypothesis $N\subseteq S_1$ one obtains that $f^{-1}(f(S_1))=S_1$. Hence

(a) $\quad (\psi\circ\varphi)(S_1)=S_1,\quad$ for all $S_1\in\mathcal{VS}_f(V_1,V_{1,0})$.

Also,we have $(\varphi\circ\psi)(S_2)=\varphi(\psi(S_2))=f(f^{-1}(S_2))$. Applying now (i)(a), it follows that $f(f^{-1}(S_2))=S_2$, since f is surjective. Hence

(b) $\quad (\varphi\circ\psi)(S_2)=S_2,\quad$ for all $S_2\in\mathcal{VS}(V_2,V_{2,0})$.

From (a) and (b) it follows that φ is bijective and $\psi=\varphi^{-1}$. The equivalence (4.1) holds, since φ and ψ are increasing maps.

(iii)(b) Applying (ii)(b) and (i)(b), one obtains the equivalence (4.2). □

Corollary 4.2. *Let $(f,f_0):(V_1,V_{1,0})\to(V_2,V_{2,0})$ be a vector groupoid homomorphism. Then, $f(\mathrm{Is}(V_1))$ is a normal vector subgroupoid of $f(V_1)$.*

Proof. We apply Proposition 4.5(i) and Theorem 4.1(ii)(a). □

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Received June 11, 2012
Accepted September 17, 2012