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Research Article

Spherical Cubic Bi-Ideals of Gamma Near-Ring

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Abstract. The purpose of the article is to study about spherical cubic sets and spherical cubic bi-ideals of Gamma near-ring $\mathcal{R}$. We define spherical internal and external cubic sets and their properties. We discuss P-order and R-order, P-union, P-intersection, R-union and R-intersection of spherical cubic sets. We define spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$ and prove that P-union, P-intersection, R-union and R-intersection of spherical cubic bi-ideals of Gamma near-ring $\mathcal{R}$ are also spherical cubic bi-ideals of Gamma near-ring $\mathcal{R}$.

Keywords. Spherical set, Cubic set, Gamma near-ring, Bi-ideal

Mathematics Subject Classification (2020). 16Y30; 03E72; 16D25

1. Introduction

The notion of fuzzy set was introduced by Zadeh [16] in 1965. It is identified as a better tool for the scientific study of uncertainty, and came as a boost to the researchers working in the field of uncertainty. Many extensions and generalizations of fuzzy set was conceived by a number of researchers and a large number of real-life applications were developed in a variety of areas. In addition to this, parallel analysis of the classical results of many branches of Mathematics were also carried out in the fuzzy settings. It was initiated by Rosenfeld [13], who coined the idea of fuzzy subgroup of a group in 1971 and studied some basic properties of this structure.Properties of fuzzy ideals in near-rings was studied by Hong *et al.* [5]. Fuzzy ideals in Gamma near-ring $\mathcal{R}$ was discussed by Jun *et al.* [7,8] and gamma near-rings studied by Satyanarayana [1]. Meenakumari and Chelvam [12] have defined fuzzy bi-ideals in gamma near-rings and

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established some properties of this structure. Srinivas and Nagaiah [15] have proved some results on T-fuzzy ideals of Γ-near-rings. Jun [9] introduced a new notion called a cubic set and investigated several properties. Kahraman and Gundogdu [4] introduced spherical fuzzy sets as an extension of picture fuzzy sets. Debabrata Mandal [11] defined the characterizations of semi-rings by their cubic ideals. Jana *et al*. [6] discussed different types of cubic ideals in BCI-algebras based on fuzzy points. Chinnadurai *et al*. [2,3] discussed spherical fuzzy and spherical interval-valued fuzzy bi-ideals of gamma near-rings. In this research work, we introduce the notion of spherical cubic bi-ideal of Gamma near-ring $\mathcal{R}$, establish some of its properties and study the relationship between bi-ideal and spherical cubic bi-ideal of Gamma near-ring $\mathcal{R}$.

2. Preliminaries

In this section we present some definitions which are used in this research.

Let $\mathcal{R}$ be a near-ring and Γ be a non-empty set such that $\mathcal{R}$ is a Gamma near-ring.

A subgroup H of $(\mathcal{R},+)$ is a bi-ideal if and only if $H\Gamma\mathcal{R}\Gamma H \subseteq H$.

Let $\mathcal{R}$ be a nonempty set. By a cubic set in $\mathcal{R}$ we mean a structure $\mathscr{A} = \{u, A(u), \lambda(u) \mid u \in \mathcal{R}\}$ in which A is an interval-valued fuzzy set in $\mathcal{R}$ and λ is a fuzzy set in $\mathcal{R}$. A cubic set is simply denoted by $\mathscr{A} = \langle A, \lambda\rangle$.

A spherical fuzzy set $\widetilde{A}_s$ of the universe of discourse U is given by,

$$\widetilde{A}_s = \{u, (\widetilde{\mu}(u), \widetilde{\nu}(u), \widetilde{\xi}(u)) \mid u \in U\},$$

where $\widetilde{\mu}(u) : U \to [0,1]$, $\widetilde{\nu}(u) : U \to [0,1]$ and $\widetilde{\xi}(u) : U \to [0,1]$ and $0 \le \widetilde{\mu}^2(u) + \widetilde{\nu}^2(u) + \widetilde{\xi}^2(u) \le 1$, $u \in U$.

For each u, the numbers $\widetilde{\mu}(u)$, $\widetilde{\nu}(u)$ and $\widetilde{\xi}(u)$ are the degrees of membership, non-membership and hesitancy of u to $\widetilde{A}_s$, respectively.

A spherical fuzzy set $A_s = (\mu, \nu, \xi)$, where $\mu : \mathcal{R} \to [0,1]$, $\nu : \mathcal{R} \to [0,1]$ and $\xi : \mathcal{R} \to [0,1]$ of $\mathcal{R}$ is said to be a spherical fuzzy bi-ideal of $\mathcal{R}$ if the following conditions are satisfied:

(i) $\mu(u-v) \ge \min\{\mu(u), \mu(v)\}$,

(ii) $\nu(u-v) \ge \min\{\nu(u), \nu(v)\}$,

(iii) $\xi(u-v) \le \max\{\xi(u), \xi(v)\}$,

(iv) $\mu(u\alpha v\beta w) \ge \min\{\mu(u), \mu(w)\}$,

(v) $\nu(u\alpha v\beta w) \ge \min\{\nu(u), \nu(w)\}$,

(vi) $\xi(u\alpha v\beta w) \le \max\{\xi(u), \xi(w)\}$,

for all $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

3. Spherical Cubic Set

Definition 3.1. Let $\mathcal{R}$ be a non-empty set. A spherical cubic set in $\mathcal{R}$ is defined by $\mathscr{CU}_s = \{\langle u, \mathscr{A}_s(u), \mu(u)\rangle, \langle u, \mathscr{B}_s(u), \nu(u)\rangle, \langle u, \mathscr{C}_s(u), \xi(u)\rangle \mid u \in \mathcal{R}\}$, where $\mathscr{A}_s$, $\mathscr{B}_s$, $\mathscr{C}_s$ are interval-valued spherical sets in $\mathcal{R}$ and μ, ν, ξ are spherical fuzzy sets in $\mathcal{R}$.

A spherical cubic set $\mathscr{CU}_s = \{\langle u, \mathscr{A}_s(u), \mu(u)\rangle, \langle u, \mathscr{B}_s(u), \nu(u)\rangle, \langle u, \mathscr{C}_s(u), \xi(u)\rangle \mid u \in \mathcal{R}\}$ is simply denoted by $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$.

Definition 3.2. Let $\mathcal{R}$ be a non-empty set. A spherical cubic set $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ in $\mathcal{R}$ is said to be a spherical internal cubic set if

$$\mathscr{A}_s^-(u) \le \mu(u) \le \mathscr{A}_s^+(u),\ \mathscr{B}_s^-(u) \le \nu(u) \le \mathscr{B}_s^+(u) \text{ and } \mathscr{C}_s^-(u) \le \xi(u) \le \mathscr{C}_s^+(u),$$

for all $u \in \mathcal{R}$.

Example 3.3. Let $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ be a spherical cubic set in $\mathcal{R}$. If

$$\mathscr{A}_s(u) = [0.4, 0.8] \text{ and } \mu(u) = 0.6,\ \mathscr{B}_s(u) = [0.3, 0.5] \text{ and } \nu(u) = 0.4, \text{ and}$$
$$\mathscr{C}_s(u) = [0.2, 0.6] \text{ and } \xi(u) = 0.4,$$

for all $u \in \mathcal{R}$, then $\mathscr{CU}_s$ is a spherical internal cubic set in $\mathcal{R}$.

Definition 3.4. Let $\mathcal{R}$ be a non-empty set. A spherical cubic set $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ in $\mathcal{R}$ is said to be a spherical external cubic set if

$$\mu(u) \notin (\mathscr{A}_s^-(u), \mathscr{A}_s^+(u)),\ \nu(u) \notin (\mathscr{B}_s^-(u), \mathscr{B}_s^+(u)) \text{ and } \xi(u) \notin (\mathscr{C}_s^-(u), \mathscr{C}_s^+(u)),$$

for all $u \in \mathcal{R}$.

Definition 3.5. Let $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ be a spherical cubic set in $\mathcal{R}$. If

$$\mathscr{A}_s(u) = [0.4, 0.8] \text{ and } \mu(u) = 0.2,\ \mathscr{B}_s(u) = [0.3, 0.5] \text{ and } \nu(u) = 0.6, \text{ and}$$
$$\mathscr{C}_s(u) = [0.2, 0.6] \text{ and } \xi(u) = 0.7,$$

for all $u \in \mathcal{R}$, then $\mathscr{CU}_s$ is a spherical external cubic set in $\mathcal{R}$.

Theorem 3.6. *Let $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ be a spherical cubic set in $\mathcal{R}$, which is not a spherical external cubic set in $\mathcal{R}$. Then there exist $u \in \mathcal{R}$ such that*

$$\mu(u) \in (\mathscr{A}_s^-(u), \mathscr{A}_s^+(u)),\ \nu(u) \in (\mathscr{B}_s^-(u), \mathscr{B}_s^+(u)) \textit{ and } \xi(u) \in (\mathscr{C}_s^-(u), \mathscr{C}_s^+(u)).$$

Proof. Given $\mathscr{CU}_s$ is not a spherical external cubic set in $\mathcal{R}$. Then it is a spherical internal cubic set in $\mathcal{R}$. Thus there exist $u \in \mathcal{R}$ such that

$$\mu(u) \in (\mathscr{A}_s^-(u), \mathscr{A}_s^+(u)),\ \nu(u) \in (\mathscr{B}_s^-(u), \mathscr{B}_s^+(u)) \text{ and } \xi(u) \in (\mathscr{C}_s^-(u), \mathscr{C}_s^+(u)).$$ □

Theorem 3.7. *Let $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ be a spherical cubic set in $\mathcal{R}$. If $\mathscr{CU}_s$ is both a spherical internal and external cubic set, then*

$$\mu(u) \in U(\mathscr{A}_s) \cup L(\mathscr{A}_s),\ \nu(u) \in U(\mathscr{B}_s) \cup L(\mathscr{B}_s) \textit{ and } \xi(u) \in U(\mathscr{C}_s) \cup L(\mathscr{C}_s)$$

for all $u \in \mathcal{R}$, where

$$U(\mathscr{A}_s) = \{\mathscr{A}_s^+(u) \mid u \in \mathcal{R}\} \textit{ and } L(\mathscr{A}_s) = \{\mathscr{A}_s^-(u) \mid u \in \mathcal{R}\},$$
$$U(\mathscr{B}_s) = \{\mathscr{B}_s^+(u) \mid u \in \mathcal{R}\} \textit{ and } L(\mathscr{B}_s) = \{\mathscr{B}_s^-(u) \mid u \in \mathcal{R}\},$$
$$U(\mathscr{C}_s) = \{\mathscr{C}_s^+(u) \mid u \in \mathcal{R}\} \textit{ and } L(\mathscr{C}_s) = \{\mathscr{C}_s^-(u) \mid u \in \mathcal{R}\}.$$

Proof. Given the condition that $\mathscr{CU}_s = \{\langle \mathscr{A}_s, \mu\rangle, \langle \mathscr{B}_s, \nu\rangle, \langle \mathscr{C}_s, \xi\rangle\}$ is both a spherical internal and external cubic set, then

$$\mathscr{A}_s^-(u) \le \mu(u) \le \mathscr{A}_s^+(u),\ \mathscr{B}_s^-(u) \le \nu(u) \le \mathscr{B}_s^+(u) \text{ and } \mathscr{C}_s^-(u) \le \xi(u) \le \mathscr{C}_s^+(u),$$

$$\mu(u)\notin(\mathscr{A}_s^-(u),\mathscr{A}_s^+(u)),\ \nu(u)\notin(\mathscr{B}_s^-(u),\mathscr{B}_s^+(u)) \text{ and } \xi(u)\notin(\mathscr{C}_s^-(u),\mathscr{C}_s^+(u))$$

for all $u\in\mathcal{R}$. Now,

$$\mu(u)=\mathscr{A}_s^-(u) \text{ or } \mathscr{A}_s^+(u),\ \nu(u)=\mathscr{B}_s^-(u) \text{ or } \mathscr{B}_s^+(u) \text{ and } \xi(u)=\mathscr{C}_s^-(u) \text{ or } \mathscr{C}_s^+(u).$$

Hence

$$\mu(u)\in U(\mathscr{A}_s)\cup L(\mathscr{A}_s),\ \nu(u)\in U(\mathscr{B}_s)\cup L(\mathscr{B}_s) \text{ and } \xi(u)\in U(\mathscr{C}_s)\cup L(\mathscr{C}_s).$$ □

Definition 3.8. Let $\mathscr{CU}_{s_1}=\{\langle\mathscr{A}_{s_1},\mu_1\rangle,\langle\mathscr{B}_{s_1},\nu_1\rangle,\langle\mathscr{C}_{s_1},\xi_1\rangle\}$ and $\mathscr{CU}_{s_2}=\{\langle\mathscr{A}_{s_2},\mu_2\rangle,\langle\mathscr{B}_{s_2},\nu_2\rangle,\langle\mathscr{C}_{s_2},\xi_2\rangle\}$ be spherical cubic sets in $\mathcal{R}$. Then the equality of spherical cubic set in $\mathcal{R}$ is defined as $\mathscr{CU}_{s_1}=\mathscr{CU}_{s_2}$ if and only if $\mathscr{A}_{s_1}=\mathscr{A}_{s_2}$, $\mathscr{B}_{s_1}=\mathscr{B}_{s_2}$, $\mathscr{C}_{s_1}=\mathscr{C}_{s_2}$ and $\mu_1=\mu_2$, $\nu_1=\nu_2$, $\xi_1=\xi_2$.

Definition 3.9. Let $\mathscr{CU}_{s_1}=\{\langle\mathscr{A}_{s_1},\mu_1\rangle,\langle\mathscr{B}_{s_1},\nu_1\rangle,\langle\mathscr{C}_{s_1},\xi_1\rangle\}$ and $\mathscr{CU}_{s_2}=\{\langle\mathscr{A}_{s_2},\mu_2\rangle,\langle\mathscr{B}_{s_2},\nu_2\rangle,\langle\mathscr{C}_{s_2},\xi_2\rangle\}$ be spherical cubic sets in $\mathcal{R}$. Then P-order is defined as $\mathscr{CU}_{s_1}\subseteq_P\mathscr{CU}_{s_2}$ if and only if $\mathscr{A}_{s_1}\subseteq\mathscr{A}_{s_2}$, $\mathscr{B}_{s_1}\subseteq\mathscr{B}_{s_2}$, $\mathscr{C}_{s_1}\subseteq\mathscr{C}_{s_2}$ and $\mu_1\le\mu_2$, $\nu_1\le\nu_2$, $\xi_1\le\xi_2$.

Definition 3.10. Let $\mathscr{CU}_{s_1}=\{\langle\mathscr{A}_{s_1},\mu_1\rangle,\langle\mathscr{B}_{s_1},\nu_1\rangle,\langle\mathscr{C}_{s_1},\xi_1\rangle\}$ and $\mathscr{CU}_{s_2}=\{\langle\mathscr{A}_{s_2},\mu_2\rangle,\langle\mathscr{B}_{s_2},\nu_2\rangle,\langle\mathscr{C}_{s_2},\xi_2\rangle\}$ be spherical cubic sets in $\mathcal{R}$. Then R-order is defined as $\mathscr{CU}_{s_1}\subseteq_R\mathscr{CU}_{s_2}$ if and only if $\mathscr{A}_{s_1}\subseteq\mathscr{A}_{s_2}$, $\mathscr{B}_{s_1}\subseteq\mathscr{B}_{s_2}$, $\mathscr{C}_{s_1}\subseteq\mathscr{C}_{s_2}$ and $\mu_1\ge\mu_2$, $\nu_1\ge\nu_2$, $\xi_1\ge\xi_2$.

Definition 3.11. Let $\mathscr{CU}_{s_i}=\{\langle u,\mathscr{A}_{s_i(u)},\mu_i(u)\rangle,\langle u,\mathscr{B}_{s_i(u)},\nu_i(u)\rangle,\langle u,\mathscr{C}_{s_i(u)},\xi_i(u)\rangle|u\in\mathcal{R}\}$, where $i\in\Lambda$ be a family of spherical cubic sets in $\mathcal{R}$. Now we define P-union, P-intersection, R-union and R-intersection respectively as

$$\bigcup_{i\in\Lambda}{}_P\mathscr{CU}_{s_i}=\{\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\vee}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\vee}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\vee}\xi_i(u)\rangle|u\in\mathcal{R}\}$$

$$\bigcap_{i\in\Lambda}{}_P\mathscr{CU}_{s_i}=\{\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\xi_i(u)\rangle|u\in\mathcal{R}\}$$

$$\bigcup_{i\in\Lambda}{}_R\mathscr{CU}_{s_i}=\{\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\xi_i(u)\rangle|u\in\mathcal{R}\}$$

$$\bigcap_{i\in\Lambda}{}_R\mathscr{CU}_{s_i}=\{\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\vee}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\vee}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\vee}\xi_i(u)\rangle|u\in\mathcal{R}\}$$

Theorem 3.12. *Let $\mathscr{CU}_s=\{\langle\mathscr{A}_s,\mu\rangle,\langle\mathscr{B}_s,\nu\rangle,\langle\mathscr{C}_s,\xi\rangle\}$ be a spherical cubic set in $\mathcal{R}$. If $\mathscr{CU}_s$ is a spherical internal cubic set (resp. spherical external cubic set), then $\mathscr{CU}_s{}^c$ is also a spherical internal cubic set (resp. spherical external cubic set).*

Proof. Since $\mathscr{CU}_s$ is a spherical internal cubic set in $\mathcal{R}$, we have

$$\mathscr{A}_s^-(u)\le\mu(u)\le\mathscr{A}_s^+(u),\ \mathscr{B}_s^-(u)\le\nu(u)\le\mathscr{B}_s^+(u) \text{ and}$$
$$\mathscr{C}_s^-(u)\le\xi(u)\le\mathscr{C}_s^+(u),\ \mu(u)\notin(\mathscr{A}_s^-(u),\mathscr{A}_s^+(u)),\ \nu(u)\notin(\mathscr{B}_s^-(u),\mathscr{B}_s^+(u)) \text{ and}$$
$$\xi(u)\notin(\mathscr{C}_s^-(u),\mathscr{C}_s^+(u)),$$

for all $u\in\mathcal{R}$.

$1-\mathscr{A}_s^-(u)\le1-\mu(u)\le1-\mathscr{A}_s^+(u)$, $1-\mathscr{B}_s^-(u)\le1-\nu(u)\le1-\mathscr{B}_s^+(u)$ and

$1-\mathscr{C}_s^-(u)\le1-\xi(u)\le1-\mathscr{C}_s^+(u)$, $1-\mu(u)\notin(1-\mathscr{A}_s^-(u),1-\mathscr{A}_s^+(u))$, $\nu(u)\notin(1-\mathscr{B}_s^-(u),\ 1-\mathscr{B}_s^+(u))$ and

$1-\xi(u)\notin(1-\mathscr{C}_s^-(u),1-\mathscr{C}_s^+(u))$.

Then $\mathscr{CU}_s{}^c = \{\langle u, \mathscr{A}_s^c(u), \mu^c(u)\rangle, \langle u, \mathscr{B}_s^c(u), \nu^c(u)\rangle, \langle u, \mathscr{C}_s^c(u), \xi^c(u)\rangle \mid u \in \mathcal{R}\}$ is a spherical internal cubic set in $\mathcal{R}$. A similar proof holds for the external cubic sets. □

Theorem 3.13. *Let $\mathscr{CU}_{s_i} = \{\langle \mathscr{A}_{s_i}, \mu_i\rangle, \langle \mathscr{B}_{s_i}, \nu_i\rangle, \langle \mathscr{C}_{s_i}, \xi_i\rangle \mid\in \Lambda\}$ be a family of spherical internal cubic sets in $\mathcal{R}$. Then P-union and P-intersection of $\mathscr{CU}_{s_i} = \{\langle \mathscr{A}_{s_i}, \mu_i\rangle, \langle \mathscr{B}_{s_i}, \nu_i\rangle, \langle \mathscr{C}_{s_i}, \xi_i\rangle \mid i \in \Lambda\}$ are also spherical internal cubic sets in $\mathcal{R}$.*

Proof. Since $\mathscr{CU}_{s_i}$ is a spherical internal cubic sets in $\mathcal{R}$, we have

$$\mathscr{A}_{s_i}^-(u) \le \mu_i(u) \le \mathscr{A}_{s_i}^+(u),\ \mathscr{B}_{s_i}^-(u) \le \nu_i(u) \le \mathscr{B}_{s_i}^+(u) \text{ and}$$
$$\mathscr{C}_{s_i}^-(u) \le \xi_i(u) \le \mathscr{C}_{s_i}^+(u)$$

for $i \in \Lambda$, which gives

$$(\bigcup_{i\in\Lambda} \mathscr{A}_{s_i})^-(u) \le \bigvee_{i\in\Lambda} \mu_i(u) \le (\bigcup_{i\in\Lambda} \mathscr{A}_{s_i})^+(u),\ (\bigcup_{i\in\Lambda} \mathscr{B}_{s_i})^-(u) \le \bigvee_{i\in\Lambda} \nu_i(u) \le (\bigcup_{i\in\Lambda} \mathscr{B}_{s_i})^+(u),$$
$$(\bigcup_{i\in\Lambda} \mathscr{C}_{s_i})^-(u) \le \bigvee_{i\in\Lambda} \xi_i(u) \le (\bigcup_{i\in\Lambda} \mathscr{C}_{s_i})^+(u)$$

and also

$$(\bigcap_{i\in\Lambda} \mathscr{A}_{s_i})^-(u) \le \bigwedge_{i\in\Lambda} \mu_i(u) \le (\bigcap_{i\in\Lambda} \mathscr{A}_{s_i})^+(u),\ (\bigcap_{i\in\Lambda} \mathscr{B}_{s_i})^-(u) \le \bigwedge_{i\in\Lambda} \nu_i(u) \le (\bigcap_{i\in\Lambda} \mathscr{B}_{s_i})^+(u),$$
$$(\bigcap_{i\in\Lambda} \mathscr{C}_{s_i})^-(u) \le \bigwedge_{i\in\Lambda} \xi_i(u) \le (\bigcap_{i\in\Lambda} \mathscr{C}_{s_i})^+(u).$$

Then $\underset{i\in\Lambda}{\cup_P} \mathscr{CU}_{s_i}$ and $\underset{i\in\Lambda}{\cap_P} \mathscr{CU}_{s_i}$ are spherical internal cubic sets in $\mathcal{R}$. □

Example 3.14. Let $\mathscr{CU}_{s_1} = \{\langle \mathscr{A}_{s_1}, \mu_1\rangle, \langle \mathscr{B}_{s_1}, \nu_1\rangle, \langle \mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle \mathscr{A}_{s_2}, \mu_2\rangle, \langle \mathscr{B}_{s_2}, \nu_2\rangle, \langle \mathscr{C}_{s_2}, \xi_2\rangle\}$ be spherical external cubic sets in $I = [0,1]$ in which

$\mathscr{A}_{s_1}(u) = [0.2, 0.4],\ \mu_1(u) = [0.7];\ \mathscr{A}_{s_2}(u) = [0.6, 0.9],\ \mu_2(u) = [0.3];\ \mathscr{B}_{s_1}(u) = [0.1, 0.4],\ \nu_1(u) = [0.6];$
$\mathscr{B}_{s_2}(u) = [0.5, 0.8],\ \nu_2(u) = [0.2];\ \mathscr{C}_{s_1}(u) = [0.3, 0.6],\ \xi_1(u) = [0.8];\ \mathscr{C}_{s_1}(u) = [0.7, 0.9],\ \xi_1(u) = [0.5];$

for $u \in I$. Then

$$\mathscr{CU}_{s_1} \cup_P \mathscr{CU}_{s_2} = \{\langle u, \mathscr{A}_{s_2}, \mu_1\rangle, \langle u, \mathscr{B}_{s_2}, \nu_1\rangle, \langle u, \mathscr{C}_{s_2}, \xi_1\rangle \mid u \in I\}$$

and hence

$$\mu_1(u) \in (\mathscr{A}_{s_2}^-(u), \mathscr{A}_{s_2}^+(u)),\ \nu_1(u) \in (\mathscr{B}_{s_2}^-(u), \mathscr{B}_{s_2}^+(u)) \text{ and } \xi_1(u) \in (\mathscr{C}_{s_2}^-(u), \mathscr{C}_{s_2}^+(u))$$

for $u \in I$. Thus $\mathscr{CU}_{s_1} \cup_P \mathscr{CU}_{s_2}$ is not a spherical external cubic set in I. Now,

$$\mathscr{CU}_{s_1} \cap_P \mathscr{CU}_{s_2} = \{\langle u, \mathscr{A}_{s_1(u)}, \mu_2(u)\rangle, \langle u, \mathscr{B}_{s_1}(u), \nu_2(u)\rangle, \langle u, \mathscr{C}_{s_1}(u), \xi_2(u)\rangle \mid u \in I\}$$

and hence

$$\mu_2(u) \in (\mathscr{A}_{s_1}^-(u), \mathscr{A}_{s_1}^+(u)),\ \nu_2(u) \in (\mathscr{B}_{s_1}^-(u), \mathscr{B}_{s_1}^+(u)) \text{ and } \xi_1(u) \in (\mathscr{C}_{s_1}^-(u), \mathscr{C}_{s_1}^+(u))$$

for $u \in I$. Thus $\mathscr{CU}_{s_1} \cap_P \mathscr{CU}_{s_2}$ is not a spherical external cubic set in I.

Example 3.15. Let $\mathscr{CU}_{s_1} = \{\langle \mathscr{A}_{s_1}, \mu_1\rangle, \langle \mathscr{B}_{s_1}, \nu_1\rangle, \langle \mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle \mathscr{A}_{s_2}, \mu_2\rangle, \langle \mathscr{B}_{s_2}, \nu_2\rangle, \langle \mathscr{C}_{s_2}, \xi_2\rangle\}$ be spherical internal cubic sets in $I = [0,1]$ in which

$\mathscr{A}_{s_1}(u) = [0.2, 0.4],\ \mu_1(u) = [0.3]; \mathscr{A}_{s_2}(u) = [0.6, 0.9],\ \mu_2(u) = [0.8]; \mathscr{B}_{s_1}(u) = [0.1, 0.4],\ \nu_1(u) = [0.2];$
$\mathscr{B}_{s_2}(u) = [0.5, 0.8],\ \nu_2(u) = [0.6]; \mathscr{C}_{s_1}(u) = [0.3, 0.6],\ \xi_1(u) = [0.5]; \mathscr{C}_{s_1}(u) = [0.7, 0.9],\ \xi_1(u) = [0.8];$

for $u \in I$. Then

$$\mathscr{CU}_{s_1} \cup_R \mathscr{CU}_{s_2} = \{\langle u, \mathscr{A}_{s_2}(u), \mu_1(u)\rangle, \langle u, \mathscr{B}_{s_2}(u), \nu_1(u)\rangle, \langle u, \mathscr{C}_{s_2}(u), \xi_1(u)\rangle \mid u \in I\}$$

and hence

$$\mu_1(u) \notin (\mathscr{A}^-_{s_2}(u), \mathscr{A}^+_{s_2}(u)),\ \nu_1(u) \notin (\mathscr{B}^-_{s_2}(u), \mathscr{B}^+_{s_2}(u)) \text{ and } \xi_1(u) \in (\mathscr{C}^-_{s_2}(u), \mathscr{C}^+_{s_2}(u))$$

for $u \in I$. Thus $\mathscr{CU}_{s_1} \cup_R \mathscr{CU}_{s_2}$ is not a spherical internal cubic set in I.

Example 3.16. Let $\mathscr{CU}_{s_1} = \{\langle \mathscr{A}_{s_1}, \mu_1\rangle, \langle \mathscr{B}_{s_1}, \nu_1\rangle, \langle \mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle \mathscr{A}_{s_2}, \mu_2\rangle, \langle \mathscr{B}_{s_2}, \nu_2\rangle, \langle \mathscr{C}_{s_2}, \xi_2\rangle\}$ be spherical external cubic sets in $I = [0,1]$ in which

$\mathscr{A}_{s_1}(u) = [0.1, 0.5]$, $\mu_1(u) = [0.6]$; $\mathscr{A}_{s_2}(u) = [0.5, 0.8]$, $\mu_2(u) = [0.9]$; $\mathscr{B}_{s_1}(u) = [0.2, 0.4]$, $\nu_1(u) = [0.5]$; $\mathscr{B}_{s_2}(u) = [0.4, 0.7]$, $\nu_2(u) = [0.8]$; $\mathscr{C}_{s_1}(u) = [0.2, 0.6]$, $\xi_1(u) = [0.8]$; $\mathscr{C}_{s_1}(u) = [0.7, 0.9]$, $\xi_1(u) = [1]$;

for $u \in I$. Then

$$\mathscr{CU}_{s_1} \cup_R \mathscr{CU}_{s_2} = \{\langle u, \mathscr{A}_{s_2}(u), \mu_1(u)\rangle, \langle u, \mathscr{B}_{s_2}(u), \nu_1(u)\rangle, \langle u, \mathscr{C}_{s_2}(u), \xi_1(u)\rangle \mid u \in I\}$$

and hence

$$\mu_1(u) \in (\mathscr{A}^-_{s_2}(u), \mathscr{A}^+_{s_2}(u)),\ \nu_1(u) \in (\mathscr{B}^-_{s_2}(u), \mathscr{B}^+_{s_2}(u)) \text{ and } \xi_1(u) \in (\mathscr{C}^-_{s_2}(u), \mathscr{C}^+_{s_2}(u))$$

for $u \in I$. Thus $\mathscr{CU}_{s_1} \cup_R \mathscr{CU}_{s_2}$ is not a spherical external cubic set in I.

Example 3.17. Let $\mathscr{CU}_{s_1} = \{\langle \mathscr{A}_{s_1}, \mu_1\rangle, \langle \mathscr{B}_{s_1}, \nu_1\rangle, \langle \mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle \mathscr{A}_{s_2}, \mu_2\rangle, \langle \mathscr{B}_{s_2}, \nu_2\rangle, \langle \mathscr{C}_{s_2}, \xi_2\rangle\}$ be spherical external cubic sets in $I = [0,1]$ in which

$\mathscr{A}_{s_1}(u) = [0.3, 0.5]$, $\mu_1(u) = [0.2]$; $\mathscr{A}_{s_2}(u) = [0.6, 0.8]$, $\mu_2(u) = [0.4]$; $\mathscr{B}_{s_1}(u) = [0.2, 0.4]$, $\nu_1(u) = [0.1]$; $\mathscr{B}_{s_2}(u) = [0.6, 0.9]$, $\nu_2(u) = [0.3]$; $\mathscr{C}_{s_1}(u) = [0.4, 0.7]$, $\xi_1(u) = [0.3]$; $\mathscr{C}_{s_2}(u) = [0.8, 1]$, $\xi_2(u) = [0.5]$;

for $u \in I$. Then

$$\mathscr{CU}_{s_1} \cap_R \mathscr{CU}_{s_2} = \{\langle u, \mathscr{A}_{s_1}(u), \mu_2(u)\rangle, \langle u, \mathscr{B}_{s_1}(u), \nu_2(u)\rangle, \langle u, \mathscr{C}_{s_1}(u), \xi_2(u)\rangle \mid u \in I\}$$

and hence

$$\mu_2(u) \in (\mathscr{A}^-_{s_1}(u), \mathscr{A}^+_{s_1}(u)), \nu_2(u) \in (\mathscr{B}^-_{s_1}(u), \mathscr{B}^+_{s_1}(u)) \text{ and } \xi_2(u) \in (\mathscr{C}^-_{s_1}(u), \mathscr{C}^+_{s_1}(u))$$

for $u \in I$. Thus $\mathscr{CU}_{s_1} \cap_R \mathscr{CU}_{s_2}$ is not a spherical external cubic set in I.

Theorem 3.18. *Let $\mathscr{CU}_{s_1} = \{\langle \mathscr{A}_{s_1}, \mu_1\rangle, \langle \mathscr{B}_{s_1}, \nu_1\rangle, \langle \mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle \mathscr{A}_{s_2}, \mu_2\rangle, \langle \mathscr{B}_{s_2}, \nu_2\rangle, \langle \mathscr{C}_{s_2}, \xi_2\rangle\}$ be spherical internal cubic sets in $\mathcal{R}$, such that*

$$\max\{\mathscr{A}^-_{s_1}(u), \mathscr{A}^-_{s_2}(u)\} \le (\mu_1 \wedge \mu_2)(u),\ \max\{\mathscr{B}^-_{s_1}(u), \mathscr{B}^-_{s_2}(u)\} \le (\nu_1 \wedge \nu_2)(u),$$
$$\max\{\mathscr{C}^-_{s_1}(u), \mathscr{C}^-_{s_2}(u)\} \le (\xi_1 \wedge \xi_2)(u)$$

for all $u \in \mathcal{R}$. Then R-union of $\mathscr{CU}_{s_1}$ and $\mathscr{CU}_{s_2}$ is also a spherical internal cubic set in $\mathcal{R}$.

Proof. Let $\mathscr{CU}_{s_1} = \{\langle \mathscr{A}_{s_1}, \mu_1\rangle, \langle \mathscr{B}_{s_1}, \nu_1\rangle, \langle \mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle \mathscr{A}_{s_2}, \mu_2\rangle, \langle \mathscr{B}_{s_2}, \nu_2\rangle, \langle \mathscr{C}_{s_2}, \xi_2\rangle\}$ be spherical internal cubic sets in $\mathcal{R}$ which satisfy the given conditions. Then

$$\mathscr{A}^-_{s_1}(u) \le \mu_1(u) \le \mathscr{A}^+_{s_1}(u),\ \mathscr{A}^-_{s_2}(u) \le \mu_2(u) \le \mathscr{A}^+_{s_2}(u);$$
$$\mathscr{B}^-_{s_1}(u) \le \nu_1(u) \le \mathscr{B}^+_{s_1}(u),\ \mathscr{B}^-_{s_2}(u) \le \nu_2(u) \le \mathscr{B}^+_{s_2}(u),$$
$$\mathscr{C}^-_{s_1}(u) \le \xi_1(u) \le \mathscr{C}^+_{s_1}(u),\ \mathscr{C}^-_{s_2}(u) \le \xi_2(u) \le \mathscr{C}^+_{s_2}(u).$$

This gives

$$(\mu_1 \wedge \mu_2)(u) \leq (\mathscr{A}_{s_1} \cup \mathscr{A}_{s_2})^+(u),\ (\nu_1 \wedge \nu_2)(u) \leq (\mathscr{B}_{s_1} \cup \mathscr{B}_{s_2})^+(u),\ (\xi_1 \wedge \xi_2)(u) \leq (\mathscr{C}_{s_1} \cup \mathscr{C}_{s_2})^+(u).$$

Then

$$(\mathscr{A}_{s_1} \cup \mathscr{A}_{s_2})^-(u) = \max\{\mathscr{A}_{s_1}^-(u), \mathscr{A}_{s_2}^-(u)\} \leq (\mu_1 \wedge \mu_2)(u) \leq (\mathscr{A}_{s_1} \cup \mathscr{A}_{s_2})^+(u),$$
$$(\mathscr{B}_{s_1} \cup \mathscr{B}_{s_2})^-(u) = \max\{\mathscr{B}_{s_1}^-(u), \mathscr{B}_{s_2}^-(u)\} \leq (\nu_1 \wedge \nu_2)(u) \leq (\mathscr{B}_{s_1} \cup \mathscr{B}_{s_2})^+(u),$$
$$(\mathscr{C}_{s_1} \cup \mathscr{C}_{s_2})^-(u) = \max\{\mathscr{C}_{s_1}^-(u), \mathscr{C}_{s_2}^-(u)\} \leq (\xi_1 \wedge \xi_2)(u) \leq (\mathscr{C}_{s_1} \cup \mathscr{C}_{s_2})^+(u)$$

and so

$$\mathscr{CU}_{s_1} \cup_R \mathscr{CU}_{s_2} = \{\langle(\mathscr{A}_{s_1} \cup \mathscr{A}_{s_2})(u), (\mu_1 \wedge \mu_2)(u)\rangle, \langle(\mathscr{B}_{s_1} \cup \mathscr{B}_{s_2})(u), (\nu_1 \wedge \nu_2)(u)\rangle,$$
$$\langle(\mathscr{C}_{s_1} \cup \mathscr{C}_{s_2})(u), (\xi_1 \wedge \xi_2)(u)\rangle \mid u \in \mathcal{R}\}$$

is a spherical internal cubic set in $\mathcal{R}$. □

Theorem 3.19. *Let $\mathscr{CU}_{s_1} = \{\langle\mathscr{A}_{s_1}, \mu_1\rangle, \langle\mathscr{B}_{s_1}, \nu_1\rangle, \langle\mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle\mathscr{A}_{s_2}, \mu_2\rangle, \langle\mathscr{B}_{s_2}, \nu_2\rangle, \langle\mathscr{C}_{s_2}, \xi_2\rangle\}$ are spherical internal cubic sets in $\mathcal{R}$, such that*

$$\min\{\mathscr{A}_{s_1}^+(u), \mathscr{A}_{s_2}^+(u)\} \geq (\mu_1 \vee \mu_2)(u),$$
$$\min\{\mathscr{B}_{s_1}^+(u), \mathscr{B}_{s_2}^+(u)\} \geq (\nu_1 \vee \nu_2)(u),$$
$$\min\{\mathscr{C}_{s_1}^+(u), \mathscr{C}_{s_2}^+(u)\} \geq (\xi_1 \vee \xi_2)(u)$$

for all $u \in \mathcal{R}$. Then R-intersection of $\mathscr{CU}_{s_1}$ and $\mathscr{CU}_{s_2}$ are spherical internal cubic sets in $\mathcal{R}$.

Proof. Let $\mathscr{CU}_{s_1} = \{\langle\mathscr{A}_{s_1}, \mu_1\rangle, \langle\mathscr{B}_{s_1}, \nu_1\rangle, \langle\mathscr{C}_{s_1}, \xi_1\rangle\}$ and $\mathscr{CU}_{s_2} = \{\langle\mathscr{A}_{s_2}, \mu_2\rangle, \langle\mathscr{B}_{s_2}, \nu_2\rangle, \langle\mathscr{C}_{s_2}, \xi_2\rangle\}$ are spherical internal cubic sets in $\mathcal{R}$ which satisfy the given condition. Then

$$\mathscr{A}_{s_1}^-(u) \leq \mu_1(u) \leq \mathscr{A}_{s_1}^+(u),\ \mathscr{A}_{s_2}^-(u) \leq \mu_2(u) \leq \mathscr{A}_{s_2}^+(u);$$
$$\mathscr{B}_{s_1}^-(u) \leq \nu_1(u) \leq \mathscr{B}_{s_1}^+(u),\ \mathscr{B}_{s_2}^-(u) \leq \nu_2(u) \leq \mathscr{B}_{s_2}^+(u),$$
$$\mathscr{C}_{s_1}^-(u) \leq \xi_1(u) \leq \mathscr{C}_{s_1}^+(u), \mathscr{C}_{s_2}^-(u) \leq \xi_2(u) \leq \mathscr{C}_{s_2}^+(u).$$

This gives

$$(\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^-(u) \leq (\mu_1 \vee \mu_2)(u),\ (\mathscr{B}_{s_1} \cap \mathscr{B}_{s_2})^-(u) \leq (\nu_1 \vee \nu_2)(u),\ (\mathscr{C}_{s_1} \cap \mathscr{C}_{s_2})^-(u) \leq (\xi_1 \vee \xi_2)(u).$$

Then

$$(\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^-(u) \leq (\mu_1 \vee \mu_2)(u) \leq \min\{\mathscr{A}_{s_1}^+(u), \mathscr{A}_{s_2}^+(u)\} = (\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u),$$
$$(\mathscr{B}_{s_1} \cap \mathscr{B}_{s_2})^-(u) \leq (\nu_1 \vee \nu_2)(u) \leq \min\{\mathscr{B}_{s_1}^+(u), \mathscr{B}_{s_2}^+(u)\} = (\mathscr{B}_{s_1} \cap \mathscr{B}_{s_2})^+(u),$$
$$(\mathscr{C}_{s_1} \cap \mathscr{C}_{s_2})^-(u) \leq (\xi_1 \vee \xi_2)(u) \leq \min\{\mathscr{C}_{s_1}^+(u), \mathscr{C}_{s_2}^+(u)\} = (\mathscr{C}_{s_1} \cap \mathscr{C}_{s_2})^+(u)$$

and so

$$\mathscr{CU}_{s_1} \cap_R \mathscr{CU}_{s_2} = \{\langle(\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})(u), (\mu_1 \vee \mu_2)(u)\rangle, \langle(\mathscr{B}_{s_1} \cap \mathscr{B}_{s_2})(u), (\nu_1 \vee \nu_2)(u)\rangle,$$
$$\langle(\mathscr{C}_{s_1} \cap \mathscr{C}_{s_2})(u), (\xi_1 \vee \xi_2)(u)\rangle \mid u \in \mathcal{R}\}$$

is a spherical internal cubic set in $\mathcal{R}$. □

Theorem 3.20. *Let $\mathscr{CU}_{s_1}$ and $\mathscr{CU}_{s_2}$ be any two spherical external cubic sets in $\mathcal{R}$ such that*

$$\min\{\max\{\mathscr{A}_{s_1}^+(u), \mathscr{A}_{s_2}^-(u)\}, \max\{\mathscr{A}_{s_1}^-(u), \mathscr{A}_{s_2}^+(u)\}\} \geq (\mu_1 \wedge \mu_2)(u)$$

$$> \max\{\min\{\mathscr{A}^+_{s_1}(u),\mathscr{A}^-_{s_2}(u)\},\min\{\mathscr{A}^-_{s_1}(u),\mathscr{A}^+_{s_2}(u)\}\}$$
$$\min\{\max\{\mathscr{B}^+_{s_1}(u),\mathscr{B}^-_{s_2}(u)\},\max\{\mathscr{B}^-_{s_1}(u),\mathscr{B}^+_{s_2}(u)\}\} \geq (\nu_1 \wedge \nu_2)(u)$$
$$> \max\{\min\{\mathscr{B}^+_{s_1}(u),\mathscr{B}^-_{s_2}(u)\},\min\{\mathscr{B}^-_{s_1}(u),\mathscr{B}^+_{s_2}(u)\}\}$$
$$\min\{\max\{\mathscr{C}^+_{s_1}(u),\mathscr{C}^-_{s_2}(u)\},\max\{\mathscr{C}^-_{s_1}(u),\mathscr{C}^+_{s_2}(u)\}\} \geq (\mu_1 \wedge \mu_2)(u)$$
$$> \max\{\min\{\mathscr{C}^+_{s_1}(u),\mathscr{C}^-_{s_2}(u)\},\min\{\mathscr{C}^-_{s_1}(u),\mathscr{C}^+_{s_2}(u)\}\}$$

for all $u \in \mathcal{R}$. Then P-intersection of $\mathscr{CU}_{s_1}$ and $\mathscr{CU}_{s_2}$ is also a spherical external cubic set in $\mathcal{R}$.

Proof. For each $u \in \mathcal{R}$, we take

$$a_x = \min\{\max\{\mathscr{A}^+_{s_1}(u),\mathscr{A}^-_{s_2}(u)\},\max\{\mathscr{A}^-_{s_1}(u),\mathscr{A}^+_{s_2}(u)\}\},$$
$$b_x = \max\{\min\{\mathscr{A}^+_{s_1}(u),\mathscr{A}^-_{s_2}(u)\},\min\{\mathscr{A}^-_{s_1}(u),\mathscr{A}^+_{s_2}(u)\}\}.$$

Then a_x is one of $\mathscr{A}^-_{s_1}(u)$, $\mathscr{A}^+_{s_1}(u)$, $\mathscr{A}^-_{s_2}(u)$, $\mathscr{A}^+_{s_2}(u)$.
We consider $a_x = \mathscr{A}^-_{s_1}(u)$ or $a_x = \mathscr{A}^+_{s_1}(u)$ only.
If $a_x = \mathscr{A}^-_{s_1}(u)$, then $\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^+_{s_2}(u) \leq \mathscr{A}^-_{s_1}(u) \leq \mathscr{A}^+_{s_1}(u)$.
And so $b_x = \mathscr{A}^+_{s_2}(u)$. Thus

$$\mathscr{A}^-_{s_2}(u) = (\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^-(u) \leq (\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u) = \mathscr{A}^+_{s_2}(u) = b_x < (\mu_1 \wedge \mu_2)(u).$$

Hence $(\mu_1 \wedge \mu_2)(u) \notin ((\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^-(u),(\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u))$.
If $a_x = \mathscr{A}^+_{s_1}(u)$, then $\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u)$.
And so $b_x = \max\{\mathscr{A}^-_{s_1}(u),\mathscr{A}^-_{s_2}(u)\}$.
Assume that $b_x = \mathscr{A}^-_{s_1}(u)$. Then

$$\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^-_{s_1}(u) < (\mu_1 \wedge \mu_2)(u) \leq \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u).$$

From this it follows that

$$\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^-_{s_1}(u) < (\mu_1 \wedge \mu_2)(u) < \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u)$$

or

$$\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^-_{s_1}(u) < (\mu_1 \wedge \mu_2)(u) = \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u).$$

But $\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^-_{s_1}(u) < (\mu_1 \wedge \mu_2)(u) < \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u)$ is a contradiction to the fact that $\mathscr{CU}_{s_1}$ and $\mathscr{CU}_{s_2}$ are spherical external cubic sets in $\mathcal{R}$.
For the case $\mathscr{A}^-_{s_2}(u) \leq \mathscr{A}^-_{s_1}(u) < (\mu_1 \wedge \mu_2)(u) = \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u)$, we have

$$(\mu_1 \wedge \mu_2)(u) \notin ((\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^-(u),(\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u))$$

since $(\mu_1 \wedge \mu_2)(u) = \mathscr{A}^+_{s_1}(u) = (\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u)$.
Assume that $b_x = \mathscr{A}^-_{s_2}(u)$. Then

$$\mathscr{A}^-_{s_1}(u) \leq \mathscr{A}^-_{s_2}(u) < (\mu_1 \wedge \mu_2)(u) \leq \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u).$$

From this we get

$$\mathscr{A}^-_{s_1}(u) \leq \mathscr{A}^-_{s_2}(u) < (\mu_1 \wedge \mu_2)(u) < \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u)$$

or

$$\mathscr{A}^-_{s_1}(u) \leq \mathscr{A}^-_{s_2}(u) < (\mu_1 \wedge \mu_2)(u) = \mathscr{A}^+_{s_1}(u) \leq \mathscr{A}^+_{s_2}(u).$$

But $\mathscr{A}^-_{s_1}(u) \le \mathscr{A}^-_{s_2}(u) < (\mu_1 \wedge \mu_2)(u) < \mathscr{A}^+_{s_1}(u) \le \mathscr{A}^+_{s_2}(u)$ is a contradiction to the fact that $\mathscr{C}\mathscr{U}_{s_1}$ and $\mathscr{C}\mathscr{U}_{s_2}$ are spherical external cubic sets in $\mathfrak{R}$.

For the case $\mathscr{A}^-_{s_1}(u) \le \mathscr{A}^-_{s_2}(u) < (\mu_1 \wedge \mu_2)(u) = \mathscr{A}^+_{s_1}(u) \le \mathscr{A}^+_{s_2}(u)$, we have

$$(\mu_1 \wedge \mu_2)(u) \notin ((\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^-(u), (\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u))$$

since $(\mu_1 \wedge \mu_2)(u) = \mathscr{A}^+_{s_1}(u) = (\mathscr{A}_{s_1} \cap \mathscr{A}_{s_2})^+(u)$.

Similar proof holds for all other cases. Hence P-intersection of $\mathscr{C}\mathscr{U}_{s_1}$ and $\mathscr{C}\mathscr{U}_{s_2}$ is a spherical external cubic set in $\mathfrak{R}$. □

4. Spherical Cubic Bi-Ideals of Gamma Near-Rings

In this section we define spherical cubic bi-ideals of gamma near-rings in $\mathfrak{R}$ and study some of it properties.

Definition 4.1. A spherical cubic set $\mathscr{C}\mathscr{U}_s = \{\langle u, \mathscr{A}_s(u), \mu(u)\rangle, \langle u, \mathscr{B}_s(u), \nu(u)\rangle, \langle u, \mathscr{C}_s(u), \xi(u)\rangle \mid u \in \mathfrak{R}\}$ is said to be a spherical cubic bi-ideal of gamma near-ring if the following conditions are satisfied

(i) $\mathscr{A}_s(u-v) \ge \min^i\{\mathscr{A}_s(u), \mathscr{A}_s(v)\}$, $\mu(u-v) \le \max\{\mu(u), \mu(v)\}$,

(ii) $\mathscr{B}_s(u-v) \ge \min^i\{\mathscr{B}_s(u), \mathscr{B}_s(v)\}$, $\nu(u-v) \le \max\{\nu(u), \nu(v)\}$,

(iii) $\mathscr{C}_s(u-v) \le \max^i\{\mathscr{C}_s(u), \mathscr{C}_s(v)\}$, $\xi(u-v) \ge \min\{\xi(u), \xi(v)\}$,

(iv) $\mathscr{A}_s(u\alpha v\beta w) \ge \min^i\{\mathscr{A}_s(u), \mathscr{A}_s(w)\}$, $\mu(u\alpha v\beta w) \le \max\{\mu(u), \mu(w)\}$,

(v) $\mathscr{B}_s(u\alpha v\beta w) \ge \min^i\{\mathscr{B}_s(u), \mathscr{B}_s(w)\}$, $\nu(u\alpha v\beta w) \le \max\{\nu(u), \nu(w)\}$,

(vi) $\mathscr{C}_s(u\alpha v\beta w) \le \max^i\{\mathscr{C}_s(u), \mathscr{C}_s(w)\}$, $\xi(u\alpha v\beta w) \ge \min\{\xi(u), \xi(w)\}$,

for all $u, v, w \in \mathfrak{R}$ and $\alpha, \beta \in \Gamma$, where $\mathscr{A}_s : \mathfrak{R} \to D[0,1]$, $\mathscr{B}_s : \mathfrak{R} \to D[0,1]$ and $\mathscr{C}_s : \mathfrak{R} \to D[0,1]$. Here $D[0,1]$ denotes the family of closed subintervals of $[0,1]$ and $\mu : \mathfrak{R} \to [0,1]$, $\nu : \mathfrak{R} \to [0,1]$ and $\xi : \mathfrak{R} \to [0,1]$.

Example 4.2. Let $\mathfrak{R} = \{0,1,2,3\}$ with binary operation "+" on $\mathfrak{R}$, $\Gamma = \{0,1\}$ and $\mathfrak{R} \times \Gamma \times \mathfrak{R} \to \mathfrak{R}$ be a mapping. From the Cayley table,

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

0	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

1	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Clearly $(\mathfrak{R}, +)$ is a group. Now we define spherical cubic set in $\mathfrak{R}$ as

$\mathfrak{R}$	$\mathscr{A}_s$	μ
0	(0.2, 0.6)	0.8
1	(0.5, 0.7)	0.7
2	(0.6, 0.8)	0.5
3	(0.7, 0.9)	0.5

$\mathfrak{R}$	$\mathscr{B}_s$	ν
0	(0.1, 0.4)	0.9
1	(0.5, 0.6)	0.7
2	(0.2, 0.5)	0.6
3	(0.3, 0.7)	0.6

$\mathfrak{R}$	$\mathscr{C}_s$	ξ
0	(0.1,0.3)	0.4
1	(0.6, 0.8)	0.2
2	(0.3, 0.5)	0.6
3	(0.4, 0.7)	0.8

Then $\mathscr{C}\mathscr{U}_s$ is a spherical cubic bi-ideal of gamma near-ring.

Theorem 4.3. *If* $\mathscr{CU}_{s_i} = \{\langle \mathscr{A}_{s_i}, \mu_i\rangle, \langle \mathscr{B}_{s_i}, \nu_i\rangle, \langle \mathscr{C}_{s_i}, \xi_i\rangle \mid i \in \Lambda\}$ *be a family of spherical cubic bi-ideals of gamma near-ring* $\mathcal{R}$*, then*

$$\bigcap_{\substack{P\\ i\in\Lambda}} \mathscr{CU}_{s_i} = \{\langle u, (\underset{i\in\Lambda}{\cap} \mathscr{A}_{s_i})(u), \underset{i\in\Lambda}{\wedge} \mu_i(u)\rangle, \langle u, (\underset{i\in\Lambda}{\cap} \mathscr{B}_{s_i})(u), \underset{i\in\Lambda}{\wedge} \nu_i(u)\rangle, \langle u, (\underset{i\in\Lambda}{\cap} \mathscr{C}_{s_i})(u), \underset{i\in\Lambda}{\wedge} \xi_i(u)\rangle \mid u \in \mathcal{R}\}$$

is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$*, where* Λ *is an index set.*

Proof. Let $\mathscr{CU}_{s_i} = \{\langle \mathscr{A}_{s_i}, \mu_i\rangle, \langle \mathscr{B}_{s_i}, \nu_i\rangle, \langle \mathscr{C}_{s_i}, \xi_i\rangle \mid i \in \Lambda\}$ be a family of spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$. For any $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

(i)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(u-v) &= \inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(u-v)\\
&\ge \inf^i_{i\in\Lambda} \min^i\{\mathscr{A}_{s_i}(u), \mathscr{A}_{s_i}(v)\}\\
&= \min^i\{\inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(v)\}\\
&= \min^i\{\bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(v)\},\\
\underset{i\in\Lambda}{\wedge} \mu_i(u-v) &= \inf^i_{i\in\Lambda} \mu_i(u-v)\\
&\le \inf^i_{i\in\Lambda} \max^i\{\mu_i(u), \mu_i(v)\}\\
&= \max^i\{\inf^i_{i\in\Lambda} \mu_i(u), \inf^i_{i\in\Lambda} \mu_i(v)\}\\
&= \max^i\{\underset{i\in\Lambda}{\wedge} \mu_i(u), \underset{i\in\Lambda}{\wedge} \mu_i(v)\},
\end{aligned}$$

(ii)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{B}_{s_i}(u-v) &= \inf^i_{i\in\Lambda} \mathscr{B}_{s_i}(u-v)\\
&\ge \inf^i_{i\in\Lambda} \min^i\{\mathscr{B}_{s_i}(u), \mathscr{B}_{s_i}(v)\}\\
&= \min^i\{\inf^i_{i\in\Lambda} \mathscr{B}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{B}_{s_i}(v)\}\\
&= \min^i\{\bigcap_{i\in\Lambda} \mathscr{B}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{B}_{s_i}(v)\},\\
\underset{i\in\Lambda}{\wedge} \nu_i(u-v) &= \inf^i_{i\in\Lambda} \nu_i(u-v)\\
&\le \inf^i_{i\in\Lambda} \max^i\{\nu_i(u), \nu_i(v)\}\\
&= \max^i\{\inf^i_{i\in\Lambda} \nu_i(u), \inf^i_{i\in\Lambda} \nu_i(v)\}\\
&= \max^i\{\underset{i\in\Lambda}{\wedge} \nu_i(u), \underset{i\in\Lambda}{\wedge} \nu_i(v)\},
\end{aligned}$$

(iii)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{C}_{s_i}(u-v) &= \inf^i_{i\in\Lambda} \mathscr{C}_{s_i}(u-v)\\
&\le \inf^i_{i\in\Lambda} \max^i\{\mathscr{C}_{s_i}(u), \mathscr{C}_{s_i}(v)\}\\
&= \max^i\{\inf^i_{i\in\Lambda} \mathscr{C}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{C}_{s_i}(v)\}\\
&= \max^i\{\bigcap_{i\in\Lambda} \mathscr{C}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{C}_{s_i}(v)\},\\
\underset{i\in\Lambda}{\wedge} \xi_i(u-v) &= \inf^i_{i\in\Lambda} \xi_i(u-v)\\
&\ge \inf^i_{i\in\Lambda} \min^i\{\xi_i(u), \xi_i(v)\}\\
&= \min^i\{\inf^i_{i\in\Lambda} \xi_i(u), \inf^i_{i\in\Lambda} \xi_i(v)\}\\
&= \min^i\{\underset{i\in\Lambda}{\wedge} \xi_i(u), \underset{i\in\Lambda}{\wedge} \xi_i(v)\},
\end{aligned}$$

$$
\begin{aligned}
\text{(iv)}\quad \bigcap_{i\in\Lambda}\mathscr{A}_{s_i}(u\alpha v\beta w) &= \inf^{i}_{i\in\Lambda}\mathscr{A}_{s_i}(u\alpha v\beta w)\\
&\geq \inf^{i}_{i\in\Lambda}\min{}^{i}\{\mathscr{A}_{s_i}(u),\mathscr{A}_{s_i}(w)\}\\
&= \min{}^{i}\{\inf^{i}_{i\in\Lambda}\mathscr{A}_{s_i}(u),\inf^{i}_{i\in\Lambda}\mathscr{A}_{s_i}(w)\}\\
&= \min{}^{i}\{\bigcap_{i\in\Lambda}\mathscr{A}_{s_i}(u),\bigcap_{i\in\Lambda}\mathscr{A}_{s_i}(w)\},\\
\underset{i\in\Lambda}{\wedge}\mu_i(u\alpha v\beta w) &= \inf^{i}_{i\in\Lambda}\mu_i(u\alpha v\beta w)\\
&\leq \inf^{i}_{i\in\Lambda}\max{}^{i}\{\mu_i(u),\mu_i(w)\}\\
&= \max{}^{i}\{\inf^{i}_{i\in\Lambda}\mu_i(u),\inf^{i}_{i\in\Lambda}\mu_i(w)\}\\
&= \max{}^{i}\{\underset{i\in\Lambda}{\wedge}\mu_i(u),\underset{i\in\Lambda}{\wedge}\mu_i(w)\},\\
\text{(v)}\quad \bigcap_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w) &= \inf^{i}_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w)\\
&\geq \inf^{i}_{i\in\Lambda}\min{}^{i}\{\mathscr{B}_{s_i}(u),\mathscr{B}_{s_i}(w)\}\\
&= \min{}^{i}\{\inf^{i}_{i\in\Lambda}\mathscr{B}_{s_i}(u),\inf^{i}_{i\in\Lambda}\mathscr{B}_{s_i}(w)\}\\
&= \min{}^{i}\{\bigcap_{i\in\Lambda}\mathscr{B}_{s_i}(u),\bigcap_{i\in\Lambda}\mathscr{B}_{s_i}(w)\},\ \underset{i\in\Lambda}{\wedge}\nu_i(u\alpha v\beta w)\\
&= \inf^{i}_{i\in\Lambda}\nu_i(u\alpha v\beta w)\\
&\leq \inf^{i}_{i\in\Lambda}\max{}^{i}\{\nu_i(u),\nu_i(w)\}\\
&= \max{}^{i}\{\inf^{i}_{i\in\Lambda}\nu_i(u),\inf^{i}_{i\in\Lambda}\nu_i(w)\}\\
&= \max{}^{i}\{\underset{i\in\Lambda}{\wedge}\nu_i(u),\underset{i\in\Lambda}{\wedge}\nu_i(w)\},\ \text{and}\\
\text{(vi)}\quad \bigcap_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w) &= \inf^{i}_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w)\\
&\leq \inf^{i}_{i\in\Lambda}\max{}^{i}\{\mathscr{C}_{s_i}(u),\mathscr{C}_{s_i}(w)\}\\
&= \max{}^{i}\{\inf^{i}_{i\in\Lambda}\mathscr{C}_{s_i}(u),\inf^{i}_{i\in\Lambda}\mathscr{C}_{s_i}(w)\}\\
&= \max{}^{i}\{\bigcap_{i\in\Lambda}\mathscr{C}_{s_i}(u),\bigcap_{i\in\Lambda}\mathscr{C}_{s_i}(w)\},\\
\underset{i\in\Lambda}{\wedge}\xi_i(u\alpha v\beta w) &= \inf^{i}_{i\in\Lambda}\xi_i(u\alpha v\beta w)\\
&\geq \inf^{i}_{i\in\Lambda}\min{}^{i}\{\xi_i(u),\xi_i(w)\}\\
&= \min{}^{i}\{\inf^{i}_{i\in\Lambda}\xi_i(u),\inf^{i}_{i\in\Lambda}\xi_i(w)\}\\
&= \min{}^{i}\{\underset{i\in\Lambda}{\wedge}\xi_i(u),\underset{i\in\Lambda}{\wedge}\xi_i(w)\}.
\end{aligned}
$$

Hence $\bigcap_{P\, i\in\Lambda}\mathscr{C}\mathscr{U}_{s_i}$ is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$. □

Theorem 4.4. *If $\mathscr{C}\mathscr{U}_{s_i} = \{\langle\mathscr{A}_{s_i},\mu_i\rangle,\langle\mathscr{B}_{s_i},\nu_i\rangle,\langle\mathscr{C}_{s_i},\xi_i\rangle \mid i\in\Lambda\}$ be a family of spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$, then*

$$\bigcap_{\substack{R\\ i\in\Lambda}}\mathscr{C}\mathscr{U}_{s_i} = \{\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\vee}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\vee}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cap}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\vee}\xi_i(u)\rangle \mid u\in\mathcal{R}\}$$

is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$, where Λ is an index set.

Proof. Let $\mathscr{CU}_{s_i} = \{\langle \mathscr{A}_{s_i}, \mu_i\rangle, \langle \mathscr{B}_{s_i}, \nu_i\rangle, \langle \mathscr{C}_{s_i}, \xi_i\rangle \mid i \in \Lambda\}$ be a family of spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$. For any $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

(i)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(u-v) &= \inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(u-v) \\
&\geq \inf^i_{i\in\Lambda} \min{}^i\{\mathscr{A}_{s_i}(u), \mathscr{A}_{s_i}(v)\} \\
&= \min{}^i\{\inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(v)\} \\
&= \min{}^i\{\bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(v)\},
\end{aligned}$$
$$\begin{aligned}
\bigvee_{i\in\Lambda} \mu_i(u-v) &= \sup^i_{i\in\Lambda} \mu_i(u-v) \\
&\leq \sup^i_{i\in\Lambda} \max{}^i\{\mu_i(u), \mu_i(v)\} \\
&= \max{}^i\{\sup^i_{i\in\Lambda} \mu_i(u), \sup^i_{i\in\Lambda} \mu_i(v)\} \\
&= \max{}^i\{\bigvee_{i\in\Lambda} \mu_i(u), \bigvee_{i\in\Lambda} \mu_i(v)\},
\end{aligned}$$

(ii)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{B}_{s_i}(u-v) &= \inf^i_{i\in\Lambda} \mathscr{B}_{s_i}(u-v) \\
&\geq \inf^i_{i\in\Lambda} \min{}^i\{\mathscr{B}_{s_i}(u), \mathscr{B}_{s_i}(v)\} \\
&= \min{}^i\{\inf^i_{i\in\Lambda} \mathscr{B}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{B}_{s_i}(v)\} \\
&= \min{}^i\{\bigcap_{i\in\Lambda} \mathscr{B}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{B}_{s_i}(v)\},
\end{aligned}$$
$$\begin{aligned}
\bigvee_{i\in\Lambda} \nu_i(u-v) &= \sup^i_{i\in\Lambda} \nu_i(u-v) \\
&\leq \sup^i_{i\in\Lambda} \max{}^i\{\nu_i(u), \nu_i(v)\} \\
&= \max{}^i\{\sup^i_{i\in\Lambda} \nu_i(u), \sup^i_{i\in\Lambda} \nu_i(v)\} \\
&= \max{}^i\{\bigvee_{i\in\Lambda} \nu_i(u), \bigvee_{i\in\Lambda} \nu_i(v)\},
\end{aligned}$$

(iii)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{C}_{s_i}(u-v) &= \inf^i_{i\in\Lambda} \mathscr{C}_{s_i}(u-v) \\
&\leq \inf^i_{i\in\Lambda} \max{}^i\{\mathscr{C}_{s_i}(u), \mathscr{C}_{s_i}(v)\} \\
&= \max{}^i\{\inf^i_{i\in\Lambda} \mathscr{C}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{C}_{s_i}(v)\} \\
&= \max{}^i\{\bigcap_{i\in\Lambda} \mathscr{C}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{C}_{s_i}(v)\},
\end{aligned}$$
$$\begin{aligned}
\bigvee_{i\in\Lambda} \xi_i(u-v) &= \sup^i_{i\in\Lambda} \xi_i(u-v) \\
&\geq \sup^i_{i\in\Lambda} \min{}^i\{\xi_i(u), \xi_i(v)\} \\
&= \min{}^i\{\sup^i_{i\in\Lambda} \xi_i(u), \sup^i_{i\in\Lambda} \xi_i(v)\} \\
&= \min{}^i\{\bigvee_{i\in\Lambda} \xi_i(u), \bigvee_{i\in\Lambda} \xi_i(v)\},
\end{aligned}$$

(iv)
$$\begin{aligned}
\bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(u\alpha v\beta w) &= \inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(u\alpha v\beta w) \\
&\geq \inf^i_{i\in\Lambda} \min{}^i\{\mathscr{A}_{s_i}(u), \mathscr{A}_{s_i}(w)\} \\
&= \min{}^i\{\inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(u), \inf^i_{i\in\Lambda} \mathscr{A}_{s_i}(w)\} \\
&= \min{}^i\{\bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(u), \bigcap_{i\in\Lambda} \mathscr{A}_{s_i}(w)\},
\end{aligned}$$

$$\begin{aligned}
\underset{i\in\Lambda}{\vee}\mu_i(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\mu_i(u\alpha v\beta w)\\
&\le \sup^i_{i\in\Lambda}\max{}^i\{\mu_i(u),\mu_i(w)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda}\mu_i(u),\sup^i_{i\in\Lambda}\mu_i(w)\}\\
&= \max{}^i\{\underset{i\in\Lambda}{\vee}\mu_i(u),\underset{i\in\Lambda}{\vee}\mu_i(w)\},
\end{aligned}$$

(v) $$\begin{aligned}
\bigcap_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w) &= \inf^i_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w)\\
&\ge \inf^i_{i\in\Lambda}\min{}^i\{\mathscr{B}_{s_i}(u),\mathscr{B}_{s_i}(w)\}\\
&= \min{}^i\{\inf^i_{i\in\Lambda}\mathscr{B}_{s_i}(u),\inf^i_{i\in\Lambda}\mathscr{B}_{s_i}(w)\}\\
&= \min{}^i\{\bigcap_{i\in\Lambda}\mathscr{B}_{s_i}(u),\bigcap_{i\in\Lambda}\mathscr{B}_{s_i}(w)\},\\
\underset{i\in\Lambda}{\vee}\nu_i(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\nu_i(u\alpha v\beta w)\\
&\le \sup^i_{i\in\Lambda}\max{}^i\{\nu_i(u),\nu_i(w)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda}\nu_i(u),\sup^i_{i\in\Lambda}\nu_i(w)\}\\
&= \max{}^i\{\underset{i\in\Lambda}{\vee}\nu_i(u),\underset{i\in\Lambda}{\vee}\nu_i(w)\}, \text{ and}
\end{aligned}$$

(vi) $$\begin{aligned}
\bigcap_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w) &= \inf^i_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w)\\
&\le \inf^i_{i\in\Lambda}\max{}^i\{\mathscr{C}_{s_i}(u),\mathscr{C}_{s_i}(w)\}\\
&= \max{}^i\{\inf^i_{i\in\Lambda}\mathscr{C}_{s_i}(u),\inf^i_{i\in\Lambda}\mathscr{C}_{s_i}(w)\}\\
&= \max{}^i\{\bigcap_{i\in\Lambda}\mathscr{C}_{s_i}(u),\bigcap_{i\in\Lambda}\mathscr{C}_{s_i}(w)\},\\
\underset{i\in\Lambda}{\vee}\xi_i(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\xi_i(u\alpha v\beta w)\\
&\ge \sup^i_{i\in\Lambda}\min{}^i\{\xi_i(u),\xi_i(w)\}\\
&= \min{}^i\{\sup^i_{i\in\Lambda}\xi_i(u),\sup^i_{i\in\Lambda}\xi_i(w)\}\\
&= \min{}^i\{\underset{i\in\Lambda}{\vee}\xi_i(u),\underset{i\in\Lambda}{\vee}\xi_i(w)\}.
\end{aligned}$$

Hence $\bigcap_{R\, i\in\Lambda}\mathscr{C}\mathscr{U}_{s_i}$ is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$. □

Theorem 4.5. *If $\mathscr{C}\mathscr{U}_{s_i} = \{\langle\mathscr{A}_{s_i},\mu_i\rangle,\langle\mathscr{B}_{s_i},\nu_i\rangle,\langle\mathscr{C}_{s_i},\xi_i\rangle \mid i\in\Lambda\}$ be a family of spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$, then*

$$\bigcup_{\substack{P\\ i\in\Lambda}}\mathscr{C}\mathscr{U}_{s_i} = \{\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\vee}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\vee}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\vee}\xi_i(u)\rangle \mid u\in\mathcal{R}\}$$

is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$, where Λ is an index set.

Proof. Let $\mathscr{C}\mathscr{U}_{s_i} = \{\langle\mathscr{A}_{s_i},\mu_i\rangle,\langle\mathscr{B}_{s_i},\nu_i\rangle,\langle\mathscr{C}_{s_i},\xi_i\rangle \mid i\in\Lambda\}$ be a family of spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$. For any $u,v,w\in\mathcal{R}$ and $\alpha,\beta\in\Gamma$.

(i) $$\begin{aligned}
\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(u-v) &= \sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(u-v)\\
&\ge \sup^i_{i\in\Lambda}\min{}^i\{\mathscr{A}_{s_i}(u),\mathscr{A}_{s_i}(v)\}
\end{aligned}$$

$$
\begin{aligned}
&= \min{}^i\{\sup^i_{i\in\Lambda} \mathscr{A}_{s_i}(u), \sup^i_{i\in\Lambda} \mathscr{A}_{s_i}(v)\}\\
&= \min{}^i\{\bigcup_{i\in\Lambda} \mathscr{A}_{s_i}(u), \bigcup_{i\in\Lambda} \mathscr{A}_{s_i}(v)\},\\
\vee_{i\in\Lambda} \mu_i(u-v) &= \sup^i_{i\in\Lambda} \mu_i(u-v)\\
&\le \sup^i_{i\in\Lambda} \max{}^i\{\mu_i(u), \mu_i(v)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda} \mu_i(u), \sup^i_{i\in\Lambda} \mu_i(v)\}\\
&= \max{}^i\{\vee_{i\in\Lambda} \mu_i(u), \vee_{i\in\Lambda} \mu_i(v)\},
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\bigcup_{i\in\Lambda} \mathscr{B}_{s_i}(u-v) &= \sup^i_{i\in\Lambda} \mathscr{B}_{s_i}(u-v)\\
&\ge \sup^i_{i\in\Lambda} \min{}^i\{\mathscr{B}_{s_i}(u), \mathscr{B}_{s_i}(v)\}\\
&= \min{}^i\{\sup^i_{i\in\Lambda} \mathscr{B}_{s_i}(u), \sup^i_{i\in\Lambda} \mathscr{B}_{s_i}(v)\}\\
&= \min{}^i\{\bigcup_{i\in\Lambda} \mathscr{B}_{s_i}(u), \bigcup_{i\in\Lambda} \mathscr{B}_{s_i}(v)\},\\
\vee_{i\in\Lambda} \nu_i(u-v) &= \sup^i_{i\in\Lambda} \nu_i(u-v)\\
&\le \sup^i_{i\in\Lambda} \max{}^i\{\nu_i(u), \nu_i(v)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda} \nu_i(u), \sup^i_{i\in\Lambda} \nu_i(v)\}\\
&= \max{}^i\{\vee_{i\in\Lambda} \nu_i(u), \vee_{i\in\Lambda} \nu_i(v)\},
\end{aligned}
$$

(iii)
$$
\begin{aligned}
\bigcup_{i\in\Lambda} \mathscr{C}_{s_i}(u-v) &= \sup^i_{i\in\Lambda} \mathscr{C}_{s_i}(u-v)\\
&\le \sup^i_{i\in\Lambda} \max{}^i\{\mathscr{C}_{s_i}(u), \mathscr{C}_{s_i}(v)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda} \mathscr{C}_{s_i}(u), \sup^i_{i\in\Lambda} \mathscr{C}_{s_i}(v)\}\\
&= \max{}^i\{\bigcup_{i\in\Lambda} \mathscr{C}_{s_i}(u), \bigcup_{i\in\Lambda} \mathscr{C}_{s_i}(v)\},\\
\vee_{i\in\Lambda} \xi_i(u-v) &= \sup^i_{i\in\Lambda} \xi_i(u-v)\\
&\ge \sup^i_{i\in\Lambda} \min{}^i\{\xi_i(u), \xi_i(v)\}\\
&= \min{}^i\{\sup^i_{i\in\Lambda} \xi_i(u), \sup^i_{i\in\Lambda} \xi_i(v)\}\\
&= \min{}^i\{\vee_{i\in\Lambda} \xi_i(u), \vee_{i\in\Lambda} \xi_i(v)\},
\end{aligned}
$$

(iv)
$$
\begin{aligned}
\bigcup_{i\in\Lambda} \mathscr{A}_{s_i}(u\alpha v\beta w) &= \sup^i_{i\in\Lambda} \mathscr{A}_{s_i}(u\alpha v\beta w)\\
&\ge \sup^i_{i\in\Lambda} \min{}^i\{\mathscr{A}_{s_i}(u), \mathscr{A}_{s_i}(w)\}\\
&= \min{}^i\{\sup^i_{i\in\Lambda} \mathscr{A}_{s_i}(u), \sup^i_{i\in\Lambda} \mathscr{A}_{s_i}(w)\}\\
&= \min{}^i\{\bigcup_{i\in\Lambda} \mathscr{A}_{s_i}(u), \bigcup_{i\in\Lambda} \mathscr{A}_{s_i}(w)\},\\
\vee_{i\in\Lambda} \mu_i(u\alpha v\beta w) &= \sup^i_{i\in\Lambda} \mu_i(u\alpha v\beta w)\\
&\le \sup^i_{i\in\Lambda} \max{}^i\{\mu_i(u), \mu_i(w)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda} \mu_i(u), \sup^i_{i\in\Lambda} \mu_i(w)\}
\end{aligned}
$$

$$= \max^i\{\underset{i\in\Lambda}{\vee}\mu_i(u), \underset{i\in\Lambda}{\vee}\mu_i(w)\},$$

(v)
$$\begin{aligned}\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w)\\ &\geq \sup^i_{i\in\Lambda}\min^i\{\mathscr{B}_{s_i}(u),\mathscr{B}_{s_i}(w)\}\\ &= \min^i\{\sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(w)\}\\ &= \min^i\{\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(w)\},\\ \underset{i\in\Lambda}{\vee}\nu_i(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\nu_i(u\alpha v\beta w)\\ &\leq \sup^i_{i\in\Lambda}\max^i\{\nu_i(u),\nu_i(w)\}\\ &= \max^i\{\sup^i_{i\in\Lambda}\nu_i(u),\sup^i_{i\in\Lambda}\nu_i(w)\}\\ &= \max^i\{\underset{i\in\Lambda}{\vee}\nu_i(u),\underset{i\in\Lambda}{\vee}\nu_i(w)\}, \text{ and}\end{aligned}$$

(vi)
$$\begin{aligned}\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w)\\ &\leq \sup^i_{i\in\Lambda}\max^i\{\mathscr{C}_{s_i}(u),\mathscr{C}_{s_i}(w)\}\\ &= \max^i\{\sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(w)\}\\ &= \max^i\{\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(w)\},\\ \underset{i\in\Lambda}{\vee}\xi_i(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\xi_i(u\alpha v\beta w)\\ &\geq \sup^i_{i\in\Lambda}\min^i\{\xi_i(u),\xi_i(w)\}\\ &= \min^i\{\sup^i_{i\in\Lambda}\xi_i(u),\sup^i_{i\in\Lambda}\xi_i(w)\}\\ &= \min^i\{\underset{i\in\Lambda}{\vee}\xi_i(u),\underset{i\in\Lambda}{\vee}\xi_i(w)\}.\end{aligned}$$

Hence $\bigcup_{P, i\in\Lambda}\mathscr{CU}_{s_i}$ is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$. □

Theorem 4.6. *If* $\mathscr{CU}_{s_i} = \{\langle\mathscr{A}_{s_i},\mu_i\rangle,\langle\mathscr{B}_{s_i},\nu_i\rangle,\langle\mathscr{C}_{s_i},\xi_i\rangle \mid i\in\Lambda\}$ *be a family of spherical cubic bi-ideals of gamma near-ring* $\mathcal{R}$*, then*

$$\bigcup_{R, i\in\Lambda}\mathscr{CU}_{s_i} = \{\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{A}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\mu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{B}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\nu_i(u)\rangle,\langle u,(\underset{i\in\Lambda}{\cup}\mathscr{C}_{s_i})(u),\underset{i\in\Lambda}{\wedge}\xi_i(u)\rangle \mid u\in\mathcal{R}\}$$

is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$*, where* Λ *is an index set.*

Proof. Let $\mathscr{CU}_{s_i} = \{\langle\mathscr{A}_{s_i},\mu_i\rangle,\langle\mathscr{B}_{s_i},\nu_i\rangle,\langle\mathscr{C}_{s_i},\xi_i\rangle \mid i\in\Lambda\}$ be a family of spherical cubic bi-ideals of gamma near-ring $\mathcal{R}$. For any $u,v,w\in\mathcal{R}$ and $\alpha,\beta\in\Gamma$.

(i)
$$\begin{aligned}\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(u-v) &= \sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(u-v)\\ &\geq \sup^i_{i\in\Lambda}\min^i\{\mathscr{A}_{s_i}(u),\mathscr{A}_{s_i}(v)\}\\ &= \min^i\{\sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(v)\}\\ &= \min^i\{\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(v)\},\end{aligned}$$

$$\begin{aligned}
\wedge_{i\in\Lambda}\mu_i(u-v) &= \inf^i_{i\in\Lambda}\mu_i(u-v)\\
&\le \inf^i_{i\in\Lambda}\max{}^i\{\mu_i(u),\mu_i(v)\}\\
&= \max{}^i\{\inf^i_{i\in\Lambda}\mu_i(u),\inf^i_{i\in\Lambda}\mu_i(v)\}\\
&= \max{}^i\{\wedge_{i\in\Lambda}\mu_i(u),\wedge_{i\in\Lambda}\mu_i(v)\},
\end{aligned}$$

(ii)
$$\begin{aligned}
\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(u-v) &= \sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(u-v)\\
&\ge \sup^i_{i\in\Lambda}\min{}^i\{\mathscr{B}_{s_i}(u),\mathscr{B}_{s_i}(v)\}\\
&= \min{}^i\{\sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(v)\}\\
&= \min{}^i\{\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(v)\},
\end{aligned}$$

$$\begin{aligned}
\wedge_{i\in\Lambda}\nu_i(u-v) &= \inf^i_{i\in\Lambda}\nu_i(u-v)\\
&\le \inf^i_{i\in\Lambda}\max{}^i\{\nu_i(u),\nu_i(v)\}\\
&= \max{}^i\{\inf^i_{i\in\Lambda}\nu_i(u),\inf^i_{i\in\Lambda}\nu_i(v)\}\\
&= \max{}^i\{\wedge_{i\in\Lambda}\nu_i(u),\wedge_{i\in\Lambda}\nu_i(v)\},
\end{aligned}$$

(iii)
$$\begin{aligned}
\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(u-v) &= \sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(u-v)\\
&\le \sup^i_{i\in\Lambda}\max{}^i\{\mathscr{C}_{s_i}(u),\mathscr{C}_{s_i}(v)\}\\
&= \max{}^i\{\sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(v)\}\\
&= \max{}^i\{\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(v)\},
\end{aligned}$$

$$\begin{aligned}
\wedge_{i\in\Lambda}\xi_i(u-v) &= \inf^i_{i\in\Lambda}\xi_i(u-v)\\
&\ge \inf^i_{i\in\Lambda}\min{}^i\{\xi_i(u),\xi_i(v)\}\\
&= \min{}^i\{\inf^i_{i\in\Lambda}\xi_i(u),\inf^i_{i\in\Lambda}\xi_i(v)\}\\
&= \min{}^i\{\wedge_{i\in\Lambda}\xi_i(u),\wedge_{i\in\Lambda}\xi_i(v)\},
\end{aligned}$$

(iv)
$$\begin{aligned}
\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(u\alpha v\beta w)\\
&\ge \sup^i_{i\in\Lambda}\min{}^i\{\mathscr{A}_{s_i}(u),\mathscr{A}_{s_i}(w)\}\\
&= \min{}^i\{\sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{A}_{s_i}(w)\}\\
&= \min{}^i\{\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{A}_{s_i}(w)\},
\end{aligned}$$

$$\begin{aligned}
\wedge_{i\in\Lambda}\mu_i(u\alpha v\beta w) &= \inf^i_{i\in\Lambda}\mu_i(u\alpha v\beta w)\\
&\le \inf^i_{i\in\Lambda}\max{}^i\{\mu_i(u),\mu_i(w)\}\\
&= \max{}^i\{\inf^i_{i\in\Lambda}\mu_i(u),\inf^i_{i\in\Lambda}\mu_i(w)\}\\
&= \max{}^i\{\wedge_{i\in\Lambda}\mu_i(u),\wedge_{i\in\Lambda}\mu_i(w)\},
\end{aligned}$$

$$
\begin{aligned}
\text{(v)}\quad \bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(u\alpha v\beta w)\\
&\geq \sup^i_{i\in\Lambda}\min^i\{\mathscr{B}_{s_i}(u),\mathscr{B}_{s_i}(w)\}\\
&= \min^i\{\sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{B}_{s_i}(w)\}\\
&= \min^i\{\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{B}_{s_i}(w)\},\\
\bigwedge_{i\in\Lambda}\nu_i(u\alpha v\beta w) &= \inf^i_{i\in\Lambda}\nu_i(u\alpha v\beta w)\\
&\leq \inf^i_{i\in\Lambda}\max^i\{\nu_i(u),\nu_i(w)\}\\
&= \max^i\{\inf^i_{i\in\Lambda}\nu_i(u),\inf^i_{i\in\Lambda}\nu_i(w)\}\\
&= \max^i\{\bigwedge_{i\in\Lambda}\nu_i(u),\bigwedge_{i\in\Lambda}\nu_i(w)\}, \text{ and}\\
\text{(vi)}\quad \bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w) &= \sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(u\alpha v\beta w)\\
&\leq \sup^i_{i\in\Lambda}\max^i\{\mathscr{C}_{s_i}(u),\mathscr{C}_{s_i}(w)\}\\
&= \max^i\{\sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(u),\sup^i_{i\in\Lambda}\mathscr{C}_{s_i}(w)\}\\
&= \max^i\{\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(u),\bigcup_{i\in\Lambda}\mathscr{C}_{s_i}(w)\},\\
\bigwedge_{i\in\Lambda}\xi_i(u\alpha v\beta w) &= \inf^i_{i\in\Lambda}\xi_i(u\alpha v\beta w)\\
&\geq \inf^i_{i\in\Lambda}\min^i\{\xi_i(u),\xi_i(w)\}\\
&= \min^i\{\inf^i_{i\in\Lambda}\xi_i(u),\inf^i_{i\in\Lambda}\xi_i(w)\}\\
&= \min^i\{\bigwedge_{i\in\Lambda}\xi_i(u),\bigwedge_{i\in\Lambda}\xi_i(w)\}.
\end{aligned}
$$

Hence $\bigcup_{R\, i\in\Lambda}\mathscr{C}\mathscr{U}_{s_i}$ is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$. □

Lemma 4.7. *Let C be a bi-ideal of gamma near-ring $\mathcal{R}$. For any $0<m<1$, there exists $\mathscr{C}\mathscr{U}_s$, a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$ such that $\mathscr{C}\mathscr{U}_{s_m}=C$.*

Proof. Let C be a bi-ideal of gamma near ring $\mathcal{R}$. Define $\mathscr{C}\mathscr{U}_s$ by

$$
\mathscr{C}\mathscr{U}_s(u)=\begin{cases} m, & \text{if } u\in C\\ 0, & \text{if } u\notin C,\end{cases}
$$

where m be a constant in $(0,1)$. Clearly, $\mathscr{C}\mathscr{U}_{s_m}=C$.

Let $u,v\in\mathcal{R}$. If $u,v\in C$, then $\mathscr{A}_s(u-v)=m\geq\min^i\{\mathscr{A}_s(u),\mathscr{A}_s(v)\}$, $\mathscr{B}_s(u-v)=m\geq\min^i\{\mathscr{B}_s(u),\mathscr{B}_s(v)\}$ and $\mathscr{C}_s(u-v)=m\leq\max^i\{\mathscr{C}_s(u),\mathscr{C}_s(v)\}$, $\mu(u-v)=m\leq\max\{\mu(u),\mu(v)\}$, $\nu(u-v)=m\leq\max\{\nu(u),\nu(v)\}$ and $\xi(u-v)=m\geq\min\{\xi(u),\xi(v)\}$.

If at least one of u and v is not in C, then $u-v\notin C$ and so $\mathscr{A}_s(u-v)=0=\min^i\{\mathscr{A}_s(u),\mathscr{A}_s(v)\}$, $\mathscr{B}_s(u-v)=0=\min^i\{\mathscr{B}_s(u),\mathscr{B}_s(v)\}$ and $\mathscr{C}_s(u-v)=0=\max^i\{\mathscr{C}_s(u),\mathscr{C}_s(v)\}$, $\mu(u-v)=0=\max\{\mu(u),\mu(v)\}$, $\nu(u-v)=0=\max\{\nu(u),\nu(v)\}$ and $\xi(u-v)=0=\min\{\xi(u),\xi(v)\}$.

Let $u,v,w\in\mathcal{R}$ and $\alpha,\beta\in\Gamma$. If $u,w\in C$, then $\mathscr{A}_s(u),\mathscr{B}_s(u),\mathscr{C}_s(u)=m;\mathscr{A}_s(w),\mathscr{B}_s(w),\mathscr{C}_s(w)=m$, $\mu(u),\nu(u),\xi(u)=m;\mu(w),\nu(w),\xi(w)=m$.

Also, $\mathscr{A}_s(u\alpha v\beta w) = m \geq \min^i\{\mathscr{A}_s(u), \mathscr{A}_s(w)\}$, $\mathscr{B}_s(u\alpha v\beta w) = m \geq \min^i\{\mathscr{B}_s(u), \mathscr{B}_s(w)\}$ and $\mathscr{C}_s(u\alpha v\beta w) = m \leq \max^i\{\mathscr{C}_s(u), \mathscr{C}_s(w)\}$, $\mu(u\alpha v\beta w) = m \leq \max\{\mu(u), \mu(w)\}$, $\nu(u\alpha v\beta w) = m \leq \max\{\nu(u), \nu(w)\}$ and $\xi(u\alpha v\beta w) = m \geq \min\{\xi(u), \xi(w)\}$.
If at least one of u and w is not in C, then $\mathscr{A}_s(u\alpha v\beta w) \geq 0 = \min^i\{\mathscr{A}_s(u), \mathscr{A}_s(w)\}$, $\mathscr{B}_s(u\alpha v\beta w) \geq 0 = \min^i\{\mathscr{B}_s(u), \mathscr{B}_s(w)\}$ and $\mathscr{C}_s(u\alpha v\beta w) \leq 0 = \max^i\{\mathscr{C}_s(u), \mathscr{C}_s(w)\}, \mu(u\alpha v\beta w) \leq 0 = \max\{\mu(u), \mu(w)\}$, $\nu(u\alpha v\beta w) \leq 0 = \max\{\nu(u), \nu(w)\}$ and $\xi(u\alpha v\beta w) \geq 0 = \min\{\xi(u), \xi(w)\}$.
Thus $\mathscr{CU}_s$ is a spherical cubic bi-ideal of gamma near-ring $\mathfrak{R}$. □

Theorem 4.8. *If $\mathscr{CU}_s$ be a spherical cubic bi-ideal of gamma near-ring $\mathfrak{R}$, then the complement ${\mathscr{CU}_s}^c$ is also a spherical cubic bi-ideal of gamma near-ring $\mathfrak{R}$.*

Proof. For $u, v, w \in \mathfrak{R}$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}
{\mathscr{A}_s}^c(u-v) &= 1 - \mathscr{A}_s(u-v) \\
&\geq 1 - \min^i\{\mathscr{A}_s(u), \mathscr{A}_s(v)\} \\
&= \min^i\{1-\mathscr{A}_s(u), 1-\mathscr{A}_s(v)\} \\
&= \min^i\{{\mathscr{A}_s}^c(u), {\mathscr{A}_s}^c(v)\}, \\
{\mathscr{B}_s}^c(u-v) &= 1 - \mathscr{B}_s(u-v) \\
&\geq 1 - \min^i\{\mathscr{B}_s(u), \mathscr{B}_s(v)\} \\
&= \min^i\{1-\mathscr{B}_s(u), 1-\mathscr{B}_s(v)\} \\
&= \min^i\{{\mathscr{B}_s}^c(u), {\mathscr{B}_s}^c(v)\}, \\
{\mathscr{C}_s}^c(u-v) &= 1 - \mathscr{C}_s(u-v) \\
&\leq 1 - \max^i\{\mathscr{C}_s(u), \mathscr{C}_s(v)\} \\
&= \max^i\{1-\mathscr{C}_s(u), 1-\mathscr{A}_s(v)\} \\
&= \max^i\{{\mathscr{C}_s}^c(u), {\mathscr{C}_s}^c(v)\}, \\
\mu^c(u-v) &= 1 - \mu(u-v) \\
&\leq 1 - \max\{\mu(u), \mu(v)\} \\
&= \max\{1-\mu(u), 1-\mu(v)\} \\
&= \max\{\mu^c(u), \mu^c(v)\}, \\
\nu^c(u-v) &= 1 - \nu(u-v) \\
&\leq 1 - \max\{\nu(u), \nu(v)\} \\
&= \max\{1-\nu(u), 1-\nu(v)\} \\
&= \max\{\nu^c(u), \nu^c(v)\}, \\
\xi^c(u-v) &= 1 - \xi(u-v) \\
&\geq 1 - \min\{\xi(u), \xi(v)\} \\
&= \min\{1-\xi(u), 1-\xi(v)\} \\
&= \min\{\xi^c(u), \xi^c(v)\}, \\
{\mathscr{A}_s}^c(u\alpha v\beta w) &= 1 - \mathscr{A}_s(u\alpha v\beta w) \\
&\geq 1 - \min^i\{\mathscr{A}_s(u), \mathscr{A}_s(w)\}
\end{aligned}$$

$$
\begin{aligned}
&= \min{}^{i}\{1-\mathscr{A}_s(u), 1-\mathscr{A}_s(w)\}\\
&= \min{}^{i}\{\mathscr{A}_s{}^{c}(u), \mathscr{A}_s{}^{c}(w)\},\\
\mathscr{B}_s{}^{c}(u\alpha v\beta w) &= 1-\mathscr{B}_s(u\alpha v\beta w)\\
&\geq 1-\min{}^{i}\{\mathscr{B}_s(u), \mathscr{B}_s(w)\}\\
&= \min{}^{i}\{1-\mathscr{B}_s(u), 1-\mathscr{B}_s(w)\}\\
&= \min{}^{i}\{\mathscr{B}_s{}^{c}(u), \mathscr{B}_s{}^{c}(w)\},\\
\mathscr{C}_s{}^{c}(u\alpha v\beta w) &= 1-\mathscr{C}_s(u\alpha v\beta w)\\
&\leq 1-\max{}^{i}\{\mathscr{C}_s(u), \mathscr{C}_s(w)\}\\
&= \max{}^{i}\{1-\mathscr{C}_s(u), 1-\mathscr{C}_s(w)\}\\
&= \max{}^{i}\{\mathscr{C}_s{}^{c}(u), \mathscr{C}_s{}^{c}(w)\},\\
\mu^{c}(u\alpha v\beta w) &= 1-\mu(u\alpha v\beta w)\\
&\leq 1-\max\{\mu(u), \mu(w)\}\\
&= \max\{1-\mu(u), 1-\mu(w)\}\\
&= \max\{\mu^{c}(u), \mu^{c}(w)\},\\
\nu^{c}(u\alpha v\beta w) &= 1-\nu(u\alpha v\beta w)\\
&\leq 1-\max\{\nu(u), \nu(w)\}\\
&= \max\{1-\nu(u), 1-\nu(w)\}\\
&= \max\{\nu^{c}(u), \nu^{c}(w)\},\\
\xi^{c}(u\alpha v\beta w) &= 1-\xi(u\alpha v\beta w)\\
&\geq 1-\min\{\xi(u), \xi(w)\}\\
&= \min\{1-\xi(u), 1-\xi(w)\}\\
&= \min\{\xi^{c}(u), \xi^{c}(w)\}.
\end{aligned}
$$

Hence $\mathscr{C}\mathscr{U}_s{}^{c}$ is also a spherical cubic bi-ideal of gamma near-ring $\mathcal{R}$. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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