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A Class of Univalent Functions with Negative Coefficients defined by Hadamard Product with Komatu Integral Operator

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Abstract In our paper, we study a class $SRA(\lambda,\beta,\alpha,\mu,\theta)$ which consists of analytic and univalent functions with negative coefficients in the open unit disk $\mathbb{U}=\{z:|z|<1\}$ defined by Hadamard product (or convolution) with Komatu integral operator, we obtain coefficient bounds and exterior points for this class. Also Distortion Theorem using Fractional Calculus techniques and some results for this class are obtained.

1. Introduction

Let $\mathscr{R}$ denote the class of functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n, \quad a_n\ge 0,\ n\in\mathbb{N}=\{1,2,3,\dots\} \tag{1.1}$$

which are *analytic* and *univalent* in the unit disk $\mathbb{U}=\{z:|z|<1\}$. If $f\in\mathscr{R}$ is given by (1.1) and $g\in\mathscr{R}$ is given by,

$$g(z)=z-\sum_{n=2}^{\infty}b_nz^n, \quad b_n\ge 0$$

then the Hadamard product $f*g$ of f and g is defined by

$$(f*g)(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z). \tag{1.2}$$

Recently, T.N. Shanmugam and C. Ramachandran [3] have studied the certain sub class of the class $\mathscr{A}$ for which the Komatu Integral Transform has some properties.

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Lemma 1.1. *The Komatu integral operator of $f \in \mathscr{R}$ for $0 \leq \theta \leq 1$, $0 \leq \mu \leq 1$ is denoted by I_μ^θ and defined as following:*

$$I_\mu^\theta(f(z)) = \frac{(\mu+1)^\theta}{z^\mu \Gamma(\theta)} \int_0^z \left(\log \frac{z}{t}\right)^{\theta-1} f(t) t^{\mu-1} dt$$
$$= z - \sum_{n=2}^{\infty} K(\theta,\mu,n) a_n z^n \tag{1.3}$$

where $K(\theta,\mu,n) = \left(\frac{\mu+1}{n+\mu}\right)^\theta$.

Proof.

$$I_\mu^\theta(f(z)) = \frac{(\mu+1)^\theta}{z^\mu \Gamma(\theta)} \int_0^z \left(\log \frac{z}{t}\right)^{\theta-1} f(t) t^{\mu-1} dt$$
$$= z - \sum_{n=2}^{\infty} K(\theta,\mu,n) a_n z^n. \qquad \square$$

Definition 1.1. *A function $f \in \mathscr{R}$, $z \in \mathbb{U}$ is said to be in the class $SRA(\lambda,\beta,\alpha,\mu,\theta)$ if and only if the following inequality is satisfied:*

$$Re\left\{\frac{z(I_\mu^\theta((f*g)(z)))' + \lambda z^2 (I_\mu^\theta((f*g)(z)))''}{(1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z (I_\mu^\theta((f*g)(z)))'}\right\}$$
$$\geq \beta \left| \frac{z(I_\mu^\theta((f*g)(z)))' + \lambda z^2 (I_\mu^\theta((f*g)(z)))''}{(1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z (I_\mu^\theta((f*g)(z)))'} - 1 \right| + \alpha \tag{1.4}$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $\beta \geq 0$, $z \in \mathbb{U}$, $0 \leq \mu < 1$, $0 < \theta \leq 1$ and $g \in \mathscr{R}$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0.$$

Lemma 1.2. *Let $w = u + iv$. Then $Re(w) \geq \sigma$ if and only if $|w-(1+\sigma)| \leq |w+(1-\sigma)|$.*

Proof. If $Re(w) \geq \sigma$, then

$$\begin{aligned}
|w-1-\sigma|^2 &= [w-(1+\sigma)][\overline{w}-(1+\sigma)] \\
&= |w|^2 - 2(1+\sigma)Re(w) + (1+\sigma)^2 \\
&= |w|^2 - 2Re(w) + (1-\sigma)^2 + 4\sigma \\
&\leq |w|^2 - 2\sigma + (1-\sigma)^2 + 4\sigma \\
&\leq |w|^2 - (2\sigma)Re(w) + (1-\sigma)^2 + 2Re(w) \\
&= |w|^2 + 2Re(w)(1-\sigma) + (1-\sigma)^2 \\
&= |w+1-\sigma|^2.
\end{aligned}$$

Hence,

$$|w-(1+\sigma)| \leq |w+(1-\sigma)|\,.$$

Conversely, if

$$|w-(1+\sigma)| \leq |w+(1-\sigma)|$$

then

$$\begin{aligned}
|(u+iv-1-\sigma)| &\leq |(u+iv+1-\sigma)| \\
(u-1-\sigma)^2+v^2 &\leq (u+1-\sigma)^2+v^2 \\
-4u+4\sigma &\leq 0 \\
u-\sigma &\geq 0 \\
u &\geq \sigma\,.
\end{aligned}$$

Hence,

$$Re(w) \geq \sigma\,. \qquad \square$$

Lemma 1.3 ([1]). *Let $w = u+iv$ and $\sigma \geq 0$, γ is a real number. Then $Re(w) > \sigma|w-1+\gamma|$ if and only if $Re[w(1+\sigma e^{i\phi})-\sigma e^{i\phi}] > \gamma$.*

We aim to study the Coefficient bounds, Extreme points, Application of Fractional calculus and Hadamard product of the class $SRA(\lambda,\beta,\alpha,\mu,\theta)$.

2. Coefficient Bounds and Extreme Points

We obtain the necessary and sufficient condition and extreme points for the functions $f(z)$ in the class $SRA(\lambda,\beta,\alpha,\mu,\theta)$.

Theorem 2.1. *The function $f(z)$ defined by equation (1.1) is in the class $SRA(\lambda,\beta,\alpha,\mu,\theta)$ if and only if*

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(\theta,\mu,n)a_nb_n \leq 1-\alpha\,. \tag{2.1}$$

Proof. From the definition, we have

$$\begin{aligned}
&Re\left\{\frac{z(I_\mu^\theta((f*g)(z)))'+\lambda z^2(I_\mu^\theta((f*g)(z)))''}{(1-\lambda)I_\mu^\theta((f*g)(z))+\lambda z(I_\mu^\theta((f*g)(z)))'}\right\} \\
&\quad\geq \beta\left|\frac{z(I_\mu^\theta((f*g)(z)))'+\lambda z^2(I_\mu^\theta((f*g)(z)))''}{(1-\lambda)I_\mu^\theta((f*g)(z))+\lambda z(I_\mu^\theta((f*g)(z)))'}-1\right|+\alpha\,.
\end{aligned}$$

From Lemma 1.3, we have

$$Re\left\{\frac{z(I_\mu^\theta((f*g)(z)))'+\lambda z^2(I_\mu^\theta((f*g)(z)))''}{(1-\lambda)I_\mu^\theta((f*g)(z))+\lambda z(I_\mu^\theta((f*g)(z)))'}(1+\beta e^{i\phi})-\beta e^{i\phi}\right\} \geq \alpha,$$

$-\pi \leq \phi \leq \pi$, or equivalently,

$$Re\left\{\frac{zI_\mu^\theta((f*g)(z))' + \lambda z^2(I_\mu^\theta((f*g)(z)))''(1+\beta e^{i\phi})}{(1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z(I_\mu^\theta((f*g)(z)))'}\right.$$
$$\left. - \frac{\beta e^{i\phi}(1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z(I_\mu^\theta((f*g)(z)))'}{(1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z(I_\mu^\theta((f*g)(z)))'}\right\} \geq \alpha\,. \tag{2.2}$$

Let

$$F(z) = [z(I_\mu^\theta((f*g)(z)))' + \lambda z^2(I_\mu^\theta((f*g)(z)))''](1+\beta e^{i\phi})$$
$$- \beta e^{i\phi}[(1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z(I_\mu^\theta((f*g)(z)))']$$

and

$$E(z) = (1-\lambda)I_\mu^\theta((f*g)(z)) + \lambda z(I_\mu^\theta(f*g)(z))'.$$

By Lemma 1.2, and equation (2.2) is equivalent to $|F(z) + (1-\alpha)E(z)| \geq |F(z) - (1+\alpha)E(z)|$, for $0 \leq \alpha < 1$. But

$$|F(Z) + (1-\alpha)E(Z)|$$
$$= \left|(2-\alpha)z - \sum_{n=2}^{\infty}[n + n\lambda(n-1) + (1-\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n z^n\right.$$
$$\left. - \beta e^{i\phi}\sum_{n=2}^{\infty}[n + n\lambda(n-1) - (1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n z^n\right|$$
$$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}[n + \lambda(n-1) + (1-\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n |z|^n$$
$$- \beta\sum_{n=2}^{\infty}[n + \lambda n(n-2) - 1 + \lambda]K(n,\mu,\theta)a_n b_n |z|^n\,.$$

Now,

$$|F(Z) - (1+\alpha)E(Z)|$$
$$= \left|-\alpha z - \sum_{n=2}^{\infty}[n + n\lambda(n-1) - (1+\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n z^n\right.$$
$$\left. - \beta e^{i\phi}\sum_{n=2}^{\infty}[n + n\lambda(n-1) - (1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n z^n\right|$$
$$\leq \alpha|z| + \sum_{n=2}^{\infty}[n + n\lambda(n-1) - (1+\alpha)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n |z|^n$$
$$+ \beta\sum_{n=2}^{\infty}[n + n\lambda(n-1) - (1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n |z|^n\,.$$

Hence,

$$\begin{aligned}
&|F(z)+(1-\alpha)E(z)|-|F(z)-(1+\alpha)E(z)| \\
&\quad \geq 2(1-\alpha)|z|-\sum_{n=2}^{\infty}[(2n+2n(n-1)\lambda-2\alpha(1-\lambda+n\lambda) \\
&\qquad -\beta(2n+2n\lambda)(n-1)-2(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n|z|^n \\
&\quad \geq 0
\end{aligned}$$

or

$$\sum_{n=2}^{\infty}[n(1+\beta)+n\lambda(n-1)(1+\beta)-(1-\lambda+n\lambda)(\alpha+\beta)]K(n,\mu,\theta)a_n b_n \leq 1-\alpha$$

which is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)a_n b_n \leq 1-\alpha\,.$$

Conversely suppose that the equation(2.1) holds good, then we have to prove that

$$Re\left\{\frac{zI_\mu^\theta((f*g)(z))'+\lambda z^2(I_\mu^\theta((f*g)(z)))''(1+\beta e^{i\phi})}{(1-\lambda)I_\mu^\theta((f*g)(z))+\lambda z(I_\mu^\theta((f*g)(z)))'}\right.$$
$$\left.-\frac{\beta e^{i\phi}(1-\lambda)I_\mu^\theta((f*g)(z))+\lambda z(I_\mu^\theta((f*g)(z)))'}{(1-\lambda)I_\mu^\theta((f*g)(z))+\lambda z(I_\mu^\theta((f*g)(z)))'}\right\}\geq\alpha\,.$$

Now choosing the value of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$Re\left\{\frac{\begin{pmatrix}(1-\alpha)-\sum_{n=2}^{\infty}[n(1+\beta e^{i\phi})(1-\lambda+n\lambda)\\ -(\alpha+\beta e^{i\phi})(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n r^{n-1}\end{pmatrix}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n r^{n-1}}\right\}\geq 0\,.$$

Since $Re(e^{-i\phi}) \geq -|e^{i\phi}| = -1$, the above inequality reduces to

$$Re\left\{\frac{\begin{pmatrix}(1-\alpha)-\sum_{n=2}^{\infty}[n(1+\beta)(1-\lambda+n\lambda)\\ -(\alpha+\beta)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n b_n r^{n-1}\end{pmatrix}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n r^{n-1}}\right\}\geq 0\,.$$

Letting $r \to 1^-$, we get the desired result. Hence the proof. □

Corollary 2.1. *If $f \in SRA(\lambda,\beta,\alpha,\mu,\theta)$, then*

$$a_n \leq \frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n},$$

where $0 \leq \alpha < 1,\ \beta \geq 0,\ 0 \leq \lambda \leq 1,\ 0 \leq \mu < 1,\ 0 < \theta \leq 1$.

Theorem 2.2. *If $f_1(z)=z$ and*

$$f_n(z)=z-\frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)}z^n,$$

where $n\geq 2$, $n\in\mathbb{N}$, $0\leq\alpha<1$, $\beta\geq 0$, $0\leq\lambda\leq 1$, $0\leq\mu<1$, $0<\theta\leq 1$. Then $f\in SRA(\lambda,\beta,\alpha,\mu,\theta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=2}^{\infty}\sigma_n f_n(z),$$

where $\sigma_n\geq 0$ and $\sum_{n=2}^{\infty}\sigma_n=1$ or $1=\sigma_1+\sum_{n=2}^{\infty}\sigma_n$.

Proof. let $f(z)=\sum_{n=2}^{\infty}\sigma_n f_n(z)$, where $\sigma_n\geq 0$ and $\sum_{n=2}^{\infty}\sigma_n=1$ or $1=\sigma_1+\sum_{n=2}^{\infty}\sigma_n$. Then

$$f(z)=z-\frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)}\sigma_n z^n.$$

But

$$\begin{aligned} f(z)&=\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)}{1-\alpha}b_n\right\}\\ &\qquad\times\left\{\frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)}\sigma_n\right\}\\ &=\sum_{n=2}^{\infty}\sigma_n\\ &=1-\sigma_1\leq 1\qquad\text{(from Theorem 2.1).}\end{aligned}$$

Using Theorem 2.1, we have $f\in SRA(\lambda,\beta,\alpha,\mu,\theta)$. Conversely, let us assume that $f(z)$ of the form (1.1) belongs to $SRA(\lambda,\beta,\alpha,\mu,\theta)$. Then

$$a_n\leq\frac{1-\alpha}{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n},\quad n\in\mathbb{N},\ n\geq 2.$$

Setting

$$\sigma_n=\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)a_n b_n}{1-\alpha}$$

and

$$\sigma_1=1-\sum_{n=2}^{\infty}\sigma_n$$

we have

$$f(z)=\sum_{n=2}^{\infty}\sigma_n f_n(z)=\sigma_1 f_1(z)+\sum_{n=2}^{\infty}\sigma_n f_n(z).$$

Hence the proof. □

3. **Hadamard Product**

Theorem 3.1. *Let*

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \quad \textit{and} \quad g(z)=z-\sum_{n=2}^{\infty}b_nz^n$$

belongs to $SRA(\lambda,\beta,\alpha,\mu,\theta)$. *Then the Hadamard Product of* $f(z)$ *and* $g(z)$ *given by*

$$(f*g)(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n \textit{ belongs to } SRA(\lambda,\beta,\alpha,\mu,\theta).$$

Proof. Since $f(z)$ and $g(z)$ belongs to $SRA(\lambda,\beta,\alpha,\mu,\theta)$, we have

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n}{1-\alpha}\right\}a_n\le 1$$

and

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)a_n}{1-\alpha}\right\}b_n\le 1$$

and by applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)\sqrt{a_nb_n}}{1-\alpha}\right\}\sqrt{a_nb_n}\\ &\le\left(\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)b_n}{1-\alpha}\right\}a_n\right)^{\frac{1}{2}}\\ &\quad\times\left(\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)a_n}{1-\alpha}\right\}b_n\right)^{\frac{1}{2}}.\end{aligned}$$

However, we obtain

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)\sqrt{a_nb_n}}{1-\alpha}\right\}\sqrt{a_nb_n}\le 1\,.$$

Now, we have to prove that

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)}{1-\alpha}\right\}a_nb_n\le 1\,.$$

Since

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)}{1-\alpha}\right\}a_nb_n\\ &=\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)(n(1+\beta)-(\beta+\alpha))K(n,\mu,\theta)\sqrt{a_nb_n}}{1-\alpha}\right\}\sqrt{a_nb_n}\;.\end{aligned}$$

Hence the proof. □

4. Application of the Fractional Calculus

Various operators of fractional calculus (i.e. fractional derivative and fractional integral) have been rather extensively studied by many researchers ([4, 5, 6]). Each of these theorems would involve certain operator of fractional calculus which are defined as follows ([2]).

Definition 4.1. *The fractional integral operator of order δ is defined, for a function $f(z)$, by*

$$D_z^{\delta}(f(z)) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}}\, dt, \quad \delta > 0 \tag{4.1}$$

where $f(z)$ is analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be read when $(z-t) > 0$.

Definition 4.2. *The fractional derivative of order δ is defined for a function $f(z)$ by*

$$D_z^{\delta}(f(z)) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}}\, dt, \quad 0 \le \delta < 1 \tag{4.2}$$

where $f(z)$ is analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be read when $(z-t) > 0$.

Definition 4.3. *The fractional derivative of order $k+\delta$ is defined by*

$$D_z^{k+\delta}(f(z)) = \frac{d^k}{dz^k} D_z^{\delta} f(z), \quad 0 \le \delta < 1. \tag{4.3}$$

From Definition 4.1 and 4.2, after a simple computation we obtain

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta}, \tag{4.4}$$

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}. \tag{4.5}$$

Now using equations (4.4) and (4.5). Let us prove the following theorems:

Theorem 4.1. *Let $f \in SRA(\lambda, \beta, \alpha, \mu, \theta)$. Then*

$$|D_z^{-\delta} f(z)| \le \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[1 + \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2} |z| \right], \tag{4.6}$$

$$|D_z^{-\delta} f(z)| \ge \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[1 - \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2} |z| \right]. \tag{4.7}$$

The inequalities (4.6) and (4.7) are attained for the function f given by

$$f(z) = z - \frac{1-\alpha}{(1+\alpha)(2+\beta-\alpha)(\theta+1)b_2} z^2. \tag{4.8}$$

Proof. From Theorem 2.1, we obtain

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2} \,. \tag{4.9}$$

Using equation (4.4), we obtain

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z) = z - \sum_{n=2}^{\infty} l(n,\delta)a_n z^n \tag{4.10}$$

such that

$$l(n,\delta) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}, \quad n \geq 2$$

where $l(n,\delta)$ is a decreasing function of n and $0 < l(n,\delta) \leq (2,\delta) = \dfrac{2}{2+\delta}$.

Using equations (4.9) and (4.10), we obtain

$$\begin{aligned} |\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| &\leq |z| + l(2,\delta)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{2(1-\alpha)}{(2+\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|^2, \end{aligned}$$

which is an equation (4.6).
Similarly we can get equation (4.7). □

Theorem 4.2. *Let* $f \in SRA(\lambda,\beta,\alpha,\mu,\theta)$. *Then*

$$|D_z^{\delta}f(z)| \leq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1 + \frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|\right] \tag{4.11}$$

and

$$|D_z^{\delta}f(z)| \geq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1 - \frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2+\beta-\alpha)(\theta+1)b_2}|z|\right]. \tag{4.12}$$

The inequalities (4.11) *and* (4.12) *are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{1-\alpha}{(1+\alpha)(2+\beta-\alpha)(\theta+1)b_2}z^2 \,. \tag{4.13}$$

Proof. From equation (4.5), we obtain

$$\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z) = z - \sum_{n=2}^{\infty} \zeta(n,\delta)a_n z^n$$

such that

$$\zeta(n,\delta) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}, \quad n \geq 2 \tag{4.14}$$

where $\zeta(n,\delta)$ is a decreasing function and $0 < \zeta(n,\delta) \leq \zeta(2,\delta) = \dfrac{2}{2-\delta}$. Using equations (4.9) and (4.14), we obtain

$$\begin{aligned}|\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)| &\leq |z| + \zeta(2,\delta)|z|^2\sum_{n=2}^{\infty}a_n \\ &\leq |z| + \frac{2(1-\alpha)}{(2-\delta)(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|^2\end{aligned}$$

which is nothing but equation (4.11).
Similarly we can get equation (4.12). □

Corollary 4.1. *For every $f \in SRA(\lambda,\beta,\alpha,\mu,\theta)$, we have*

$$\begin{aligned}&\frac{|z|^2}{2}\left[1-\frac{2(1-\alpha)}{3(1+\lambda)(2-\beta+\alpha)(1+\theta)b_2}|z|\right] \\ &\leq \left|\int_0^z f(t)dt\right| \\ &\leq \frac{|z|^2}{2}\left[1+\frac{2(1-\alpha)}{3(1+\lambda)(2-\beta+\alpha)(1+\theta)b_2}|z|\right]\end{aligned}$$

and

$$\begin{aligned}&|z|\left[1-\frac{1-\alpha}{(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right] \\ &\leq |f(z)| \\ &\leq |z|\left[1+\frac{1-\alpha}{(1+\lambda)(2-\beta+\alpha)(\theta+1)b_2}|z|\right].\end{aligned}$$

Proof. By Definition 4.1 and Theorem 4.1 for $\delta = 1$, we have $D_z^{-}1f(z) = \int_0^z f(t)dt$, the result is true. Also by Definition 4.2 and Theorem 4.2 for $\delta = 0$, we have

$$D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)dt = f(z).$$

Hence the result is true. □

References

[1] E.S. Aqlan, *Some Problems Connected with Geometric Function Theory*, Ph.D. Thesis, Pune University, Pune (unpublished), 2004.

[2] S. Owa, On the distortion theorems, *Kyungpook Math. J.* **18**(1978), 53–59.

[3] T.N. Shanmugam and C. Ramachandran, Komatu integral transforms of analytic functions subordinate to convex functions, *AJMAA* **7**(2007), 1–9.

[4] H.M. Srivatsava and R.G. Buschmann, *Convolution Integral Equation with Special function Kernels*, John Wiley and Sons, New York — London — Sydney — Toronto, 1977.

[5] H.M. Srivatsava and S. Owa, An application of the fractional derivative, *Math. Japan* **29**(1984), 384–389.

[6] H.M. Srivatsava and S. Owa (editors), *Univalent Functions, Fractional Calculus and Their Applications*, Halsted Press (Ellis Harwood Limited, Chichester), John Wiley and Sons, New York — Chichester — Brisbane — Toronto, 1989.

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