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Research Article

Partial Ordering of Block Matrices in Minkowski Space

D. Krishnaswamy * and M. Velmurugan

Department of Mathematics, Annamalai University, Annamalainagar 608002, Tamil Nadu, India

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Abstract. In this paper, we study the partial orderings of block matrices and the submatrix partial orderings, we also present the results of star orderings in Minkowski space.

Keywords. Matrix partial orderings, Moore-Penrose inverse, Block matrix, Minkowski adjoint, Minkowski inverse, Minkowski space

Mathematics Subject Classification (2020). 15A09; 15A45; 15A57

1. Introduction

Throughout this paper, let us denote the set of complex matrices as $C^{m\times n}$ and C^n represent complex n-tuples. The symbols P_1^*, $P_1^\dagger$, $P_1^\sim$, $P_1^{ⓜ}$, $R(P_1)$ and $N(P_1)$ denote the conjugate transpose, Moore-Penrose inverse, Minkowski adjoint, Minkowski inverse, range space and null space of a matrix P_1, respectively. The components of this complex vector in C^n is represented as $u=(u_0,u_1,u_2,\dots,u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu=(u_0,-u_1,-u_2,\dots,-u_{n-1})$. Clearly, the Minkowski metric matrix is given by

$$G=\begin{pmatrix}1 & 0\\ 0 & -I_{n-1}\end{pmatrix}, \tag{1.1}$$

$G=G^*$ and $G^2=I_n$. In [11], defined Minkowski inner product on C^n by $(u,v)=[u,Gv]$, where $[\cdot,\cdot]$ denotes the conventional Hilbert space inner product, $\mathcal{M}$ denotes the Minkowski space, which is a space with Minkowski inner product.

*Corresponding author: mpvelmurugan18@gmail.com

In the year 2000 Meenakshi [7] presented the concept of Minkowski inverse of a matrix represented as $A \in C^{m\times n}$. Also, presented a unique solution to the following four matrix equations:

$$AXA = A,\ XAX = X,\ (AX)^{\sim} = AX,\ (XA)^{\sim} = XA, \tag{1.2}$$

where $A^{\sim}$ denotes the Minkowski adjoint of the matrix A in $\mathcal{M}$.

However, the Minkowski inverse of a matrix does not exists always as in Moore-Penrose inverse of a matrix. The proved that the Minkowski inverse of a matrix $A \in C^{m\times n}$ exists if and only if $rk(AA^{\sim}) = rk(A^{\sim}A) = rk(A)$. A matrix $A \in C^{n}$ is said to be m-symmetric if $A = A^{\sim}$. Also, presented the notion of range symmetric matrices in Minkowski space. Further developed the concept of Minkowski inverse of the range symmetric matrices and its equivalent conditions.

Many authors show interest on partial orders on matrices. Most of the authors present different kinds of generalized inverses following mainly on Moore-Penrose inverses. [1, 2, 10, 12] present the result involving partial orders on matrices. Drazin [5] presented the concept of Star partial ordering $\overset{*}{\leq}$, Hartwig [6] introduced the notion of minus partial order $\leq^{-}$, Mitra [9] presented the concept of Sharp partial order $\leq^{\#}$, left star ordering $*\leq$ and right star ordering $\leq *$.

In this paper, we consider matrix partial orderings in $C^{m\times n}$. First, we discuss on star ordering in Minkowski space is defined by

$$P_1 \overset{\sim}{\leq} Q_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1 \ \text{ and } \ P_1G(GP_1)^{\sim} = (Q_1G)(GP_1)^{\sim} \tag{1.3}$$

and

$$P_1 \overset{\sim}{\leq} Q_1 \Leftrightarrow (GP_1)^{\textcircled{m}}GP_1 = (GP_1)^{\textcircled{m}}GQ_1 \ \text{ and } \ P_1G(P_1G)^{\textcircled{m}} = Q_1G(P_1G)^{\textcircled{m}}. \tag{1.4}$$

The left, right star orderings in Minkowski space is defined by

$$P_1 \sim\leq Q_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1 \ (\text{or } (GP_1)^{\textcircled{m}}GP_1 = (GP_1)^{\textcircled{m}}GQ_1) \ \text{ and } \ R(P_1) \subseteq R(Q_1), \tag{1.5}$$

$$P_1 \leq\sim Q_1 \Leftrightarrow P_1G(GP_1)^{\sim} = Q_1G(GP_1)^{\sim} \ (\text{or } P_1G(P_1G)^{\textcircled{m}} = Q_1G(P_1G)^{\textcircled{m}}) \ \text{ and } \ R((P_1G)^{\sim}) \subseteq R((Q_1G)^{\sim}). \tag{1.6}$$

The reverse order law and matrix partial ordering were investigated by Benitez *et al*. [4]

2. Star Partial Ordering in Minkowski Space

In this section, we present the results on the star partial orderings in Minkowski space.

Theorem 2.1. *Let $P_1, R_1 \in C^{m\times n}$ and $Q_1, S_1 \in C^{m\times k}$ be star-ordered as $P_1 \overset{\sim}{\leq} R_1, Q_1 \overset{\sim}{\leq} S_1$. If $R(P_1) = R(Q_1)$, then $G\begin{pmatrix} P_1 & Q_1 \end{pmatrix} \overset{\sim}{\leq} G\begin{pmatrix} R_1 & S_1 \end{pmatrix}$.*

Proof. On account of eqs. (1.3) and (1.4), since $P_1 \overset{\sim}{\leq} R_1$, $Q_1 \overset{\sim}{\leq} S_1$ and $R(P_1) = R(Q_1)$, so

(i) $P_1 \overset{\sim}{\leq} R_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GR_1$ and $P_1G(GP_1)^{\sim} = R_1G(GP_1)^{\sim}$,

(ii) $Q_1 \overset{\sim}{\leq} S_1 \Leftrightarrow (Q_1G)^{\sim}GQ_1 = (Q_1G)^{\sim}GS_1$ and $Q_1G(GQ_1)^{\sim} = S_1G(GQ_1)^{\sim}$.

$$(P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GR_1$$

$$\begin{aligned}
GP_1 &= ((P_1G)^{\sim})^{ⓜ}(P_1G)^{\sim}GR_1 \\
&= ((P_1G)^{ⓜ})^{\sim}(P_1G)^{\sim}GR_1 \\
GP_1 &= ((P_1G)(P_1G)^{ⓜ})^{\sim}GR_1, \qquad (2.1)\\
(Q_1G)^{\sim}GQ_1 &= (Q_1G)^{\sim}GS_1 \\
GQ_1 &= ((Q_1G)^{\sim})^{ⓜ}(Q_1G)^{\sim}GS_1 \\
&= ((Q_1G)^{ⓜ})^{\sim}(Q_1G)^{\sim}GS_1 \\
GQ_1 &= ((Q_1G)(Q_1G)^{ⓜ})^{\sim}GS_1 \qquad (2.2)\\
P_1G(GP_1)^{\sim} &= R_1G(GP_1)^{\sim} \\
P_1G &= R_1G(GP_1)^{\sim}((GP_1)^{\sim})^{ⓜ} \\
&= R_1G(GP_1)^{\sim}((GP_1)^{ⓜ})^{\sim} \\
P_1G &= R_1G((GP_1)^{ⓜ}GP_1)^{\sim} \qquad (2.3)\\
Q_1G(GQ_1)^{\sim} &= S_1G(GQ_1)^{\sim} \\
Q_1G &= S_1G(GQ_1)^{\sim}((GQ_1)^{\sim})^{ⓜ} \\
&= S_1G(GQ_1)^{\sim}((GQ_1)^{ⓜ})^{\sim} \\
Q_1G &= S_1G((GQ_1)^{ⓜ}(GQ_1))^{\sim}. \qquad (2.4)
\end{aligned}$$

Consider,

$$\begin{aligned}
\begin{pmatrix} GP_1^{\sim} \\ GQ_1^{\sim} \end{pmatrix}\begin{pmatrix} GP_1 & GQ_1 \end{pmatrix} &= \begin{pmatrix} GP_1^{\sim}GP_1 & GP_1^{\sim}GQ_1 \\ GQ_1^{\sim}GP_1 & GQ_1^{\sim}GQ_1 \end{pmatrix} \\
G\begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim}G\begin{pmatrix} P_1 & Q_1 \end{pmatrix} &= \begin{pmatrix} (P_1G)^{\sim}GP_1 & (P_1G)^{\sim}GQ_1 \\ (Q_1G)^{\sim}GP_1 & (Q_1G)^{\sim}GQ_1 \end{pmatrix} \qquad \text{(using (2.1) and (2.2))} \\
&= \begin{pmatrix} (P_1G)^{\sim}GR_1 & (P_1G)^{\sim}((Q_1G)(Q_1G)^{ⓜ})^{\sim}GS_1 \\ (Q_1G)^{\sim}((P_1G)(P_1G)^{ⓜ})^{\sim}GR_1 & (Q_1G)^{\sim}GS_1 \end{pmatrix} \\
&= \begin{pmatrix} (P_1G)^{\sim}GR_1 & ((Q_1G)(Q_1G)^{ⓜ}P_1G)^{\sim}GS_1 \\ ((P_1G)(P_1G)^{ⓜ}Q_1G)^{\sim}GR_1 & (Q_1G)^{\sim}GS_1 \end{pmatrix} \\
&= \begin{pmatrix} (P_1G)^{\sim}GR_1 & (P_1G)^{\sim}GS_1 \\ (Q_1G)^{\sim}GR_1 & (Q_1G)^{\sim}GS_1 \end{pmatrix} \\
&= \begin{pmatrix} (P_1G)^{\sim} \\ (Q_1G)^{\sim} \end{pmatrix}\begin{pmatrix} GR_1 & GS_1 \end{pmatrix} \\
&= \begin{pmatrix} GP_1^{\sim} \\ GQ_1^{\sim} \end{pmatrix}\begin{pmatrix} GR_1 & GS_1 \end{pmatrix} \\
&= G\begin{pmatrix} P_1^{\sim} \\ Q_1^{\sim} \end{pmatrix}G\begin{pmatrix} R_1 & S_1 \end{pmatrix} \\
&= G\begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim}G\begin{pmatrix} R_1 & S_1 \end{pmatrix} \qquad (2.5)
\end{aligned}$$

Consider,

$$\begin{pmatrix} P_1G & Q_1G \end{pmatrix}\begin{pmatrix} P_1^{\sim}G \\ Q_1^{\sim}G \end{pmatrix} = \begin{pmatrix} P_1GP_1^{\sim}G & P_1GQ_1^{\sim}G \\ Q_1GP_1^{\sim}G & Q_1GQ_1^{\sim}G \end{pmatrix}$$

$$
\begin{aligned}
\begin{pmatrix} P_1 & Q_1 \end{pmatrix} G \begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim} G &= \begin{pmatrix} P_1G(GP_1)^{\sim} & P_1G(GQ_1)^{\sim} \\ Q_1G(GP_1)^{\sim} & Q_1G(GQ_1)^{\sim} \end{pmatrix} \quad \text{(using (2.3) and (2.4))} \\
&= \begin{pmatrix} R_1G(GP_1)^{\sim} & R_1G((GP_1)^{ⓜ}(GP_1))^{\sim}(GQ_1)^{\sim} \\ S_1G((GQ_1)^{ⓜ}(GQ_1))^{\sim}(GP_1)^{\sim} & S_1G(GQ_1)^{\sim} \end{pmatrix} \\
&= \begin{pmatrix} R_1G(GP_1)^{\sim} & R_1G((GQ_1)(GP_1)^{ⓜ}(GP_1))^{\sim} \\ S_1G((GP_1)(GQ_1)^{ⓜ}(GQ_1))^{\sim} & S_1G(GQ_1)^{\sim} \end{pmatrix} \\
&= \begin{pmatrix} R_1G(GP_1)^{\sim} & R_1G(GQ_1)^{\sim} \\ S_1G(GP_1)^{\sim} & S_1G(GQ_1)^{\sim} \end{pmatrix} \\
&= \begin{pmatrix} R_1G & S_1G \end{pmatrix} \begin{pmatrix} (GP_1)^{\sim} \\ (GQ_1)^{\sim} \end{pmatrix} \\
&= \begin{pmatrix} R_1G & S_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G \\ Q_1^{\sim}G \end{pmatrix} \\
&= \begin{pmatrix} R_1 & S_1 \end{pmatrix} G \begin{pmatrix} P_1^{\sim} \\ Q_1^{\sim} \end{pmatrix} G \\
&= \begin{pmatrix} R_1 & S_1 \end{pmatrix} G \begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim} G.
\end{aligned}
$$

Pre and post multiplying by G, we have

$$
= G \begin{pmatrix} R_1 & S_1 \end{pmatrix} G \begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim}. \tag{2.6}
$$

From eqs. (2.5) and (2.6), we have

$$
G \begin{pmatrix} P_1 & Q_1 \end{pmatrix} \stackrel{\sim}{\leq} G \begin{pmatrix} R_1 & S_1 \end{pmatrix}.
$$

Hence proved. □

Theorem 2.2. *Let $P_1, R_1 \in C^{m\times n}$ and $Q_1, S_1 \in C^{m\times k}$ be star-ordered as $P_1 \sim\leq R_1, Q_1 \sim\leq S_1$. If $R(P_1) = R(Q_1)$, then $G(P_1\ Q_1) \sim\leq G(R_1\ S_1)$.*

Proof. (i) $P_1 \sim\leq Q_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1$ (or $(GP_1)^{ⓜ})GP_1 = (GP_1)^{ⓜ}GQ_1$ and $R(P_1) \subseteq R(Q_1)$.

(ii) $P_1 \sim\leq R_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GR_1$ (or $(GP_1)^{ⓜ})GP_1 = (GP_1)^{ⓜ}GR_1$ and $R(P_1) \subseteq R(R_1)$.

(iii) $Q_1 \sim\leq S_1 \Leftrightarrow (Q_1G)^{\sim}GQ_1 = (Q_1G)^{\sim}GS_1$ (or $(GQ_1)^{ⓜ})GQ_1 = (GQ_1)^{ⓜ}GS_1$ and $R(Q_1) \subseteq R(S_1)$.

Consider,

$$
\begin{aligned}
\begin{pmatrix} GP_1^{\sim} \\ GQ_1^{\sim} \end{pmatrix} \begin{pmatrix} GP_1 & GQ_1 \end{pmatrix} &= \begin{pmatrix} GP_1^{\sim}GP_1 & GP_1^{\sim}GQ_1 \\ GQ_1^{\sim}GP_1 & GQ_1^{\sim}GQ_1 \end{pmatrix} \\
G \begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim} G \begin{pmatrix} P_1 & Q_1 \end{pmatrix} &= \begin{pmatrix} (P_1G)^{\sim}GP_1 & (P_1G)^{\sim}GQ_1 \\ (Q_1G)^{\sim}GP_1 & (Q_1G)^{\sim}GQ_1 \end{pmatrix} \quad \text{(using (2.1) and (2.2))} \\
&= \begin{pmatrix} (P_1G)^{\sim}GR_1 & (P_1G)^{\sim}((Q_1G)(Q_1G)^{ⓜ})^{\sim}GS_1 \\ (Q_1G)^{\sim}((P_1G)(P_1G)^{ⓜ})^{\sim}GR_1 & (Q_1G)^{\sim}GS_1 \end{pmatrix} \\
&= \begin{pmatrix} (P_1G)^{\sim}GR_1 & ((Q_1G)(Q_1G)^{ⓜ}P_1G)^{\sim}GS_1 \\ ((P_1G)(P_1G)^{ⓜ}Q_1G)^{\sim}GR_1 & (Q_1G)^{\sim}GS_1 \end{pmatrix} \\
&= \begin{pmatrix} (P_1G)^{\sim}GR_1 & (P_1G)^{\sim}GS_1 \\ (Q_1G)^{\sim}GR_1 & (Q_1G)^{\sim}GS_1 \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} (P_1G)^{\sim} \\ (Q_1G)^{\sim} \end{pmatrix} \begin{pmatrix} GR_1 & GS_1 \end{pmatrix}$$
$$= \begin{pmatrix} GP_1^{\sim} \\ GQ_1^{\sim} \end{pmatrix} \begin{pmatrix} GR_1 & GS_1 \end{pmatrix}$$
$$= G \begin{pmatrix} P_1^{\sim} \\ Q_1^{\sim} \end{pmatrix} G \begin{pmatrix} R_1 & S_1 \end{pmatrix}$$
$$= G \begin{pmatrix} P_1 & Q_1 \end{pmatrix}^{\sim} G \begin{pmatrix} R_1 & S_1 \end{pmatrix}.$$

On the otherhand, on account of eq. (1.5), from the conditions $P_1{\sim} \le R_1$ and $Q_1{\sim} \le S_1$, we have $R(P_1) \subseteq R(R_1)$ and $R(Q_1) \subseteq R(S_1)$, which imply that $R(P_1\ Q_1) \subseteq R(R_1\ S_1)$.
According to eq. (1.5), we have $G(P_1\ Q_1){\sim} \le G(R_1\ S_1)$. □

Theorem 2.3. *Let $P_1, R_1 \in C^{m\times n}$ and $Q_1, S_1 \in C^{m\times k}$ be star-ordered as $G(P_1\ Q_1) \tilde{\le} G(R_1\ S_1)$. If $P_1 \tilde{\le} R_1$ (or $Q_1 \tilde{\le} S_1$), then $Q_1 \tilde{\le} S_1$ (or $P_1 \tilde{\le} R_1$).*
Moreover, the condition $P_1 \tilde{\le} R_1$ (or $Q_1 \tilde{\le} S_1$) can be replaced by $P_1 \le {\sim}R_1$ (or $Q_1 \le {\sim}S_1$).

Proof. Proof of Theorem 2.3 follows from Theorem 2.1. □

Corollary 2.1. *Let $P_1, R_1 \in C^{m\times n}$ and $Q_1, S_1 \in C^{k\times n}$ be star-ordered as $P_1 \tilde{\le} R_1$, $Q_1 \tilde{\le} S_1$. If $R((P_1G)^{\sim}) = R((Q_1G)^{\sim})$, then $G\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \tilde{\le} G\begin{pmatrix} R_1 \\ S_1 \end{pmatrix}$.*

Proof. It is an immediate consequence of proof of Theorem 2.1. □

Corollary 2.2. *Let $P_1, R_1 \in C^{m\times n}$ and $Q_1, S_1 \in C^{k\times n}$ be star-ordered as $P_1 \le {\sim}R_1$, $Q_1 \le {\sim}S_1$. If $R((P_1G)^{\sim}) = R((Q_1G)^{\sim})$, then $G\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \le {\sim}G\begin{pmatrix} R_1 \\ S_1 \end{pmatrix}$.*

Proof. Given,

(i) $P_1 \le {\sim}R_1 \Leftrightarrow (P_1G)(GP_1)^{\sim} = R_1G(GP_1)^{\sim}$ (or $P_1G(P_1G)^{ⓜ} = Q_1G(P_1G)^{ⓜ}$) and $R((P_1G)^{\sim}) \subseteq R((R_1G)^{\sim})$.

(ii) $Q_1 \le {\sim}S_1 \Leftrightarrow (Q_1G)(GQ_1)^{\sim} = S_1G(GQ_1)^{\sim}$ (or $Q_1G(Q_1G)^{ⓜ} = S_1G(Q_1G)^{ⓜ}$ and $R((Q_1G)^{\sim}) \subseteq R((S_1G)^{\sim})$.

Consider,

$$\begin{pmatrix} P_1G \\ Q_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G & Q_1^{\sim}G \end{pmatrix} = \begin{pmatrix} P_1GP_1^{\sim}G & P_1GQ_1^{\sim}G \\ Q_1GP_1^{\sim}G & Q_1GQ_1^{\sim}G \end{pmatrix}$$

$$\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}^{\sim} G = \begin{pmatrix} P_1G(GP_1)^{\sim} & P_1G(GQ_1)^{\sim} \\ Q_1G(GP_1)^{\sim} & Q_1G(GQ_1)^{\sim} \end{pmatrix} \qquad \text{(using (2.3) and (2.4))}$$
$$= \begin{pmatrix} R_1G(GP_1)^{\sim} & R_1G((GP_1)^{ⓜ}(GP_1))^{\sim}(GQ_1)^{\sim} \\ S_1G((GQ_1)^{ⓜ}(GQ_1))^{\sim}(GP_1)^{\sim} & S_1G(GQ_1)^{\sim} \end{pmatrix}$$
$$= \begin{pmatrix} R_1G(GP_1)^{\sim} & R_1G((GQ_1)(GP_1)^{ⓜ}(GP_1))^{\sim} \\ S_1G((GP_1)(GQ_1)^{ⓜ}(GQ_1))^{\sim} & S_1G(GQ_1)^{\sim} \end{pmatrix}$$
$$= \begin{pmatrix} R_1G(GP_1)^{\sim} & R_1G(GQ_1)^{\sim} \\ S_1G(GP_1)^{\sim} & S_1G(GQ_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} R_1G \\ S_1G \end{pmatrix} \begin{pmatrix} (GP_1)^{\sim} & (GQ_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} R_1G \\ S_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G & Q_1^{\sim}G \end{pmatrix}$$

$$= \begin{pmatrix} R_1 \\ S_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}^{\sim} G.$$

Pre and post multiplying by G, we have

$$= G \begin{pmatrix} R_1 \\ S_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}^{\sim}.$$

On the otherhand, on account of eq. (1.6), from the conditions $P_1 \leq {\sim} R_1$ and $Q_1 \leq {\sim} S_1$, we have $R((P_1G)^{\sim}) \subseteq R((R_1G)^{\sim})$ and $R((Q_1G)^{\sim}) \subseteq R((S_1G)^{\sim})$, which imply that $R \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \subseteq R \begin{pmatrix} R_1 \\ S_1 \end{pmatrix}$.
According to eq. (1.6), we have

$$G \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \leq {\sim} G \begin{pmatrix} R_1 \\ S_1 \end{pmatrix}.$$

Hence the proof. □

Corollary 2.3. *Let $P_1, R_1 \in C^{m\times n}$ and $Q_1, S_1 \in C^{k\times n}$ be star-ordered as $G \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} R_1 \\ S_1 \end{pmatrix}$. If $P_1{\sim} \leq R_1$ (or $Q_1{\sim} \leq S_1$), then $Q_1 \overset{\sim}{\leq} S_1$ (or $P_1 \overset{\sim}{\leq} R_1$).*

Proof. The proof follows from Theorem 2.3. □

Theorem 2.4. *Let $P_1, Q_1 \in C^{m\times n}$, $R_1 \in C^{m\times k}$ and $S_1 \in C^{k\times n}$. Then*

(i) *If $P_1 \overset{\sim}{\leq} Q_1$ and $R(R_1) \subseteq R(P_1)$, then $G(P_1\ R_1) \overset{\sim}{\leq} G(Q_1\ R_1)$ and $G(R_1\ P_1) \overset{\sim}{\leq} G(R_1\ Q_1)$. Moreover, both $G(P_1\ R_1) \overset{\sim}{\leq} G(Q_1\ R_1)$ and $G(R_1\ P_1) \overset{\sim}{\leq} G(R_1\ Q_1)$ imply $P_1 \overset{\sim}{\leq} Q_1$, even though $R(R_1) \not\subseteq R(P_1)$.*

(ii) *$P_1{\sim} \leq Q_1$ and $R(R_1) \subseteq R(P_1)$, then $G(P_1\ R_1){\sim} \leq G(Q_1\ R_1)$ and $G(R_1\ P_1){\sim} \leq G(R_1\ Q_1)$.*

(iii) *If $P_1 \overset{\sim}{\leq} Q_1$ and $R((S_1G)^{\sim}) \subseteq R((P_1G)^{\sim})$, then $G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix}$ and $G \begin{pmatrix} S_1 \\ P_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} S_1 \\ Q_1 \end{pmatrix}$. Moreover, both $G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix}$ and $G \begin{pmatrix} S_1 \\ P_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} S_1 \\ Q_1 \end{pmatrix}$ imply $P_1 \overset{\sim}{\leq} Q_1$, even though $R((S_1G)^{\sim}) \not\subseteq R((P_1G)^{\sim})$.*

(iv) *If $P_1 \leq {\sim} Q_1$ and $R((S_1G)^{\sim}) \subseteq R((P_1G)^{\sim})$, then $G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix} \leq {\sim} G \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix}$ and $G \begin{pmatrix} S_1 \\ P_1 \end{pmatrix} \leq {\sim} G \begin{pmatrix} S_1 \\ Q_1 \end{pmatrix}$.*

Proof. (i) Given, $P_1 \overset{\sim}{\leq} Q_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1$ and $P_1G(GP_1)^{\sim} = Q_1G(GP_1)^{\sim}$.

$$(P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1$$

$$GP_1 = ((P_1G)^{\sim})^{ⓜ}(P_1G)^{\sim}GQ_1 = ((P_1G)^{ⓜ})^{\sim}(P_1G)^{\sim}GQ_1$$

$$GP_1 = ((P_1G)(P_1G)^{ⓜ})^{\sim}GQ_1 \tag{2.7}$$

$$P_1G(GP_1)^{\sim} = Q_1G(GP_1)^{\sim}$$

$$P_1G = Q_1G(GP_1)^{\sim}((GP_1)^{\sim})^{ⓜ}$$

$$= Q_1 G(GP_1)^{\sim}((GP_1)^{ⓜ})^{\sim}$$

$$P_1 G = Q_1 G((GP_1)^{ⓜ}(GP_1))^{\sim}. \tag{2.8}$$

Consider,

$$\begin{pmatrix} GP_1^{\sim} \\ GR_1^{\sim} \end{pmatrix} \begin{pmatrix} GP_1 & GR_1 \end{pmatrix} = \begin{pmatrix} GP_1^{\sim}GP_1 & GP_1^{\sim}GR_1 \\ GR_1^{\sim}GP_1 & GR_1^{\sim}GR_1 \end{pmatrix}$$

$$G\begin{pmatrix} P_1 & R_1 \end{pmatrix}^{\sim} G\begin{pmatrix} P_1 & R_1 \end{pmatrix} = \begin{pmatrix} (P_1G)^{\sim}GP_1 & (P_1G)^{\sim}GR_1 \\ (R_1G)^{\sim}GP_1 & (R_1G)^{\sim}GR_1 \end{pmatrix} \quad \text{(using (2.7))}$$

$$= \begin{pmatrix} (P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GR_1 \\ (R_1G)^{\sim}((P_1G)(P_1G)^{ⓜ})^{\sim}GQ_1 & (R_1G)^{\sim}GR_1 \end{pmatrix}$$

$$= \begin{pmatrix} (P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GR_1 \\ ((P_1G)(P_1G)^{ⓜ}(R_1G))^{\sim}GQ_1 & (R_1G)^{\sim}GR_1 \end{pmatrix}$$

$$= \begin{pmatrix} (P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GR_1 \\ (R_1G)^{\sim}GQ_1 & (R_1G)^{\sim}GR_1 \end{pmatrix}$$

$$= \begin{pmatrix} (P_1G)^{\sim} \\ (R_1G)^{\sim} \end{pmatrix} \begin{pmatrix} GQ_1 & GR_1 \end{pmatrix}$$

$$= \begin{pmatrix} GP_1^{\sim} \\ GQ_1^{\sim} \end{pmatrix} \begin{pmatrix} GQ_1 & GR_1 \end{pmatrix}$$

$$= G\begin{pmatrix} P_1 & R_1 \end{pmatrix}^{\sim} G\begin{pmatrix} Q_1 & R_1 \end{pmatrix} \tag{2.9}$$

Consider,

$$\begin{pmatrix} P_1G & R_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G \\ R_1^{\sim}G \end{pmatrix} = \begin{pmatrix} P_1GP_1^{\sim}G & P_1GR_1^{\sim}G \\ R_1GP_1^{\sim}G & R_1GR_1^{\sim}G \end{pmatrix}$$

$$\begin{pmatrix} P_1 & R_1 \end{pmatrix} G \begin{pmatrix} P_1 & R_1 \end{pmatrix}^{\sim} G = \begin{pmatrix} P_1G(GP_1)^{\sim} & P_1G(GR_1)^{\sim} \\ R_1G(GP_1)^{\sim} & R_1G(GR_1)^{\sim} \end{pmatrix} \quad \text{(using (2.8))}$$

$$= \begin{pmatrix} Q_1G(GP_1)^{\sim} & Q_1G((GP_1)^{ⓜ}GP_1)^{\sim}(GR_1)^{\sim} \\ R_1G(GP_1)^{\sim} & R_1G(GR_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G(GP_1)^{\sim} & Q_1G((GR_1)(GP_1)^{ⓜ}(GP_1))^{\sim} \\ R_1G(GP_1)^{\sim} & R_1G(GR_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G(GP_1)^{\sim} & Q_1G(GR_1)^{\sim} \\ R_1G(GP_1)^{\sim} & R_1G(GR_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G & R_1G \end{pmatrix} \begin{pmatrix} (GP_1)^{\sim} \\ (GR_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G & R_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G \\ R_1^{\sim}G \end{pmatrix}$$

$$= \begin{pmatrix} Q_1 & R_1 \end{pmatrix} G \begin{pmatrix} P_1 & R_1 \end{pmatrix}^{\sim} G.$$

Pre and post multiplying by G, we have

$$G\begin{pmatrix} P_1 & R_1 \end{pmatrix} G \begin{pmatrix} P_1 & R_1 \end{pmatrix}^{\sim} = G\begin{pmatrix} Q_1 & R_1 \end{pmatrix} G \begin{pmatrix} P_1 & R_1 \end{pmatrix}^{\sim}. \tag{2.10}$$

From eqs. (2.9) and (2.10), we have

$$G\begin{pmatrix} P_1 & R_1 \end{pmatrix} \overset{\sim}{\leq} G\begin{pmatrix} Q_1 & R_1 \end{pmatrix}. \tag{2.11}$$

Similarly,

$$G\begin{pmatrix}R_1 & P_1\end{pmatrix} \tilde{\le} G\begin{pmatrix}R_1 & Q_1\end{pmatrix}. \tag{2.12}$$

Combining eqs. (2.11) and (2.12) implies that $P_1 \tilde{\le} Q_1$, even though $R(R_1) \not\subset R(P_1)$.

(ii) Given, $P_1 \sim\le Q_1$ and $R(R_1) \subseteq R(P_1)$.
$P_1 \sim\le Q_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1$ (or $(GP_1)^{ⓜ})GP_1 = (GP_1)^{ⓜ}GQ_1$ and $R(P_1) \subseteq R(Q_1)$.
To prove that $G\begin{pmatrix}P_1 & R_1\end{pmatrix} \sim\le G\begin{pmatrix}Q_1 & R_1\end{pmatrix}$.
Consider,

$$\begin{pmatrix}GP_1^{\sim}\\ GR_1^{\sim}\end{pmatrix}\begin{pmatrix}GP_1 & GR_1\end{pmatrix} = \begin{pmatrix}GP_1^{\sim}GP_1 & GP_1^{\sim}GR_1\\ GR_1^{\sim}GP_1 & GR_1^{\sim}GR_1\end{pmatrix}$$

$$\begin{aligned}
G\begin{pmatrix}P_1 & R_1\end{pmatrix}^{\sim} G\begin{pmatrix}P_1 & R_1\end{pmatrix} &= \begin{pmatrix}(P_1G)^{\sim}GP_1 & (P_1G)^{\sim}GR_1\\ (R_1G)^{\sim}GP_1 & (R_1G)^{\sim}GR_1\end{pmatrix} \quad \text{(using (2.7))}\\
&= \begin{pmatrix}(P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GR_1\\ (R_1G)^{\sim}((P_1G)(P_1G)^{ⓜ})^{\sim}GQ_1 & (R_1G)^{\sim}GR_1\end{pmatrix}\\
&= \begin{pmatrix}(P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GR_1\\ ((P_1G)(P_1G)^{ⓜ}(R_1G))^{\sim}GQ_1 & (R_1G)^{\sim}GR_1\end{pmatrix}\\
&= \begin{pmatrix}(P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GR_1\\ (R_1G)^{\sim}GQ_1 & (R_1G)^{\sim}GR_1\end{pmatrix}\\
&= \begin{pmatrix}(P_1G)^{\sim}\\ (R_1G)^{\sim}\end{pmatrix}\begin{pmatrix}GQ_1 & GR_1\end{pmatrix}\\
&= \begin{pmatrix}GP_1^{\sim}\\ GQ_1^{\sim}\end{pmatrix}\begin{pmatrix}GQ_1 & GR_1\end{pmatrix}\\
&= G\begin{pmatrix}P_1 & R_1\end{pmatrix}^{\sim} G\begin{pmatrix}Q_1 & R_1\end{pmatrix}.
\end{aligned}$$

On the otherhand, on account of eq. (1.5), from the conditions $P_1 \sim\le Q_1$, we have $R(P_1) \subseteq R(Q_1)$ which imply that $R\begin{pmatrix}P_1 & R_1\end{pmatrix} \subseteq R\begin{pmatrix}Q_1 & R_1\end{pmatrix}$.

According to eq. (1.5), we have $G\begin{pmatrix}P_1 & R_1\end{pmatrix} \sim\le G\begin{pmatrix}Q_1 & R_1\end{pmatrix}$.
Similarly,

$$G\begin{pmatrix}R_1 & P_1\end{pmatrix} \sim\le G\begin{pmatrix}R_1 & Q_1\end{pmatrix}.$$

(iii) Given, $P_1 \tilde{\le} Q_1$ and $R((S_1G)^{\sim}) \subseteq R((P_1G)^{\sim})$.

To prove that $G\begin{pmatrix}P_1\\ S_1\end{pmatrix} \tilde{\le} G\begin{pmatrix}Q_1\\ S_1\end{pmatrix}$.

$$P_1 \tilde{\le} Q_1 \Leftrightarrow (P_1G)^{\sim}GP_1 = (P_1G)^{\sim}GQ_1 \text{ and } P_1G(GP_1)^{\sim} = Q_1G(GP_1)^{\sim}.$$

Now consider,

$$\begin{pmatrix}GP_1^{\sim} & GS_1^{\sim}\end{pmatrix}\begin{pmatrix}GP_1\\ GS_1\end{pmatrix} = \begin{pmatrix}GP_1^{\sim}GP_1 & GP_1^{\sim}GS_1\\ GS_1^{\sim}GP_1 & GS_1^{\sim}GS_1\end{pmatrix}$$

$$\begin{aligned}
G\begin{pmatrix}P_1\\ S_1\end{pmatrix}^{\sim} G\begin{pmatrix}P_1\\ S_1\end{pmatrix} &= \begin{pmatrix}(P_1G)^{\sim}GP_1 & (P_1G)^{\sim}GS_1\\ (S_1G)^{\sim}GP_1 & (S_1G)^{\sim}GS_1\end{pmatrix} \quad \text{(using (2.7))}\\
&= \begin{pmatrix}(P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GS_1\\ (S_1G)^{\sim}((P_1G)(P_1G)^{ⓜ})^{\sim}GQ_1 & (S_1G)^{\sim}GS_1\end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} (P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GS_1 \\ ((P_1G)(P_1G)^{ⓜ}(S_1G))^{\sim}GQ_1 & (S_1G)^{\sim}GS_1 \end{pmatrix}$$

$$= \begin{pmatrix} (P_1G)^{\sim}GQ_1 & (P_1G)^{\sim}GS_1 \\ (S_1G)^{\sim}GQ_1 & (S_1G)^{\sim}GS_1 \end{pmatrix}$$

$$= \begin{pmatrix} (P_1G)^{\sim} & (S_1G)^{\sim} \end{pmatrix} \begin{pmatrix} GQ_1 \\ GS_1 \end{pmatrix}$$

$$= \begin{pmatrix} GP_1^{\sim} & GQ_1^{\sim} \end{pmatrix} \begin{pmatrix} GQ_1 \\ GS_1 \end{pmatrix}$$

$$= G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix}^{\sim} G \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix} \tag{2.13}$$

Consider,

$$\begin{pmatrix} P_1G \\ S_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G & S_1^{\sim}G \end{pmatrix} = \begin{pmatrix} P_1GP_1^{\sim}G & P_1GS_1^{\sim}G \\ S_1GP_1^{\sim}G & S_1GS_1^{\sim}G \end{pmatrix}$$

$$\begin{pmatrix} P_1 \\ S_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix}^{\sim} G = \begin{pmatrix} P_1G(GP_1)^{\sim} & P_1G(GS_1)^{\sim} \\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim} \end{pmatrix} \quad \text{(using (2.8))}$$

$$= \begin{pmatrix} Q_1G(GP_1)^{\sim} & Q_1G((GP_1)^{ⓜ}GP_1)^{\sim}(GS_1)^{\sim} \\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G(GP_1)^{\sim} & Q_1G((GS_1)(GP_1)^{ⓜ}(GP_1))^{\sim} \\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G(GP_1)^{\sim} & Q_1G(GS_1)^{\sim} \\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G \\ S_1G \end{pmatrix} \begin{pmatrix} (GP_1)^{\sim} & (GS_1)^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Q_1G \\ S_1G \end{pmatrix} \begin{pmatrix} P_1^{\sim}G & S_1^{\sim}G \end{pmatrix}$$

$$= \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix}^{\sim} G.$$

Pre and post multiplying by G, we have

$$G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix}^{\sim} = G \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix} G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix}^{\sim}. \tag{2.14}$$

From eqs. (2.13) and (2.14), we have

$$G \begin{pmatrix} P_1 \\ S_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} Q_1 \\ S_1 \end{pmatrix}. \tag{2.15}$$

Similarly,

$$G \begin{pmatrix} S_1 \\ P_1 \end{pmatrix} \overset{\sim}{\leq} G \begin{pmatrix} S_1 \\ Q_1 \end{pmatrix}.$$

(iv) Given $P_1 \leq \sim Q_1$ and $R((S_1G)^{\sim}) \subseteq R((P_1G)^{\sim})$.

$P_1 \leq \sim Q_1 \Leftrightarrow (P_1G)(GP_1)^{\sim} = Q_1G(GP_1)^{\sim}$ (or $P_1G(P_1G)^{ⓜ} = Q_1G(P_1G)^{ⓜ}$) and $R((P_1G)^{\sim}) \subseteq R((Q_1G)^{\sim})$.

To prove that $G\begin{pmatrix}P_1\\S_1\end{pmatrix}\leq\sim G\begin{pmatrix}Q_1\\S_1\end{pmatrix}$:

Consider,

$$\begin{pmatrix}P_1G\\S_1G\end{pmatrix}\begin{pmatrix}P_1^{\sim}G & S_1^{\sim}G\end{pmatrix}=\begin{pmatrix}P_1GP_1^{\sim}G & P_1GS_1^{\sim}G\\ S_1GP_1^{\sim}G & S_1GS_1^{\sim}G\end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix}P_1\\S_1\end{pmatrix}G\begin{pmatrix}P_1\\S_1\end{pmatrix}^{\sim}G &= \begin{pmatrix}P_1G(GP_1)^{\sim} & P_1G(GS_1)^{\sim}\\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim}\end{pmatrix} \qquad \text{(using (2.8))}\\
&=\begin{pmatrix}Q_1G(GP_1)^{\sim} & Q_1G((GP_1)^{ⓜ}GP_1)^{\sim}(GS_1)^{\sim}\\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim}\end{pmatrix}\\
&=\begin{pmatrix}Q_1G(GP_1)^{\sim} & Q_1G((GS_1)(GP_1)^{ⓜ}(GP_1))^{\sim}\\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim}\end{pmatrix}\\
&=\begin{pmatrix}Q_1G(GP_1)^{\sim} & Q_1G(GS_1)^{\sim}\\ S_1G(GP_1)^{\sim} & S_1G(GS_1)^{\sim}\end{pmatrix}\\
&=\begin{pmatrix}Q_1G\\S_1G\end{pmatrix}\begin{pmatrix}(GP_1)^{\sim} & (GS_1)^{\sim}\end{pmatrix}\\
&=\begin{pmatrix}Q_1G\\S_1G\end{pmatrix}\begin{pmatrix}P_1^{\sim}G & S_1^{\sim}G\end{pmatrix}\\
&=\begin{pmatrix}Q_1\\S_1\end{pmatrix}G\begin{pmatrix}P_1\\S_1\end{pmatrix}^{\sim}G\,.\end{aligned}$$

Pre and post multiplying by G, we have

$$G\begin{pmatrix}P_1\\S_1\end{pmatrix}G\begin{pmatrix}P_1\\S_1\end{pmatrix}^{\sim}=G\begin{pmatrix}Q_1\\S_1\end{pmatrix}G\begin{pmatrix}P_1\\S_1\end{pmatrix}^{\sim}.$$

On the otherhand, on account of eq. (1.6), from the condition $P_1\leq\sim Q_1$, we have $R((P_1G)^{\sim})\subseteq R((Q_1G)^{\sim})$ which imply that $R\begin{pmatrix}P_1\\S_1\end{pmatrix}\subseteq R\begin{pmatrix}Q_1\\S_1\end{pmatrix}$.

According to eq. (1.6), we have

$$G\begin{pmatrix}P_1\\S_1\end{pmatrix}\leq\sim G\begin{pmatrix}Q_1\\S_1\end{pmatrix}.$$

Similarly,

$$G\begin{pmatrix}S_1\\P_1\end{pmatrix}\leq\sim G\begin{pmatrix}S_1\\Q_1\end{pmatrix}.$$

Hence the proof. □

3. Conclusion

We have concluded the algebraic structure of the star partial ordering, left and right star partial ordering of block matrices in Minkowski space.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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