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Research Article

The Combination of Bernstein Polynomials with Positive Functions Based on a Positive Parameter s

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Abstract. This paper deals with a sequence of the combination of Bernstein polynomials with a positive function τ and based on a parameter $s > -\frac{1}{2}$. These polynomials have preserved the functions 1 and τ. First, the convergence theorem for this sequence is studied for a function $f \in C[0,1]$. Next, the rate of convergence theorem for these polynomials is descript by using the first, second modulus of continuous and Ditzian-Totik modulus of smoothness. Also, the Quantitative Voronovskaja and Grüss-Voronovskaja are obtained. Finally, two numerical examples are given for these polynomials by chosen a test function $f \in C[0,1]$ and two functions for τ to show that the effect of the different values of s and the different chosen functions τ.

Keywords. Bernstein polynomials, Modulus of smoothness, Quantitative Voronovskaja, Grüss-Voronovskaja

Mathematics Subject Classification (2020). 41A10, 41A25, 41A36

1. Introduction

For a function $f \in C[0,1]$, the well-known Bernstein polynomials [1] are defined as:

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1.1}$$

where $b_n(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $x \in [0,1]$.

Voronovskaya ([11], [3]) showed that the convergence of $B_n(f;x)$ to $f(x)$ as $n\to\infty$ is slow but sure

$$\lim_{n\to\infty} n\{B_n(f;x)-f(x)\} = \frac{x(1-x)}{n} f''(x).$$

King [5] introduced a new modification of the sequence Bernstein polynomials for $f \to B_n(f)\circ r_n(x)$ for $r_n(x)\in C[0,1]$ which is preserved two functions 1 and x^2 as follows:

$$V_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}(r_n(x))^k(1-r_n(x))^{n-k} f\left(\frac{k}{n}\right), \tag{1.2}$$

where

$$r_n(x)=\begin{cases} x^2, & n=1,\\ -\frac{1}{2(n-1)}+\sqrt{\left(\frac{n}{n-1}\right)x^2+\frac{1}{4(n-1)^2}}, & n=2,3,\dots . \end{cases}$$

For the parameter $s>-1/2$, Pallini [9] introduced another modification of Bernstein polynomials as:

$$B_{n,s}(f;x)=\sum_{k=0}^{n} b_{n,k}(x) f\left(x+\frac{n^{-1}k-x}{n^{-s}}\right). \tag{1.3}$$

For a given function $\tau:[0,1]\to\mathbb{R}$, $\mathbb{R}$ is the real field, with the properties $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$, $\forall\ x\in[0,1]$. Gonska *et al.* [4] are modified the King sequence by considered the following combinations of B_n and τ:

$$B_{n,\tau}f = B_n f\circ(B_n\tau)^{-1}\circ\tau, \qquad f\in C[0,1]$$

which preserved 1 and τ. Their definition leads to the following sequence:

$$B_{n,\tau}(f;x)=\sum_{k=0}^{n} b_{n,k,\tau}(x) f(x) \tag{1.4}$$

and $b_{n,k,\tau}(x)=b_{n,k}(\tau(x))$.

Mohiuddine *et al.* [7] are dealt with the genuine Bernstein-Durrmeyer polynomials which preserved certain functions. Also, they are presented Quantitative and Grüss-Voronovskaja [8] as:

$$U_n^{\tau}(f;x)=(n-1)\sum_{k=1}^{n-1} b_{n,k,\tau}(x)\int_0^1 (f\circ\tau^{-1})(t) b_{n-1,k-1}(t)dt$$
$$+b_{n,0,\tau}(x)(f\circ\tau^{-1})(0)+b_{n,n,\tau}(x)(f\circ\tau^{-1})(1). \tag{1.5}$$

For α and β two parameters such that $0\le\alpha\le\beta$, Srivastava *et al.* [10] construct Stancu-type of λ-Bernstein operators

$$B^{\lambda}_{n,\alpha,\beta}(f;x)=\sum_{k=0}^{n} \tilde{b}_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right). \tag{1.6}$$

This paper define and study for a given function τ and a parameter $s>-\frac{1}{2}$ the following sequence:

$$B_{n,\tau,s}(f;x)=\sum_{k=0}^{n} b_{n,k,\tau}(x)(f\circ\tau^{-1})\left(\frac{n^{-1}k-\tau(x)}{n^s}+\tau(x)\right). \tag{1.7}$$

First, the convergence of $B_{n,\tau,s}(f,x)$ to the function f is shown. Then, the rate of convergence of $B_{n,\tau,s}(f;x)$ in terms of the first, second-order modulus of continuity and Ditzian-Totik modulus of smoothness are studied. Next, the Quantitative and Grüss-Voronovskaja are discussed. Finally, two numerical examples are given for these polynomials by taking a test function $f \in C[0,1]$ and two functions for τ to show that the effect of the different values of s and the different functions τ on the polynomials $B_{n,\tau,s}(f;x)$.

2. Primary Results

This section has introduced some fundamentals that help in the proofs of the coming results of this work.

Definition 2.1 ([2])**.** Let $h \in \mathbb{R}$, the k-th order of modulus of smoothness is given:

$$\omega_k(f;h) = \sup_{\substack{|\delta|\le h \\ x+j\delta\in I}} \left\{ \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j\delta) \right| \right\}.$$

The classical k-th order modulus of smoothness has the following properties:
If $f \in C^q[0,\ 1]$, then for all $0 < h < \frac{1}{2}$ there are functions $v \in C^{q+2}[0,1]$ such that

(i) $\|f^{(q)} - v^{(q)}\| \le \frac{3}{4}\omega_2(f^{(q)};h)$;

(ii) $\|v^{(q+1)}\| \le \frac{5}{h}\omega_1(f^{(q)};h)$;

(iii) $\|v^{(q+2)}\| \le \frac{3}{2h^2}\omega_2(f^{(q)};h)$.

Lemma 2.1 ([6,8])**.** *For $e_i(t) = t^i$, $i = 0,1,2$, the following properties are:*

(i) $B_{n,\tau}(e_0;x) = 1$*;*

(ii) $B_{n,\tau}(e_1;x) = \tau(x)$*;*

(iii) $B_{n,\tau}(e_2;x) = \left(1-\frac{1}{n}\right)\tau^2(x) + \frac{\tau(x)}{n}$.

Lemma 2.2. *The properties to $B_{n,\tau,s}(e_i(\tau);x)$ are:*

(i) $B_{n,\tau,s}(e_0;x) = 1$*;*

(ii) $B_{n,\tau,s}(e_1;x) = \tau(x)$*;*

(iii) $B_{n,\tau,s}(e_2;x) = \tau^2(x) - \frac{\tau^2(x)}{n^{2s+1}} + \frac{\tau(x)}{n^{2s+1}}$.

Proof. By using the properties of Bernstein polynomials, it easy to prove. □

Definition 2.2. For $m \in \mathbb{N}^0 := \{0,1,2,\dots\}$. The m-th order moments for $B_{n,\tau,s}(f;x)$ is defined as:

$$T_{n,m,\tau,s}(x) = \frac{B_{n,\tau,s}((\tau(t)-\tau(x))^m;x)}{n^{ms}}.$$

Lemma 2.3. *For the function $T_{n,m,\tau,s}(x)$, one has:*

(i) $T_{n,0,\tau,s}(x) = 1$*;*

(ii) $T_{n,1,\tau,s}(x) = 0$*;*

(iii) $T_{n,2,\tau,s}(x) = \frac{\tau(x)(1-\tau(x))}{n^{2s+1}}$;

(iv) $T_{n,4,\tau,s}(x) = \frac{(3n-6)\tau^4(x)-(6n-12)\tau^3(x)+(7n-7)\tau^2(x)+\tau(x)}{n^{4s+3}}$.

Proof. From the properties in Lemma 2.2, one gets the results above. □

Lemma 2.4. *For $n \in N$, we calculate $\frac{T_{n,4,\tau,s}(x)}{T_{n,2,\tau,s}(x)} \le \frac{3}{n^{2s+1}}$.*

Proof. For $x \in [0,\ 1]$ and using Lemma 2.2, one has

$$\begin{aligned}\frac{T_{n,4,\tau,s}(x)}{T_{n,2,\tau,s}(x)} &= \frac{(3n-6)\tau^4(x)-(6n-12)\tau^3(x)+(3n-7)\tau^2(x)+\tau(x)}{n^{4s+3}} \frac{n^{2s+1}}{\tau(x)(1-\tau(x))}\\ &= \frac{3(n-2)\tau(\tau-1)-3(n-2)\tau}{n^{2s+2}}\\ &\le \frac{3(n-2)}{n^{2s+2}}\\ &\le \frac{3}{n^{2s+1}}.\end{aligned}$$

□

Lemma 2.5. *For $f \in C^2[0,1]$ and $x \in [0,1]$. Then $|B_{n,\tau,s}(f,x)| \le \|f\|$, where $\|\cdot\|$ is the sup-norm on $[0,1]$.*

Proof. By using $B_{n,\tau,s}(1,x) = 1$, one has that

$$\begin{aligned}|B_{n,\tau,s}(f,x)| &\le \|f \circ \tau^{-1}\| B_{n,\tau,s}(1,x)\\ &= \|f\|.\end{aligned}$$

□

3. The Main Results

This section is proved some approximation properties for modulus of smoothness for $B_{n,\tau,s}(f,x)$.

Theorem 3.1. *For $f \in C^2[0,1]$ and $x \in [0,1]$. Then one has*

$$|B_{n,\tau,s}(f,x) - f(x)| \le \frac{\tau(x)(\tau(x)-1)}{n^{2s+1}} \left(\frac{\|f''\|}{\alpha^2} + \frac{\|f'\|\,\|\tau''\|}{\alpha^3} \right).$$

Proof. By Taylor's expansion formula:

$$f(t) = (f \circ \tau^{-1})\tau(x) + (\tau(t)-\tau(x))(f \circ \tau^{-1})'\tau(x) + \int_{\tau(x)}^{\tau(t)} (f \circ \tau^{-1})'(u)(\tau(t)-u)du$$

and, since

$$\begin{aligned}&\int_{\tau(x)}^{\tau(t)} (f \circ \tau^{-1})'(u)(\tau(t)-u)du\\ &\quad = \int_{\tau(x)}^{\tau(t)} (\tau(t)-u) \frac{f''(\tau^{-1}(u))}{(\tau'(\tau^{-1}(u)))^2} du - \int_{\tau(x)}^{\tau(t)} (\tau(t)-u) \frac{f'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{(\tau'(\tau^{-1}(u)))^3} du\,.\end{aligned}$$

So, one has,

$$|B_{n,\tau,s}(f,x)| = f(x) + B_{n,\tau,s}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t)-u) \frac{f''(\tau^{-1}(u))}{(\tau'(\tau^{-1}(u)))^2} du; x \right)$$

$$-B_{n,\tau,s}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{f'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{(\tau'(\tau^{-1}(u)))^3}du;x\right).$$

Put $\alpha^2=(\tau'(\tau^{-1}(u)))^2$, $\alpha^3=(\tau'(\tau^{-1}(u)))^3$, then

$$\begin{aligned}|B_{n,\tau,s}(f,x)-f(x)|&\le B_{n,\tau,s}((\tau(t)-\tau(x))^2;x)\left(\frac{\|f''\|}{\alpha^2}+\frac{\|f'\|\,\|\tau''\|}{\alpha^3}\right)\\&\le\frac{\tau(x)(\tau(x)-1)}{n^{2s+1}}\left(\frac{\|f''\|}{\alpha^2}+\frac{\|f'\|\,\|\tau''\|}{\alpha^3}\right).\end{aligned}$$

□

Theorem 3.2. *For $f\in C^1[0,1]$, $x\in[0,1]$*

$$|B_{n,\tau,s}(f,x)-f(x)|\le\rho_n(x)\omega_1((f\circ\tau^{-1});\rho_n(x)).$$

Proof. Using the Taylor's expansion

$$f(t)=(f\circ\tau^{-1})\tau(x)+(\tau(t)-\tau(x))(f\circ\tau^{-1})'\tau(x)+\int_{\tau(x)}^{\tau(t)}(f\circ\tau^{-1})'(u)(\tau(t)-u)du$$

and applying $B_{n,\tau,s}(f,x)$, one gets

$$|B_{n,\tau,s}(f,x)-f(x)|=B_{n,\tau,s}\left(\int_{\tau(x)}^{\tau(t)}(f\circ\tau^{-1})'(u)(\tau(t)-u)du.\right).$$

By the modulus of continuity

$$\begin{aligned}|B_{n,\tau,s}(f,x)-f(x)|&\le\sqrt{T_{n,,2,\tau,s}(x)}\left\{1+\frac{1}{h}\sqrt{T_{n,,2,\tau,s}(x)}\right\}\omega_1((f\circ\tau^{-1});h)\\&\le\sqrt{T_{n,,2,\tau,s}(x)}\omega_1((f\circ\tau^{-1});h)+\sqrt{T_{n,,2,\tau,s}(x)}\omega_1((f\circ\tau^{-1});h)\\&\le2\sqrt{T_{n,,2,\tau,s}(x)}\omega_1((f\circ\tau^{-1});\sqrt{T_{n,,2,\tau,s}(x)})\\&=2\rho_n(x)\omega_1((f\circ\tau^{-1});\rho_n(x)).\end{aligned}$$

□

Theorem 3.3. *Let $f\in C^2[0,1]$ then the polynomials $B_{n,\tau,s}(f,x)$ are verify:*

$$|B_{n,\tau,s}(f,x)-f(x)|\le\frac{3}{2}\omega_2((f\circ\tau^{-1},h)+\frac{\tau(x)(\tau(x)-1)}{n^{2s+1}}\left(\frac{3}{2h^2\alpha^2}\omega_2((f\circ\tau^{-1},h)+\frac{5\|\tau''\|}{h\alpha^3}\omega_1((f\circ\tau^{-1},h)\right).$$

Proof.

$$\begin{aligned}|B_{n,\tau,s}(f,x)-f(x)|&=|B_{n,\tau,s}(f,x)-B_{n,\tau,s}(v,x)+B_{n,\tau,s}(v,x)-v(x)+v(x)-f(x)|\\&\le|B_{n,\tau,s}(f-v,x)|+|B_{n,\tau,s}(v,x)-v(x)|+|v(x)-f(x)|.\end{aligned}$$

Applying Theorem 3.1,

$$\begin{aligned}&\le2\|f-v\|+\frac{\tau(x)(\tau(x)-1)}{n^{2s+1}}\left(\frac{\|v''\|}{\alpha^2}+\frac{\|v'\|\,\|\tau'\|}{\alpha^3}\right)\\&\le\frac{3}{2}\omega_2((f\circ\tau^{-1},h)+\frac{\tau(x)(\tau(x)-1)}{n^{2s+1}}\\&\quad\cdot\left(\frac{3}{2h^2\alpha^2}\omega_2((f\circ\tau^{-1},h)+\frac{5\|\tau''\|}{h\alpha^3}\omega_1((f\circ\tau^{-1},h)\right).\end{aligned}$$

□

4. Voronovskaja-Type Theorem

Voronovskaja is the most important theorem in approximation theory to pointwise convergence result, in this work Taylor's expansion formula will be dependent to prove the result.

Theorem 4.1.

$$\left|B_{n,\tau,s}(f,x)-f(x)-\frac{\tau(x)(\tau(x)-1)}{n^{2s+1}}(f\circ\tau^{1})''\tau(x)\right|$$
$$\le\frac{1}{2}T_{n,2,\tau,s}(x)\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{3}{n^{2s+1}}}\right).$$

Proof. By quantitative Voronovskaja theorem [4]

$$\mathscr{L}_n:C[0,1]\to C[0,1]$$

then

$$\left|\mathscr{L}_{n,}(f,x)-f(x)-f'(x)\mu_{n,1}(x)-\frac{1}{2}f''(x)\mu_{n,2}(x)\right|$$
$$\le\frac{1}{2}\mu_{n,2}(x)\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right),$$

by applied $B_{n,\tau,s}(f,x)$ then

$$\left|B_{n,\tau,s}(f,x)-f(x)-f'(x)T_{n,1,\tau,s}(x)-\frac{1}{2}f''(x)T_{n,2,\tau,s}(x)\right|$$
$$\le\frac{1}{2}T_{n,2,\tau,s}(x)\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{T_{n,4,\tau,s}(x)}{T_{n,2,\tau,s}(x)}}\right),$$
$$\left|B_{n,\tau,s}(f,x)-f(x)-\frac{1}{2}f''(x)\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\right|$$
$$\le\frac{1}{2}T_{n,2,\tau,s}(x)\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{3}{n^{2s+1}}}\right).$$

□

Theorem 4.2. *Let f,g be two functions such that $f,g\in C^2[0,1]$*

$$\left|B_{n,\tau,s}(fg;x)-B_{n,\tau,s}(f;x)B_{n,\tau,s}(g;x)-T_{n,2,\tau,s}(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}\right|$$
$$\le\frac{1}{2}T_{n,2,\tau,s}(x)\breve{\omega}((fg\circ\tau^{-1})'';t_n)$$
$$+\|g\|\breve{\omega}((fg\circ\tau^{-1})'';t_n)+\|f\|\breve{\omega}((fg\circ\tau^{-1})'';t_n)+\frac{1}{4}T_{n,2,\tau,s}(x).$$

Proof.

$$B_{n,\tau,s}(fg;x)-B_{n,\tau,s}(f;x)B_{n,\tau,s}(g;x)-T_{n,2,\tau,s}(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}$$
$$=B_{n,\tau,s}(fg;x)-(fg)(x)-\frac{1}{2}T_{n,2,\tau,s}(x)(fg\circ\tau^{-1})''\tau(x)$$

$$-f(x)\left[B_{n,\tau,s}(g;x)-g(x)-\frac{1}{2}T_{n,2,\tau,s}(x)(g\circ\tau^{1})''\tau(x)\right]$$
$$-g(x)\left[B_{n,\tau,s}(f;x)-f(x)-\frac{1}{2}T_{n,2,\tau,s}(x)(f\circ\tau^{1})''\tau(x)\right]$$
$$+[f(x)-B_{n,\tau,s}(f;x)][B_{n,\tau,s}(g;x)-g(x)]$$
$$=|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|,$$
$$|\gamma_1|\le\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\breve{\omega}\left((fg\circ\tau^{1})'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right),$$
$$|\gamma_2|\le\|f\|\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\breve{\omega}\left((g\circ\tau^{1})'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right),$$
$$|\gamma_3|\le\|g\|\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\breve{\omega}\left((f\circ\tau^{1})'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right).$$

Since

$$\begin{aligned}|B_{n,\tau,s}(f;x)-f(x)|&\le\frac{1}{2}B_{n,\tau,s}(f\circ\tau^{1})''\tau(\epsilon)(\tau(t)-\tau(x))^2;x)\\&\le\frac{1}{2}\|f\circ\tau^{1}\|T_{n,2,\tau,s}(x)\\&\le\frac{\tau(x)(1-\tau(x))}{2n^{2s+1}}\|f\circ\tau^{1}\|\\&:=I_n(f,x).\end{aligned}$$

$$\left|B_{n,\tau,s}(fg;x)-B_{n,\tau,s}(f;x)B_{n,\tau,s}(g;x)-T_{n,2,\tau,s}(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}\right|$$
$$\le\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\breve{\omega}\left((fg\circ\tau^{1})'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right)+\|f\|\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\breve{\omega}\left((g\circ\tau^{1})'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right)$$
$$+\|g\|\frac{\tau(x)(1-\tau(x))}{n^{2s+1}}\breve{\omega}\left((f\circ\tau^{1})'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right)+I_n(f,x)\cdot I_n(g,x).$$ □

Corollary 4.1. *For* $f\in C^1[0,1]$

$$|B_{n,\tau,s}(f,x)-f(x)|\le\frac{2\tau(x)(1-\tau(x))}{\sqrt{n}}\omega_1\left((f\circ\tau^{-1});\frac{2\tau(x)(1-\tau(x))}{\sqrt{n}}\right).$$

5. Numerical Result

This section shows that the effect of the approximation by the numerical example, the polynomials $B_{n,\tau,s}(f,x)$ will application by given $s=\{0,0.5,1\}$ and two different functions $\tau(x)$ to see the faster convergence for the same test function.

Example 5.1. Let $f=\sqrt{x}\cos(10x)$, $\tau(x)=\frac{2x^2+x}{3}$, the polynomials $B_{n,x,0}(f,x)$, $B_{n,\tau,0.3}(f,x)$ and $B_{n,\tau,1}(f,x)$ given in Figure 1, when $n=8$ and Figure 2, when $n=15$.

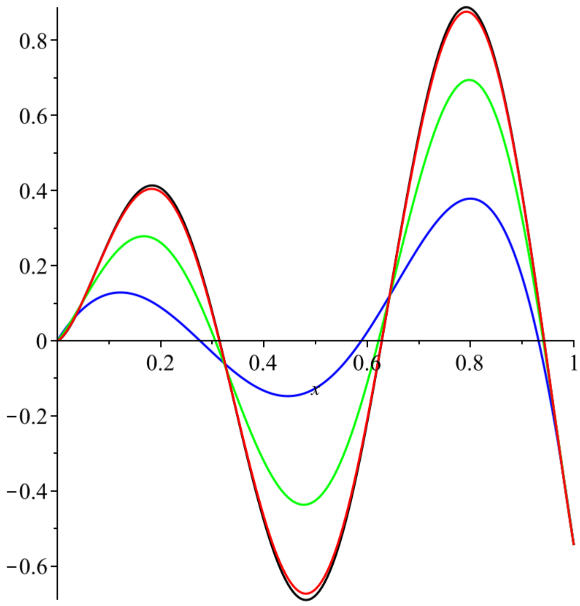

Figure 1. When $n = 8$, black$= f(x)$,
green$= B_{n,x,0.3}(f,x)$,
blue$= B_{n,x,0}(f,x)$,
red$=B_{n,x,1}(f,x)$

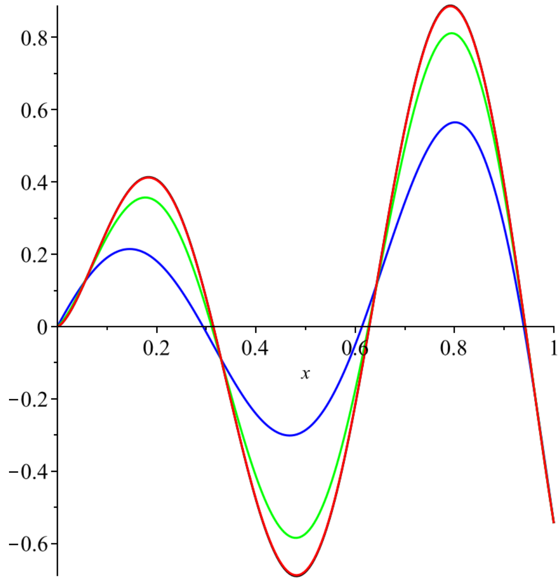

Figure 2. When $n = 15$, black$= f(x)$,
green$= B_{n,x,0.3}(f,x)$,
blue$= B_{n,x,0}(f,x)$,
red$= B_{n,x,1}(f,x)$

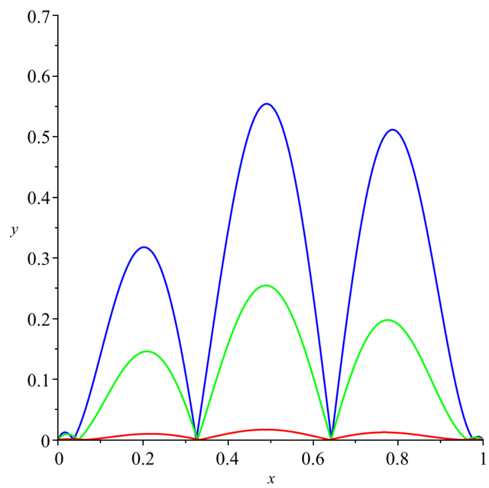

Figure 3. The error when $n = 20$,
blue$= B_{n,x,0}(f,x)$,
green$= B_{n,x,0.3}(f,x)$,
red$= B_{n,x,1}(f,x)$

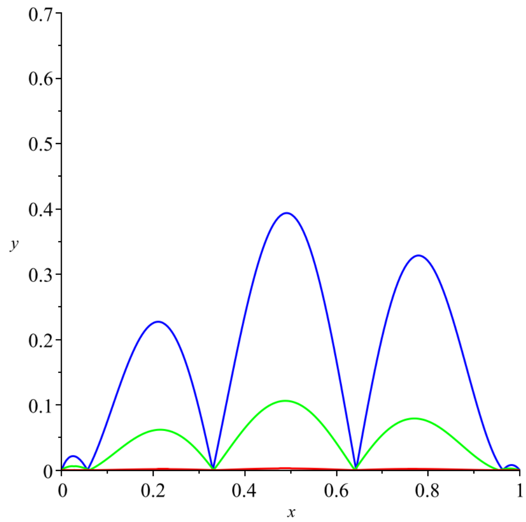

Figure 4. The error when $n = 80$,
blue$= B_{n,x,0}(f,x)$,
green$= B_{n,x,0.3}(f,x)$,
red$= B_{n,x,1}(f,x)$

The error will be interpreted by the graph in Figures 3 and 4.

Example 5.2. Let $f = \sqrt{x}\cos(10x)$, $\tau(x) = \sin\left(\frac{\pi}{2}x\right)$, the error of the convergent polynomials $B_{n,x,0}(f,x)$, $B_{n,\tau,0.3}(f,x)$ and $B_{n,\tau,1}(f,x)$ given in Figure 4, Figure 5 when $n = 8$, $n = 15$, respectively.

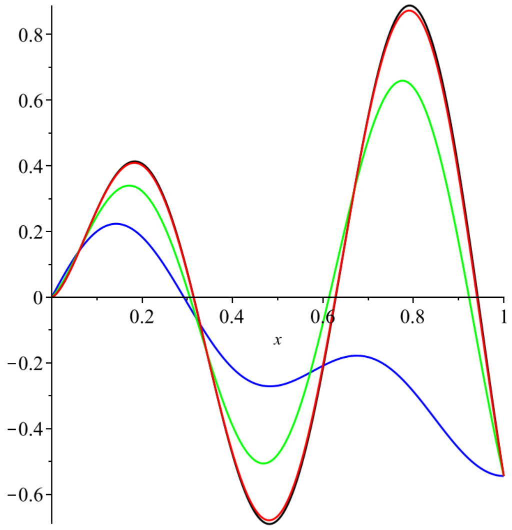

Figure 5. When $n = 20$, black$= f(x)$, green$= B_{n,x,0.3}(f,x)$, blue$= B_{n,x,0}(f,x)$, red$= B_{n,x,1}(f,x)$

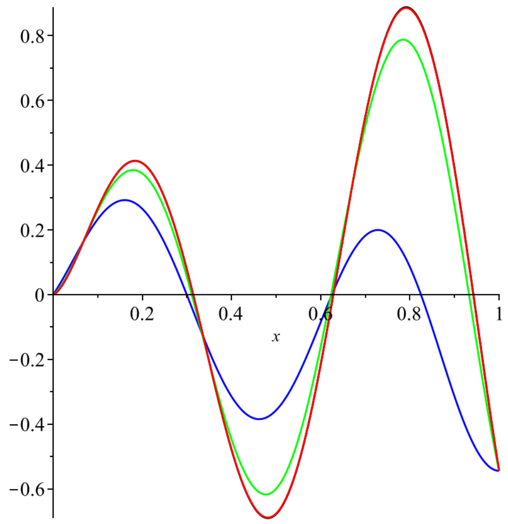

Figure 6. When $n = 80$, black$= f(x)$, green$= B_{n,x,0.3}(f,x)$, blue$= B_{n,x,0}(f,x)$, red$= B_{n,x,1}(f,x)$

Figures 7 and 8 explain the error graphically.

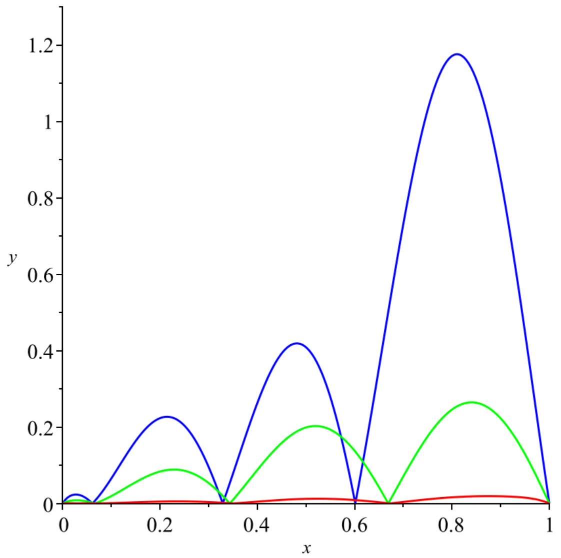

Figure 7. The error when $n = 8$, blue$= B_{n,x,0}(f,x)$, green$= B_{n,x,0.3}(f,x)$, red$= B_{n,x,1}(f,x)$

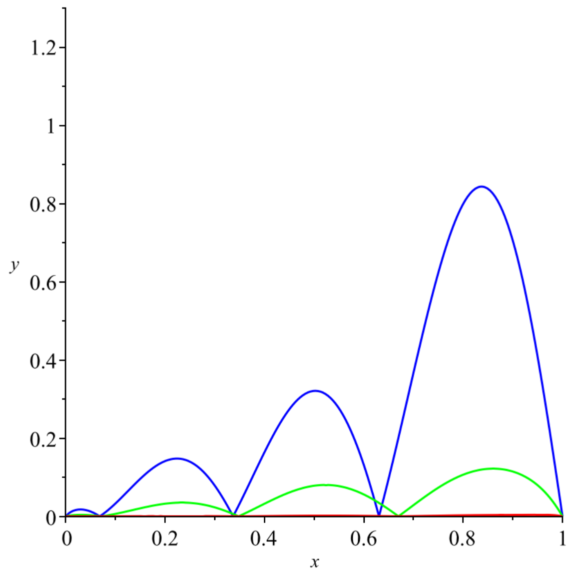

Figure 8. The error when $n = 15$, blue$= B_{n,x,0}(f,x)$, green$= B_{n,x,0.3}(f,x)$, red$= B_{n,x,1}(f,x)$

6. Conclusion

The theoretical study of this paper has constructed the polynomials $B_{n,\tau,s}(f,x)$ which are drove from Bernstein polynomials. Also, the rate of convergence of these polynomials has been studied and given by the rate of convergence of the modulus of smoothness. Next, the pointwise convergence properties have discussed. On the second hand, a numerical study of these polynomials has been done by given two numerical examples which have explained that these polynomials have faster and more accurate than Bernstein polynomials whenever $s>0$ is increasing.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités, *Communications of the Kharkov Mathematical Society* **XIII** (1912-13), 1 – 2, URL: http://nonagon.org/ExLibris/bernsteins-demonstration-du-theoreme-de-Weierstrass.

[2] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer Series in Computational Mathematics, Vol. **9**, Springer, New York, NY (1987), DOI: 10.1007/978-1-4612-4778-4.

[3] S. G. Gal and H. Gonska, Grüss and Grüss-Voronovskaya-type estimates for some Bernstein-type polynomials of real and complex variables, *Jaen Journal on Approximation* **7**(1) (2015), 97 – 122, URL: http://www.ujaen.es/revista/jja/pdf/pre/jja-0007-01-15-4.pdf.

[4] H. Gonska, P. Piţul and I. Raşa, General King-type operators, *Results in Mathematics* **53**(3-4) (2009), 279 – 286, DOI: 10.1007/s00025-008-0338-9.

[5] J. P. King, Positive linear operators which preserve x^2, *Acta Mathematica Hungarica* **99**(3) (2003), 203 – 208, DOI: 10.1023/a:1024571126455.

[6] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publishing Corp., Delhi, India (1960).

[7] S. A. Mohiuddine, T. Acar and M. A. Alghamdi, Genuine modified Bernstein-Durrmeyer operators, *Journal of Inequalities and Applications* **2018** (2018), Article number: 104, DOI: 10.1186/s13660-018-1693-z.

[8] G. Moldovan, On certain operators of Bernstein type (Romanian), *Studia Universitatis Babeș-Bolyai Mathematica-Physica* **14**(2) (1969), 59 – 64, URL: http://www.cs.ubbcluj.ro/~studia-m/old_issues/subbmath_1969_14_02.pdf.

[9] A. Pallini, Bernstein-type approximation of smooth functions, *Statistica* **65**(2) (2005), 169 – 191, DOI: 10.6092/issn.1973-2201/84.

[10] H. M. Srivastava, F. Özger and S. A. Mohiuddine, Construction of stancu-type Bernstein operators based on Bézier Bases with shape parameter λ, *Symmetry* **361** (2019), 11, DOI: 10.3390/sym11030316.

[11] E. Voronovaskaja, Détermination de la forme asymptotique d'approximation des functions par les polynômes de S.N. Bernstein, *C.R. Acad. Sci., USSR* (1932), 79 – 85.