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Research Article

Pythagorean Fuzzy Strong Bi-ideal and Direct Product of Pythagorean Fuzzy Ideals in Near Ring

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Abstract. In this paper, we introduce the notions of Pythagorean fuzzy strong bi-ideals and Direct Product Pythagorean fuzzy ideals in near ring. Also, study some of their related properties in detail.

Keywords. Pythagorean fuzzy set, Strong bi-ideal, Direct product, Near-ring

Mathematics Subject Classification (2020). 03E72; 16D25; 20K25; 16Y30

1. Introduction

The concept of fuzzy set was first proposed by Zadeh [16] in 1965 and fuzzy subgroup was presented by Rosenfeld [13]. In Liu [10] introduced the notion of fuzzy ideal of a ring. The notions of fuzzy sub near ring, fuzzy ideal and fuzzy N-subgroup of a near ring was introduced by Salah Abou-Zaid [1] and it has been studied by several authors (Kim and Jun [9], [8]; Narayanan [11]; Narayanan and Manikandan [12]; Saikia and Barthakur [14]; Kim and Kim [7], respectively). The concept of intuitionistic fuzzy set was introduced by Atanassov [2] as a generalisation of fuzzy set. This concept was further discussed by Dutta and Biswas [5]. Chinnadurai and Kadalarasi [3] discussed the direct product of n $(n = 1,2,\dots,k)$ fuzzy sub near ring. Kim [6] was introduced fuzzy ideal and fuzzy R-subgroups. Devi *et al*. [4] studied the intuitionistic fuzzy strong bi-ideal of near ring. Pythagorean fuzzy set was introduced by Yager [15] in 2013.

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In this paper, we introduce the concept of a Pythagorean fuzzy strong bi-ideal of a near ring and direct product of Pythagorean fuzzy ideal in near ring. We establish that every Pythagorean fuzzy left N-subgroup or Pythagorean fuzzy left ideal of a near ring is a Pythagorean fuzzy strong bi-ideal of a near ring and direct product of Pythagorean fuzzy ideals in the near ring and also we establish that every Pythagorean left permutable fuzzy right N-subgroup or Pythagorean left permutable fuzzy right ideal of the near ring is a Pythagorean fuzzy strong fuzzy bi-ideal of the near ring.

2. Preliminaries

Definition 2.1 ([15]). Let us consider a universal set X. A Pythagorean fuzzy set P on X is denoted and defined as $P = \{x, (W_P(x), V_P(x))/x \in X\}$ where $W_P : X \to [0,1]$ represents the membership degree and $V_P : X \to [0,1]$ represents the non-membership degree of $x \in X$ to the set P satisfying that $0 \le (W_P(x))^2 \le (V_P(x))^2 \le 1$. Hence $\pi_P(x) = \sqrt{1-(W_P(x))^2 + (V_P(x))^2}$ represents the indeterminacy of an object $x \in X$.

Definition 2.2 ([4]). An intuitionistic fuzzy set $I = (W_I, V_I)$ of a group $(G,+)$ is said to be an intuitionistic fuzzy subgroup of G if for all $x, y \in N$

(i) $W_I(x+y) \ge \min\{W_I(x), W_I(y)\}$.

(ii) $W_I(-x) = W_I(x)$, or equivalently $W_I(x-y) \ge \min\{W_I(x), W_I(y)\}$.

(iii) $V_I(x+y) \le \max\{V_I(x), V_I(y)\}$.

(iv) $V_I(-x) = V_I(x)$, or equivalently $V_I(x-y) \le \max\{V_I(x), V_I(y)\}$.

Definition 2.3 ([4]). An intuitionistic fuzzy subset $I = (W_I, V_I)$ of N is called an intuitionistic fuzzy subnear-ring of N if for all $x, y \in N$

(i) $W_I(x-y) \ge \min\{W_I(x), W_I(y)\}$.

(ii) $W_I(xy) \ge \min\{W_I(x), W_I(y)\}$.

(iii) $V_I(x-y) \le \max\{V_I(x), V_I(y)\}$.

(iv) $V_I(xy) \le \max\{V_I(x), V_I(y)\}$.

Definition 2.4 ([4]). An intuitionistic fuzzy subset $I = (W_I, V_I)$ of N is said to be an intuitionistic fuzzy two-sided N-subgroup of N if

(i) I is an intuitionistic fuzzy subgroup of $(N,+)$.

(ii) $W_I(xy) \ge W_I(x)$, for all $x, y \in N$.

(iii) $W_I(xy) \ge W_I(y)$, for all $x, y \in N$.

(iv) $V_I(xy) \le V_I(x)$, for all $x, y \in N$.

(v) $V_I(xy) \le V_I(y)$, for all $x, y \in N$.

If I satisfies (i), (ii) and (iv), then I is called an intuitionistic fuzzy right N-subgroup of N. If I satisfies (i), (iii) and (v), then I is called an intuitionistic fuzzy left N-subgroup of N.

3. Pythagorean Fuzzy Ideals in Near Ring

The aim of this study is to explore the idea of a Pythagorean fuzzy near ring and Pythagorean fuzzy ideal of a near ring.

Definition 3.1. A Pythagorean fuzzy set P in a near ring N is called a Pythagorean fuzzy subset of near ring of N if

(i) $W_P(h-k) \geq \min\{W_P(h), W_P(k)\}$, $V_P(h-k) \leq \max\{V_P(h), V_P(k)\}$.

(ii) $W_P(hk) \geq \min\{W_P(h), W_P(k)\}$, $V_P(hk) \leq \max\{V_P(h), V_P(k)\}$.

Definition 3.2. Let N be a near ring. A Pythagorean fuzzy set P in near ring N is called a Pythagorean fuzzy set of N if

(i) $W_P(h-k) \geq \min\{W_P(h), W_P(k)\}$, $V_P(h-k) \leq \max\{V_P(h), V_P(h)\}$.

(ii) $W_P(k+h-k) \geq W_P(h)$, $V_P(k+h-k) \leq V_P(h)$.

(iii) $W_P(hk) \geq W_P(k)$, $V_P(hk) \leq V_P(k)$.

(iv) $W_P((h+t)k-hk) \geq W_P(t)$, $V_P((h+t)k-hk) \leq V_P(t)$.

A Pythagorean fuzzy subset with the above conditions (i)-(iii) is called a Pythagorean fuzzy left ideal of N, where as a Pythagorean fuzzy subset with (i), (ii), and (iv) is called a Pythagorean fuzzy right ideal of N.

Definition 3.3. A Pythagorean fuzzy set $P = (W_P, V_{P_i})$ of N is said to be a Pythagorean fuzzy bi-ideal of N if for all $x, y \in N$,

(i) $W_P(h-k) \geq \min\{W_P(h), W_P(k)\}$.

(ii) $(W_P \circ N \circ W_P) \cap (W_P \circ N) \star W_P \subseteq W_P$.

(iii) $V_P(h-k) \leq \max\{V_P(h), V_P(k)\}$.

(iv) $(V_P \circ N \circ V_P) \cup (V_P \circ N) \star V_P \supseteq V_P$.

Theorem 3.4. *Let C and D be Pythagorean fuzzy ideals of N. If $C \subset D$, then $C \cup D$ is a Pythagorean fuzzy ideal of N.*

Proof. Let C and D be Pythagorean fuzzy ideals of N. Let $h, k, t \in N$; then,

$$\begin{aligned} W_{C\cup D}(h-k) &= \max(W_C(h-k), W_D(h-k)) \\ &\geq \max\{\min\{W_C(h), W_C(k)\}, \min\{W_D(h), W_D(k)\}\} \\ &= \min\{\max\{W_C(h), W_D(h)\}, \max\{W_C(k), W_D(k)\}\} \\ &= \min\{W_{C\cup D}(h), W_{C\cup D}(k)\} \end{aligned}$$

and for non-membership grade, we have

$$\begin{aligned} V_{C\cup D}(h-k) &= \min(V_C(h-k), V_D(h-k)) \\ &\leq \min\{\max\{V_C(h), V_C(k)\}, \max(V_D(h), V_D(k))\} \end{aligned}$$

$$= \max\{\min\{V_C(h), V_D(h)\} \max\{V_C(k), V_D(k)\}\}$$
$$= \max\{V_{C\cup D}(h), V_{C\cup D}(k)\}.$$

Next, we write

$$\begin{aligned} W_{C\cup D}(k+h-k) &= \max\{W_C(k+h-k), W_D(k+h-k)\} \\ &\geq \max\{W_C(h), W_D(h)\} \\ &= W_{C\cup D}(h) \end{aligned}$$

and for non-membership grade, we get

$$\begin{aligned} V_{C\cup D}(k+h-k) &= \min\{V_C(k+h-k), V_D(k+h-k)\} \\ &\leq \min\{V_C(h), V_D(h)\} \\ &= V_{C\cup D}(h). \end{aligned}$$

Furthermore, we deduce that

$$\begin{aligned} W_{C\cup D}(hk) &= \max\{W_C(hk), W_D(hk)\} \\ &\geq \max\{W_C(k), W_D(k)\} \\ &= W_{C\cup D}(k), \\ V_{C\cup D}(hk) &= \min\{V_C(hk), V_D(hk)\} \\ &\leq \min\{V_C(k), V_D(k)\} \\ &= V_{C\cup D}(k). \end{aligned}$$

At last, we obtain

$$\begin{aligned} V_{C\cup D}((h+t)k-hk) &= \max\{W_C((h+t)k-hk), W_D((h+t)k-hk)\} \\ &\geq \max\{W_C(t), W_D(t)\} \\ &= W_{C\cup D}(t), \\ V_{C\cup D}((h+t)k-hk) &= \min\{V_C((h+t)k-hk), V_D((h+t)k-hk)\} \\ &\leq \min\{V_C(t), V_D(t)\} \\ &= V_{C\cup D}(t). \end{aligned}$$

Therefore, $C \cup D$ is a Pythagorean fuzzy ideal of N. □

Theorem 3.5. *Let C and D be Pythagorean fuzzy ideals of X. If $C \subset D$, then $C \cap D$ is a Pythagorean fuzzy ideal of N.*

Proof. Let C and D be Pythagorean fuzzy ideals of N. Let $h, k, t \in N$. Then, the following are obtained.

For truth grade, we get

$$\begin{aligned} W_{C\cap D}(h-k) &= \min\{W_C(h-k), W_D(h-k)\} \\ &\geq \min\{\min\{W_C(h), W_C(k)\}, \min\{W_D(h), W_D(k)\}\} \end{aligned}$$

$$= \min\{\min\{W_C(h), W_D(h)\}, \min\{W_C(k), W_D(k)\}\}$$
$$= \min\{W_{C\cap D}(h), W_{C\cap D}(k)\}.$$

For non-membership grade, we obtain

$$\begin{aligned} V_{C\cap D}(h-k) &= \max\{V_C(h-k), V_D(h-k)\} \\ &\le \max\{\max\{V_C(h), V_C(k)\}, \max\{V_D(h), V_D(k)\}\} \\ &= \max\{\max\{V_C(h), V_D(h)\}, \max\{V_C(k), V_D(k)\}\} \\ &= \max\{V_{C\cap D}(h), V_{C\cap D}(k)\}. \end{aligned}$$

Next, we obtain

$$\begin{aligned} W_{C\cap D}(k+h-k) &= \min\{W_C(k+h-k), W_D(k+h-k)\} \\ &\ge \min\{W_C(h), W_D(h)\} \\ &= W_{C\cap D}(h). \end{aligned}$$

Similarly,

$$\begin{aligned} V_{C\cap D}(k+h-k) &= \max\{V_C(k+h-k), V_D(k+h-k)\} \\ &\le \max\{V_C(h), V_D(h)\} \\ &= V_{C\cap D}(h). \end{aligned}$$

Furthermore, we deduce that

$$\begin{aligned} W_{C\cap D}(hk) &= \min\{W_C(hk), W_D(hk)\} \\ &\ge \min\{W_C(k), W_D(k)\} \\ &= W_{C\cap D}(k), \\ V_{C\cap D}(hk) &= \max\{V_C(hk), V_D(hk)\} \\ &\le \max\{V_C(k), V_D(k)\} \\ &= V_{C\cap D}(k). \end{aligned}$$

Finally, we conclude that

$$\begin{aligned} W_{C\cap D}((h+t)k-hk) &= \min\{W_C((h+t)k-hk), W_D((h+t)k-hk)\} \\ &\ge \min\{W_C(t), W_D(t)\} \\ &= W_{C\cap D}(t), \\ V_{C\cap D}((h+t)k-hk) &= \max\{V_C((h+t)k-hk), V_D((h+t)k-hk)\} \\ &\le \max\{V_C(t), V_D(t)\} \\ &= V_{C\cap D}(t). \end{aligned}$$

Therefore, $C\cap D$ is a Pythagorean fuzzy ideal of N. □

Theorem 3.6. *Let P be a Pythagorean fuzzy ideal of N. Then,*

$$P^m = \{\langle h, W_{P^m}(h), V_{P^m}(h)\rangle : h \in N\}$$

is a Pythagorean fuzzy ideal of N, where m is a positive integer and $W_{P^m}(h) = (W_P(h))^m$ and $V_{P^m}(h) = (V_P(h))^m$.

Proof. Let P be a Pythagorean fuzzy ideal of N. Let $h, k, t \in N$. Then, the following are observed. For truth grade, we can write

$$\begin{aligned} W_{P^m}(h-k) &= (W_P(h-k))^m \\ &\geq (\min\{W_P(h), W_P(k)\})^m \\ &= \min\{(W_P(h))^m, (W_P(k))^m\} \\ &= \min\{W_{P^m}(h), W_{P^m}(k)\}. \end{aligned}$$

For non-membership grade, we obtain the following:

$$\begin{aligned} V_{P^m}(h-k) &= (V_P(h-k))^m \\ &\leq (\max\{V_P(h), V_P(k)\})^m \\ &= \max\{(V_P(h))^m, (V_P(k))^m\} \\ &= \max(V_{P^m}(h), V_{P^m}(k)). \end{aligned}$$

Next, it is obtained that

$$\begin{aligned} W_{P^m}(k+h-k) &= (W_P(k+h-k))^m \\ &\geq (W_P(h))^m \\ &= W_{P^m}(h), \\ [-2pt]V_{P^m}(k+h-k) &= (V_P(k+h-k))^m \\ &\leq (V_P(h))^m \\ &= V_{P^m}(h). \end{aligned}$$

Also, we examine that

$$\begin{aligned} W_{P^m}(hk) &= (W_P(hk))^m \\ &\geq (W_P(k))^m \\ &= W_{P^m}(k), \\ V_{P^m}(hk) &= (V_P(hk))^m \\ &\leq (V_P(k))^m \\ &= V_{P^m}(k). \end{aligned}$$

At last, we write that

$$\begin{aligned} W_{P^m}((h+t)k-hk) &= (W_P((h+t)k-hk))^m \\ &\geq (W_P(t))^m \\ &= W_{P^m}(t), \end{aligned}$$

$$\begin{aligned} V_{P^m}((h+t)k-hk) &= (V_P((h+t)k-hk))^m \\ &\le (V_P(t))^m \\ &= V_{P^m}(t). \end{aligned}$$

Therefore, P^m is a Pythagorean fuzzy ideal of N^m. □

4. Pythagorean Fuzzy Strong Bi-ideals of Near-Rings

Definition 4.1. A Pythagorean fuzzy bi-ideal $P = (W,V)$ of N is called a Pythagorean fuzzy strong bi-ideal of N, if

(i) $(N \circ W \circ W) \subseteq W$.

(ii) $(N \circ V \circ V) \supseteq V$.

Example 4.2. Let $N = \{0,a,b,c\}$ be a near-ring with two binary operations '+' and is defined as follows.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and

·	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	0	b	0
c	0	0	c	0

Define a Pythagorean fuzzy set $P = (A,B)$, where $A : N \to [0,1]$ by $A(0) = 0.8$, $A(a) = 0.6$, $A(b) = 0.3 = A(c)$. Then $(A\circ N\circ A)(0) = 0.3$, $(A\circ N\circ A)(a) = 0.3$, $(A\circ N\circ A)(b) = 0.3$, $(A\circ N\circ A)(c) = 0.3$, $(N\circ A\circ A)(0) = 0.3$, $(N\circ A\circ A)(a) = 0.3$, $(N\circ A\circ A)(b) = 0.3$, $(N\circ A\circ A)(c) = 0.3$ and so A is a Pythagorean fuzzy strong bi-ideal of N and $B : N \to [0,1]$ by $B(0) = 0.2$, $B(a) = 0.7$, $B(b) = 0.9 = B(c)$. Then $(B\circ N\circ B)(0) = 0.9$, $(B\circ N\circ B)(a) = 0.9$, $(B\circ N\circ B)(b) = 0.9$, $(B\circ N\circ B)(c) = 0.9$, $(N\circ B\circ B)(0) = 0.9$, $(N\circ B\circ B)(a) = 0.9$, $(N\circ B\circ B)(b) = 0.9$, $(N\circ B\circ B)(c) = 0.9$ and so B is a Pythagorean fuzzy strong bi-ideal of N. Thus $P = (A,B)$ is a Pythagorean fuzzy strong bi-ideal of N.

Theorem 4.3. *Let $\{P_i\} = \{(W_{P_i}, V_{P_i}) : i \in I\}$ be any family of Pythagorean fuzzy strong bi-ideals in a near-ring N. Then $\bigcap_{i\in I} P_i$ is a Pythagorean fuzzy strong bi-ideal of N, where $\bigcap_{i\in I} P_i = \left\{\left(\bigcap_{i\in I} W_{P_i}, \bigcup_{i\in I} V_{P_i}\right)\right\}$, for all $i \in I$.*

Proof. Let $\{P_i : i \in I\}$ be any family of Pythagorean fuzzy strong bi-ideals of N.
Now for all $x, y \in N$,

$$\begin{aligned} \bigcap_{i\in I} W_{P_i}(x-y) &= \min\{W_{P_i}(x-y)/i \in I\} \\ &\ge \min\{\min\{W_{P_i}(x), W_{P_i}(y)\} : i \in I\} \\ &\qquad (\text{since } W_{P_i} \text{ is a Pythagorean fuzzy subgroup of } N) \end{aligned}$$

$$= \min\left\{\bigcap_{i\in I} W(x), \bigcap_{i\in I} W(y) : i \in I\right\},$$

$$\bigcup_{i\in I} V_{P_i}(x-y) = \max\{V_{P_i}(x-y) : i \in I\}$$

$$\leq \max\{\max\{V_{P_i}(x), V_{P_i}(y)\} : i \in I\}$$

(since V_{P_i} is a Pythagorean fuzzy subgroup of N)

$$= \max\left\{\bigcup_{i\in I} V(x), \bigcup_{i\in I} V(y) : i \in I\right\}.$$

Therefore $\bigcap_{i\in I} P_i$ is a Pythagorean fuzzy subgroup of N.

To prove: $\bigcap_{i\in I} P_i$ is a Pythagorean fuzzy bi-ideal of N.

Now for all $x \in N$, since $W_{P_i} = \bigcap_{i\in I} W_{P_i} \subseteq W_{P_i}$, for every $i \in I$, we have

$$((W_{P_i} \circ N \circ W_{P_i}) \cap (W_{P_i} \circ N) \star W_{P_i})(x) \leq ((W_{P_i} \circ N \circ W_{P_i}) \cap (W_{P_i} \circ N) \star W_{P_i})(x)$$

(since W_{P_i} is a Pythagorean fuzzy bi-ideal of N)

$$\leq W_{P_i}(x), \quad \text{for every } i \in I.$$

It follows that

$$((W_{P_i} \circ N \circ W_{P_i}) \cap (W_{P_i} \circ N) \star W_{P_i}))(x) \leq \inf\{W_{P_i}(x) : i \in I\}$$

$$= \left(\bigcap_{i\in I} W_{P_i}(x)\right)$$

$$= W_{P_i}(x).$$

Thus $(W_{P_i} \circ N \circ W_{P_i}) \cap (W_{P_i} \circ N) \star W_{P_i}) \subseteq W_{P_i}$.

So W_{P_i} is a Pythagorean fuzzy bi-ideal of N.

Now for all $x \in N$, since $V_{P_i} = \bigcup_{i\in I} V_{P_i} \supseteq V_{P_i}$ for some $i \in I$, we have

$$((V_{P_i} \circ N \circ V_{P_i}) \cup (V_{P_i} \circ N) \star V_{P_i}))(x) \geq ((V_{P_i} \circ N \circ V_{P_i}) \cup (V_{P_i} \circ N) \star V_{P_i}))(x)$$

(since V_{P_i} is a Pythagorean fuzzy bi-ideal of N)

$$\geq V_{P_i}(x), \quad \text{for some } i \in I.$$

It follows that

$$((V_{P_i} \circ N \circ V_{P_i}) \cup (V_{P_i} \circ N) \star V_{P_i}))(x) \geq \sup\{V_{P_i}(x) : i \in I\}$$

$$= \left(\bigcup_{i\in I} V_{P_i}(x)\right)$$

$$= V_{P_i}(x).$$

Thus $(V_{P_i} \circ N \circ V_{P_i}) \cup (V_{P_i} \circ N) \star V_{P_i}) \supseteq V_{P_i}$.

So V_{P_i} is a Pythagorean fuzzy bi-ideal of N.

Thus $\bigcap_{i\in I} P_i$ is a Pythagorean fuzzy bi-ideal of N.

Next, we prove $\bigcap_{i\in I} P_i$ is a Pythagorean fuzzy strong bi-ideal of N.

Now for all $x \in N$, since $W_{P_i} = \bigcap_{i \in I} W_{P_i} \subseteq W_{P_i}$, for every $i \in I$, we have

$$\begin{aligned}(N \circ W_{P_i} \circ W_{P_i})(x) &\le (N \circ W_{P_i} \circ W_{P_i})(x)\\ &\le W_{P_i}(x) \text{ for every } i \in I\\ &\quad \text{(since } W_{P_i} \text{ is a Pythagorean fuzzy strong bi-ideal of } N).\end{aligned}$$

It follows that,

$$\begin{aligned}(N \circ W_{P_i} \circ W_{P_i})(x) &\le \inf\{W_{P_i}(x) \ : \ i \in I\}\\ &= \Big(\bigcap_{i \in I} W_{P_i}(x)\Big)\\ &= W_{P_i}(x).\end{aligned}$$

Thus $N \circ W_{P_i} \circ W_{P_i} \subseteq W_{P_i}$. So W_{P_i} is a Pythagorean fuzzy strong bi-ideal of N.
Now for all $x \in N$, since $V_{P_i} = \bigcup_{i \in I} V_{P_i} \supseteq V_{P_i}$, for some $i \in I$, we have

$$\begin{aligned}(N \circ V_{P_i} \circ V_{P_i})(x) &\ge (N \circ V_{P_i} \circ V_{P_i})(x)\\ &\ge V_{P_i}(x) \ \text{ for every } i \in I\\ &\quad \text{(since } V_{P_i} \text{ is a Pythagorean fuzzy strong bi-ideal of } N).\end{aligned}$$

It follows that,

$$\begin{aligned}(N \circ V_{P_i} \circ V_{P_i})(x) &\ge \sup\{V_{P_i}(x) : i \in I\}\\ &= \Big(\bigcup_{i \in I} V_{P_i}(x)\Big)\\ &= V_{P_i}(x).\end{aligned}$$

Thus $N \circ V_{P_i} \circ V_{P_i} \supseteq V_{P_i}$. So V_{P_i} is a Pythagorean fuzzy strong bi-ideal of N.
Thus $\bigcap_{i \in I} V_{P_i}$ is a Pythagorean fuzzy strong bi-ideal of N. □

Theorem 4.4. *Every left permutable Pythagorean fuzzy right N-subgroup of N is a Pythagorean fuzzy strong bi-ideal of N.*

Proof. Let $P = (W_P, V_P)$ be a left permutable Pythagorean fuzzy right N-subgroup of N.
To prove: P is a Pythagorean fuzzy strong bi-ideal of N.
First, we prove P is a Pythagorean fuzzy bi-ideal of N.
Choose $a, b, c, x, y, i, b_1, b_2, x_1, x_2, y_1, y_2 \in N$ such that $a = bc = x(y+i) - xy$, $b = b_1 b_2$, $x = x_1 x_2$ and $y = y_1 y_2$. Then

$$\begin{aligned}&(W_P \circ N \circ W_P) \cap ((W_P \circ N) \star W_P)(a)\\ &\quad = \min\{(W_P \circ N \circ W_P)(a), ((W_P \circ N) \star W_P)(a)\}\\ &\quad = \min\{\sup_{a=bc} \min\{(W_P \circ N)(b), W_P(c)\}, ((W_P \circ N) \star W_P)(x(y+i) - xy)\}\\ &\quad = \min\{\sup_{a=bc} \min\{\sup_{b=b_1 b_2} \min\{W_P(b_1), N(b_2)\}, W_P(c)\}, ((W_P \circ N) \star W_P)(x(y+i) - xy)\}\end{aligned}$$

(since $N(z)=1$, for all $z\in N$)

$$=\min\{\sup_{a=bc}\min\{\{\sup_{b=b_1b_2}\{W_P(b_1),W_P(c)\}\},((W_P\circ N)\star W_P)(x(y+i)-xy)\}\}$$

(since W_P is a Pythagorean fuzzy right N-subgroup of N,

$W_P(bc)=W_P(b_1b_2c)=W_P(b_1(b_2c))\geq W_P(b_1)$)

$$\leq\min\{\sup_{a=bc}\min\{W_P(bc),N(c)\},N(x(y+i)-xy)\}$$

$$=\min\{\sup_{a=bc}\min\{W_P(bc),N(x(y+i)-xy)\}\}=W_P(bc)=W_P(a).$$

Thus $(W_P\circ N\circ W_P)\cap((W_P\circ N)\star W_P)\subseteq W_P$.

Hence W_P is a Pythagorean fuzzy bi-ideal of N.

Choose $a,b,c,x,y,i,b_1,b_2,x_1,x_2,y_1,y_2\in N$ such that $a=bc=x(y+i)-xy$, $b=b_1,b_2$, $x=x_1x_2$ and $y=y_1y_2$. Then

$$(V_P\circ N\circ V_P)\cup((V_P\circ N)\star V_P))(a)$$

$$=\max\{(V_P\circ N\circ V_P)(a),((V_P\circ N)\star V_P)(a)\}$$

$$=\max\{\inf_{a=bc}\max\{(V_P\circ N)(b),V_P(c)\},((V_P\circ N)\star V_P)(x(y+i)-xy)\}$$

$$=\max\{\inf_{a=bc}\max\{\inf_{b=b_1b_2}\max\{V_P(b_1),N(b_2)\},V_P(c)\},((V_P\circ N)\star V_P)(x(y+i)-xy)\}$$

(since $N(z)=0$, for all $z\in N$)

$$=\max\{\inf_{a=bc}max\{\inf_{b=b_1b_2}\{V_P(b_1),V_P(c)\},((V_P\circ N)\star V_P)(x(y+i)-xy)\}$$

(since V_P is a Pythagorean fuzzy right N-subgroup of N)

$V_P(bc)=V_P(b_1b_2c)=V_P(b_1(b_2c))\leq V_P(b_1)$

$$\geq\max\{\inf_{a=bc}\max\{V_P(bc),N(c)\},N(x(y+i)-xy)\}$$

$$=\max\{\inf_{a=bc}\max\{V_P(bc),N(x(y+i)-xy)\}\}$$

$$=V_P(bc)$$

$$=V_P(a).$$

Thus $(V_P\circ N\circ V_P)\cup((V_P\circ N)\star V_P))\ (a)\supseteq V_P$.

Hence V_P is a Pythagorean fuzzy bi-ideal of N.

Thus $P=(W_P,V_P)$ is a Pythagorean fuzzy bi-ideal of N.

Next we prove: P is a Pythagorean fuzzy strong bi-ideal of N.

Choose $a,b,c,b_{12}b_2\in N$ such that $a=bc$ and $b=b_1,b_2$. Then

$$N\circ W_P\circ W_P(a)=\sup_{a=bc}\min\{(N\circ W_P)(b),W_P(c)\}$$

$$=\sup_{a=bc}\min\{\sup_{b=b_1b_2}\min\{N(b_1),W_P(b_2)\},W_P(c)\}$$

$$=\sup_{a=bc}\min\{\sup_{b=b_1b_2}\{W_P(b_2),W_P(c)\}\}$$

(since W_P is a left permutable Pythagorean fuzzy right N-subgroup of N)

$W_P(bc) = W_P((b_1b_2)c) = W_P((b_2b_1)c) \geq W_P(b_2)$ and $N(c) \geq W_P(c)$

$$\leq \sup_{a=bc} \min\{W_P(bc), N(c)\}$$
$$= \sup_{a=bc} \min\{W_P(bc), 1\}$$
$$= \sup_{a=bc} W_P(bc)$$
$$= W_P(a).$$

Therefore $N \circ W_P \circ W_P \subseteq W_P$.
Hence W_P is a Pythagorean fuzzy strong bi-ideal of N.
Choose $a, b, c, b_{12}b_2 \in N$ such that $a = bc$ and $b = b_1b_2$. Then

$$N \circ V_P \circ V_P(a) = \inf_{a=bc} \max\{(N \circ V_P)(b), V_P(c)\}$$
$$= \inf_{a=bc} \max\{\inf_{b=b_1b_2} \max\{N(b_1), V_P(b_2)\}, V_P(c)\}$$
$$= \inf_{a=bc} \max\{\inf_{b=b_1b_2} \{V_P(b_2), V_P(c)\}\}$$

(since V_P is a left permutable Pythagorean fuzzy right N-subgroup of N)

$V_P(bc) = V_P((b_1b_2)c) = V_P((b_2b_1)c) \leq V_P(b_2)$ and $N(c) \leq V_P(c)$

$$\geq \inf_{a=bc} \max\{V_P(bc), N(c)\}$$
$$= \inf_{a=bc} \max\{V_P(bc), 0\}$$
$$= \inf_{a=bc} V_P(bc)$$
$$= V_P(a).$$

Therefore $(N \circ V_P \circ V_P) \supseteq V_P$.
Hence V_P is a Pythagorean fuzzy strong bi-ideal of N. □

Theorem 4.5. *Every Pythagorean fuzzy left N-subgroup of N is a Pythagorean fuzzy strong bi-ideal of N.*

Proof. Let $P = (W_P, V_P)$ be a Pythagorean fuzzy left N-subgroup of N.
To prove: P is a Pythagorean fuzzy strong bi-ideal of N.
First, we prove: P is a Pythagorean fuzzy bi-ideal of N.
Choose $a, b, c, x, y, i, c_1, c_2, x_1, x_2, y_1, y_2 \in N$ such that $a = bc = x(y+i) - xy$, $c = c_1c_2, x = x_1x_2$ and $y = y_1y_2$. Then

$$(W_P \circ N \circ W_P) \cap ((W_P \circ N)_\star W_P))(a)$$
$$= \min\{(W_P \circ (N \circ W_P))(a), ((W_P \circ N)_\star W_P)(a)\}$$
$$= \min\{\sup_{a=bc} \min\{W_P(b), (N \circ W_P)(c)\}, ((W_P \circ N) \star W_P)(x(y+i) - xy)\}$$

$$= \min\{\sup_{a=bc} \min\{W_P(b), \sup_{c=c_1c_2} \min\{N(c_1), W_P(c_2)\}, ((W_P \circ N) \star W_P)(x(y+i)-xy)\}$$

$$= \min\{\sup_{a=bc} \min\{W_P(b), \sup_{c=c_1c_2} W_P(c_2)\}, ((W_P \circ N)_\star W_P)(x(y+i)-xy)\}$$

(since W_P is a Pythagorean fuzzy left N-subgroup of N)

$$W_P(bc) = W_P(bc_1c_2) = W_P((bc_1)c_2) \geq W_P(c_2)$$

$$\leq \min\{\sup_{a=bc} \min\{N(b), W_P(bc)\}, N(x(y+i)-xy)\}$$

$$= W_P(bc) = W_P(a).$$

Thus $(W_P \circ N \circ W_P) \cap ((W_P \circ N) \star W_P)) \subseteq W_P$.

Hence W_P is a Pythagorean fuzzy bi-ideal of N.

Choose $a, b, c, x, y, i, c_1, c_2, x_1, x_2, y_1, y_2 \in N$ such that $a = bc = x(y+i)-xy$, $c = c_1, c_2$, $x = x_1x_2$ and $y = y_1y_2$. Then

$$(V_P \circ N \circ V_P) \cup ((V_P \circ N)_\star V_P))(a)$$

$$= \max\{(V_P \circ N \circ V_P)(a), ((V_P \circ N)_\star V_P)(a)\}$$

$$= \max\{\inf_{a=bc} \max\{V_P(b), (N \circ V_P)(c)\}, ((V_P \circ N) \star V_P)(x(y+i)-xy)\}$$

$$= \max\{\inf_{a=bc} \max\{V_P(b), \inf_{c=c_1c_2} \max\{N(c_1), V_P(c_2)\}\}, ((V_P \circ N) \star V_P)(x(y+i)-xy)\}$$

$$= \max\{\inf_{a=bc} \max\{V_P(b), \inf_{c=c_1c_2} V_P(c_2)\}((V_P \circ N) \star V_P)(x(y+i)-xy)\}$$

(since V_P is a Pythagorean fuzzy left N-subgroup of N)

$$V_P(bc) = V_P(b(c_1c_2)) = V_P((bc_1)c_2) \leq V_P(c_2)$$

$$\geq \max\{\inf_{a=bc} \max\{N(b), V_P(bc)\}, N(x(y+i)-xy)\}$$

$$= V_P(bc)$$

$$= V_P(a).$$

Thus $(V_P \circ N \circ V_P) \cup ((V_P \circ N) \star V_P) \supseteq V_P$.

Hence V_P is a Pythagorean fuzzy bi-ideal of N.

Thus $P = (W_P, V_P)$ is a Pythagorean fuzzy bi-ideal of N.

Next, we prove: P is a Pythagorean fuzzy strong bi-ideal of N.

Choose $a, b, c, c_1, c_2 \in N$ such that $a = bc$ and $c = c_1, c_2$. Then

$$N \circ W_P \circ W_P(a) = \sup_{a=bc} \min\{N(b), (W_P \circ W_P)(c)\}$$

$$= \sup_{a=bc} \min\{N(b), \sup_{c=c_1c_2} \min\{W_P(c_1), W_P(c_2)\}\}$$

$$= \sup_{a=bc} \min\{1, \sup_{c=c_1c_2} \min\{W_P(c_1), W_P(c_2)\}\}$$

(since W_P is a Pythagorean fuzzy left N-subgroup of N)

$$W_P(bc) = W_P(bc_1c_2) = W_P((bc_1)c_2) \geq W_P(c_2)$$

$$\leq \sup_{a=bc} \min\{N(c_1), W_P(bc)\}$$

$$= \sup_{a=bc} \min\{1, W_P(bc)\}$$
$$= W_P(bc)$$
$$= W_P(a).$$

Therefore $N \circ W_P \circ W_P \subseteq W_P$.
Hence W_P is a Pythagorean fuzzy strong bi-ideal of N.
Choose $a, b, c, c_{12}c_2 \in N$ such that $a = bc$ and $c = c_1, c_2$. Then

$$(N \circ V_P \circ V_P)(a) = \inf_{a=bc} \max\{N(b), (V_P \circ V_P)(c)\}$$
$$= \inf_{a=bc} \max\{N(b), \inf_{c=c_1c_2} \max\{V_P(c_1), V_P(c_2)\}\}$$
$$= \inf_{a=bc} \max\{0, \inf_{c=c_1c_2} \max\{V_P(c_1), V_P(c_2)\}\}$$
$$= \inf_{a=bc} \max\{V_P(c_1), V_P(c_2)\}$$
(since V_P is a Pythagorean fuzzy left N-subgroup of N)
$$V_P(bc) = V_P(bc_1c_2) = V_P((bc_1)c_2) \leq V_P(c_2)$$
$$\geq \inf_{a=bc} \max\{N(c_1), V_P(bc)\}$$
$$= \inf_{a=bc} \max\{0, V_P(bc)\}$$
$$= V_P(bc) = V_P(a).$$

Therefore $N \circ V_P \circ V_P \supseteq V_P$.
Hence V_P is a Pythagorean fuzzy strong bi-ideal of N.
Thus $P = (W_P, V_P)$ is a Pythagorean fuzzy strong bi-ideal of N. □

Theorem 4.6. *Every Pythagorean fuzzy left ideal of N is a Pythagorean fuzzy strong bi-ideal of N.*

Proof. Let $P = (W_P, V_P)$ be a Pythagorean fuzzy left ideal of N.
To prove: P is a Pythagorean fuzzy strong bi-ideal of N.
First we prove: P is a Pythagorean fuzzy bi-ideal of N.
Choose $a, b, c, x, y, i, b_1, b_2, x_1, x_2, y_1, y_2 \in N$ such that $a = bc = x(y+i) - xy$, $b = b_1b_2$,

$x = x_1x_2$ and $y = y_1y_2$.

Then

$$(W_P \circ N \circ W_P) \cap ((W_P \circ N)_\star W_P)(a)$$
$$= \min\{(W_P \circ N \circ W_P)(a), ((W_P \circ N)_\star W_P)(a)\}$$
$$= \min\{\sup_{a=bc} \min(W_P \circ N)(b), W_P(c), ((W_P \circ N)_\star W_P)(x(y+i) - xy)\}$$
$$= \min\{\sup_{a=bc} \min\{(W_P \circ N)(b_1b_2), W_P(c)\}, \sup_{a=x(y+i)-xy} \min(W_P \circ N)(x), (W_P \circ N)(y), W_P(i)\}$$
(since $W_P \circ N \subseteq N$ and since W_P is a Pythagorean fuzzy left ideal of N

$$W_P(x(y+i)-xy) \geq W_P(i))$$

$$\leq \min\{\sup_{a=bc} \min\{N(b_1b_2), N(c)\}, \sup_{a=x(y+i)-xy} \min\{N(x), N(y),\} W_P(x(y+i)-xy)\}$$

$$= W_P(x(y+i)-xy)$$

$$= W_P(a).$$

Thus $(W_P \circ N \circ W_P) \cap ((W_P \circ N)_\star W_P) \subseteq W_P$.
Hence W_P is a Pythagorean fuzzy bi-ideal of N.
Choose $a, b, c, x, y, i, b_1, b_2, x_1, x_2, y_1, y_2 \in N$ such that $a = bc = x(y+i)-xy, b = b_1b_2, x = x_1x_2$ and $y = y_1y_2$. Then

$$(V_P \circ N \circ V_P) \cup ((V_P \circ N) \star V_P)(a)$$

$$= \max\{(V_P \circ N \circ V_P)(a), ((V_P \circ N) \star V_P)(a)\}$$

$$= \max\{\inf_{a=bc} \max\{(V_P \circ N)(b), V_P(c)\}((V_P \circ N) \star V_P)(x(y+i)-xy)\}$$

$$= \max\{\{\inf_{a=bc} \max(V_P \circ N)(b_1b_2), V_P(c)\}, \inf_{a=x(y+i)-xy} \max\{(V_P \circ N)(x), (V_P \circ N)(y), V_P(i)\}\}$$

(since $V_P \circ N \supseteq N$ and since V_P is a Pythagorean fuzzy left ideal of N)

$$V_P(x(y+i)-xy) \leq V_P(i)$$

$$\geq \max\{\inf_{a=bc} \max\{N(b_1b_2), N(c)\}, \inf_{a=x(y+i)-xy} \max\{N(x), N(y)\}, V_P(x(y+i)-xy)\}$$

$$= V_P(x(y+i)-xy)$$

$$= V_P(a).$$

Therefore $(V_P \circ N \circ V_P) \cup ((V_P \circ N) \star V_P) \supseteq V_P$.
Hence V_P is a Pythagorean fuzzy bi-ideal of N.
Thus $P = (W_P, V_P)$ is a Pythagorean fuzzy bi-ideal of N.
Next we prove: P is a Pythagorean fuzzy strong bi-ideal of N.
Choose $a, b, c, b_1b_2 \in N$ such that $a = bc = b(n+c)-bn$. Then

$$N \circ W_P \circ W_P(a) = \sup_{a=bc} \min\{(N \circ W_P)(b), W_P(c)\}$$

$$= \sup_{a=bc} \min\{\sup_{b=b_1b_2} \min\{N(b_1), W_P(b_2), W_P(c)\}\}$$

$$= \sup_{a=bc} \min\{\sup_{b=b_1b_2} \{W_P(b_2), W_P(c)\}\}$$

(since A is a Pythagorean fuzzy left ideal of N)

$W_P(a) = W_P(bc) = W_P(b(n+c)-bn) \geq W_P(c)$ and

$$= \sup_{a=bc} \min\{N(b_2), W_P(b(n+c)-bn)\}$$

$$= \min\{1, W_P(bc)\}$$

$$= W_P(bc)$$

$$= W_P(a).$$

Therefore $N \circ W_P \circ W_P \subseteq W_P$. Hence W_P is a Pythagorean fuzzy strong bi-ideal of N.

Choose $a,b,c,b_1,b_2 \in N$ such that $a = bc$ and $b = b_1b_2$. Then

$$\begin{aligned}N \circ V_P \circ V_P(a) &= \inf_{a=bc} \max\{(N \circ V_P)(b), V_P(c)\} \\ &= \inf_{a=bc} \max\{\inf_{b=b_1b_2} \max\{N(b_1), V_P(b_2)\}, V_P(c)\} \\ &= \inf_{a=bc} \max\{\inf_{b=b_1b_2} \{V_P(b_2), V_P(c)\}\} \quad \text{(since } P \text{ is an anti fuzzy left ideal of } N\text{),}\end{aligned}$$

$V_P(a) = V_P(bc) = V_P(b(n+c) - bn) \le V_P(c))$ and

$$\begin{aligned}&\ge \inf_{a=bc} \max\{N(b_2), V_P(b(n+c) - bn)\} \\ &= \inf_{a=bc} \max\{0, V_P(bc)\} \\ &= V_P(bc) \\ &= V_P(a).\end{aligned}$$

Therefore $N \circ V_P \circ V_P \supseteq V_P$. Hence V_P is a Pythagorean fuzzy strong bi-ideal of N.
Thus $P = (W_P, V_P)$ is a Pythagorean fuzzy strong bi-ideal of N. □

Theorem 4.7. *Let $P = (W_P, V_P)$ be any Pythagorean fuzzy strong bi-ideal of a near-ring N. Then $W_P(axy) \ge \min\{W_P(x), W_P(y)\}$ and $V_P(axy) \le \max\{V_P(x), V_P(y)\}$, for all $a,x,y \in N$.*

Proof. Assume that (W_P, V_P) is a Pythagorean fuzzy strong bi-ideal of N.
Then $N \circ W_P \circ W_P \subseteq W_P$ and $N \circ V_P \circ V_P \supseteq V_P$.
Let a,x,y be any element of N. Then

$$\begin{aligned}W_P(axy) \ge (N \circ W_P \circ W_P) &= \sup_{axy=pq} \min\{(N \circ W_P)(p), W_P(q)\} \\ &\ge \min\{(N \circ W_P)(ax), W_P(y)\} \\ &= \min\{\sup_{ax=z_1z_2} \min\{N(z_1), W_P(z_2)\}, W_P(y)\} \\ &\ge \min\{\min\{N(a), W_P(x)\}, W_P(y)\} \\ &= \min\{\min\{1, W_P(x), W_P(y)\}\} \\ &= \min\{W_P(x), W_P(y)\}.\end{aligned}$$

This show that $W_P(axy) \ge \min\{W_P(x), W_P(y)\}$, for all $a,x,y \in N$

$$\begin{aligned}V_P(axy) &\le (N \circ V_P \circ V_P)(axy) \\ &= \inf_{axy=pq} \max\{(N \circ V_P)(p), V_P(q)\} \\ &\le \max\{(N \circ V_P)(ax), V_P(y)\} \\ &= \max\{\inf_{ax=z_1z_2} \max\{N(z_1), V_P(z_2)\}, V_P(y)\} \\ &\le \max\{\max\{N(a), V_P(x)\}, V_P(y)\} \\ &= \max\{\max\{0, V_P(x)\}, V_P(y)\} \\ &= \max\{V_P(x), V_P(y)\}.\end{aligned}$$

This shows that $V_P(axy) \le \max\{V_P(x), V_P(y)\}$, for all $a,x,y \in N$. □

5. Direct Product of Pythagorean Fuzzy Ideals of Near-Rings

This section some basic properties such as union, intersection, homomorphic image, and pre-image of Pythagorean fuzzy ideals of near ring.

Definition 5.1. Let C and D be Pythagorean fuzzy subsets of near-rings (PFSSNR) N_1 and N_2, respectively. Then, the direct product of Pythagorean fuzzy subsets of near-rings is defined by $C \times D : N_1 \times N_2 \to [0,1]$ such that

$$C \times D = \{\langle (h,k), W_{C\times D}(h,k), V_{C\times D}(h,k; h \in N_1, k \in N_2)\},$$

where

$$W_{C\times D}(h,k) = \min\{W_C(h), W_D(k)\},$$

$$V_{C\times D}(h,k) = \max\{V_C(h), V_D(k)\}.$$

Definition 5.2. Let C and D be Pythagorean fuzzy subsets of near-rings N_1 and N_2, respectively. Then, $C \times D$ is a Pythagorean fuzzy ideal of $N_1 \times N_2$ if it satisfies the following conditions:

(i) $W_{C\times D}((h_1,h_2)-(k_1,k_2)) \geq \min\{W_{C\times D}(h_1,h_2), W_{C\times D}(k_1,k_2)\}$,

(ii) $V_{C\times D}((h_1,h_2)-(k_1,k_2)) \leq \max\{V_{C\times D}(h_1,h_2), V_{C\times D}(k_1,k_2)\}$,

(iii) $W_{C\times D}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) \geq \min(W_{C\times D}(h_1,h_2))$,

(iv) $V_{C\times D}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) \leq \max(V_{C\times D}(h_1,h_2))$,

(v) $W_{C\times D}((h_1,h_2)(k_1,k_2)) \geq W_{C\times D}(k_1,k_2)$,

(vi) $V_{C\times D}((h_1,h_2)(k_1,k_2)) \leq V_{C\times D}(k_1,k_2)$,

(vii) $W_{C\times D}([(h_1,h_2)+(t_1,t_2)](k_1,k_2)-(h_1,h_2)(k_1,k_2)) \geq W_{C\times D}(t_1,t_2)$,

(viii) $V_{C\times D}([(h_1,h_2)+(t_1,t_2)](k_1,k_2)-(h_1,h_2)(k_1,k_2)) \leq V_{C\times D}(t_1,t_2)$.

Theorem 5.3. *Let C and D be Pythagorean fuzzy ideals of N_1 and N_2, respectively. Then $C \times D$ is a Pythagorean fuzzy ideal of $N_1 \times N_2$.*

Proof. Let C and D be Pythagorean fuzzy ideals of N_1 and N_2, respectively.
Let $(h_1,h_2),(k_1,k_2),(t_1,t_2) \in N_1 \times N_2$. Then, the following are obtained.
For truth grade, we obtain the following:

$$\begin{aligned} W_{C\times D}((h_1,h_2)-(k_1,k_2)) &= W_{C\times D}(h_1-k_1, h_2-k_2) \\ &= \min\{W_C(h_1-k_1), W_D(h_2-k_2)\} \\ &\geq \min\{\min\{W_C(h_1), W_D(k_1)\}, \min\{W_D(h_2), W_D(k_2)\}\} \\ &= \min\{\min\{W_C(h_1), W_D(h_1)\}, \min\{W_C(k_1), W_D(k_2)\}\} \\ &= \min\{W_{C\times D}(h_1,h_2), W_{C\times D}(k_1,k_2)\}. \end{aligned}$$

For non-membership grade, we get

$$V_{C\times D}((h_1,h_2)-(k_1,k_2)) = V_{C\times D}(h_1-k_1, h_2-k_2)$$

$$
\begin{aligned}
&= \max\{V_C(h_1-k_1), V_D(h_2-k_2)\}\\
&\le \max\{\max\{V_C(h_1), V_C(k_1)\}, \max\{V_D(h_2), V_D(k_2)\}\}\\
&= \max\{\max\{V_C(h_1), V_D(h_2)\}, \max\{V_C(k_1), V_D(k_2)\}\}\\
&= \max\{V_{C\times D}(h_1,h_2), V_{C\times D}(k_1,k_2)\}.
\end{aligned}
$$

Next, it is clear that

$$
\begin{aligned}
W_{C\times D}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &= W_{C\times D}(k_1+h_1-k_1, k_2+h_2-k_2)\\
&= \min\{W_C(k_1+h_1-k_1), W_D(k_2+h_2-k_2)\}\\
&\ge \min\{W_C(h_1), W_D(h_2)\}\\
&= W_{C\times D}(h_1,h_2),\\
V_{C\times D}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &= V_{C\times D}(k_1+h_1-k_1, k_2+h_2-k_2)\\
&= \max\{V_C(k_1+h_1-k_1), V_D(k_2+h_2-k_2)\}\\
&\le \max\{V_C(h_1), V_D(h_2)\}\\
&= V_{C\times D}(h_1,h_2).
\end{aligned}
$$

Moreover, we deduce the following inequalities:

$$
\begin{aligned}
W_{C\times D}((h_1,h_2)(k_1,k_2)) &= W_{C\times D}(h_1k_1, h_2k_2)\\
&= \min\{W_C(h_1k_1), W_D(h_2k_2)\}\\
&\ge \min\{W_C(k_1), W_D(k_2)\}\\
&= W_{C\times D}(k_1,k_2),\\
V_{C\times D}((h_1,h_2)(k_1,k_2)) &= V_{C\times D}(h_1k_1, h_2k_2)\\
&= \max\{V_C(h_1k_1), V_D(h_2k_2)\}\\
&\le \max\{V_C(k_1), V_D(k_2)\}\\
&= V_{C\times D}(k_1,k_2).
\end{aligned}
$$

Finally, we prove that

$$
\begin{aligned}
&W_{C\times D}((h_1,h_2)+(t_1,t_2)-(k_1,k_2)-(h_1,h_2)(k_1,k_2))\\
&\quad = W_{C\times D}([h_1+t_1]k_1-h_1k_1, [h_2+t_2]k_2-h_2k_2)\\
&\quad = \min\{W_C([h_1+t_1]k_1-h_1k_1), W_D([h_2+t_2]k_2-h_2k_2)\}\\
&\quad \ge \min\{W_C(t_1), W_D(t_2)\}\\
&\quad = W_{C\times D}(t_1,t_2).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&V_{C\times D}((h_1,h_2)+(t_1,t_2)-(k_1,k_2)-(h_1,h_2)(k_1,k_2))\\
&\quad = V_{C\times D}([h_1+t_1]k_1-h_1k_1, [h_2+t_2]k_2-h_2k_2)\\
&\quad = \max\{(V_C[h_1+t_1]k_1-h_1k_1), (V_D[h_2+t_2]k_2-h_2k_2)\}
\end{aligned}
$$

$$\leq \max\{V_C(t_1), V_D(t_2)\}$$
$$= V_{C\times D}(t_1, t_2).$$

Therefore, $C \times D$ is a Pythagorean fuzzy ideal of $N_1 \times N_2$. □

6. Homomorphism of Pythagorean Fuzzy Ideals of Near-Rings

This section is concerned with the direct product of Pythagorean fuzzy ideals of near ring.

Definition 6.1. Let R and S be two near rings. Then, the mapping $f : R \to S$ is called a near-ring homomorphism if for all $h, k \in R$, the following hold:

(i) $f(h+k) = f(h) + f(k)$,

(ii) $f(hk) = f(h)f(k)$.

Definition 6.2. Let U and Y be two nonempty sets and $f : U \to Y$ be a function.

(i) If D is a Pythagorean fuzzy set in Y, then the preimage of D under f denoted by $f^{-1}(D)$, is the Pythagorean fuzzy set in U defined by

$$f^{-1}(D) = \{\langle (h), f^{-1}(W_D(h)), f^{-1}(V_D(h))\rangle : h \in U\},$$

where $f^{-1}(W_D(h)) = W_D(f(h))$ and $f^{-1}(V_D(h)) = V_D(f(h))$ and so on.

(ii) If C is a Pythagorean fuzzy set in U, then the image of C under f denoted by $f(C)$ is the Pythagorean fuzzy set in Y defined by $f(C) = \{\langle (k), f(W_C(k)), f(V_C(k))\rangle : k \in Y\}$, where

$$f(W_C(k)) = \begin{cases} \sup\limits_{h\in f^{-1}(k)} W_C(h), & \text{if } f^{-1}(k) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f(V_C(k)) = \begin{cases} \inf\limits_{h\in f^{-1}(k)} V_C(h), & \text{if } f^{-1}(k) \neq 0 \\ 1, & \text{otherwise} \end{cases}$$

where $f(V_P(k)) = (1 - f(1 - V_P))(k)$.

Theorem 6.3. *Let N and N' be two near rings and f be a homomorphism of N onto N'. If C' is a Pythagorean fuzzy ideal of N', then $f^{-1}(C)$ is a Pythagorean fuzzy ideal of N.*

Proof. Suppose $h, k, t \in N$. Then, we can deduce the following inequalities:
For membership grade,

$$f^{-1}(W_C)(h-k) = W_C(f(h-k))$$
$$= W_C(f(h) - f(k))$$
$$\geq \min(W_C(f(h), W_C(f(k))))$$
$$= \min(f^{-1}(W_C)(h), f^{-1}(W_C)(k)).$$

For non-membership grade, we write

$$f^{-1}(V_C)(h-k) = V_C(f(h-k))$$

$$
\begin{aligned}
&= V_C(f(h) - f(k)) \\
&\leq \max(V_C(f(h), V_C(f(k)))) \\
&= \max(f^{-1}(V_C)(h), f^{-1}(V_C)(k)).
\end{aligned}
$$

Also, we acquire the following:

$$
\begin{aligned}
f^{-1}(W_C)(k+h-k) &= W_C(f(k+h-k)) \\
&= W_C(f(k)+f(h)-f(k)) \\
&\geq \min(W_C(f(h))) \\
&= \min(f^{-1}(W_C)(h)), \\
f^{-1}(V_C)(k+h-k) &= V_C(f(k+h-k)) \\
&= V_C(f(k)+f(h)-f(k)) \\
&\leq \max(V_C(f(h))) \\
&= \max(f^{-1}(V_C)(h)).
\end{aligned}
$$

Furthermore, for membership grade, we obtain

$$
\begin{aligned}
f^{-1}(W_C)(hk) &= W_C(f(hk)) \\
&= W_C(f(h)f(k)) \\
&\geq \min(W_C(f(k))) \\
&= \min(f^{-1}(W_C)(k)).
\end{aligned}
$$

For non-membership grade, we note

$$
\begin{aligned}
f^{-1}(V_C)(hk) &= V_C(f(hk)) \\
&= V_C(f(h)f(k)) \\
&\leq \max(V_C(f(k))) \\
&= \max(f^{-1}(V_C)(k)).
\end{aligned}
$$

Finally, for truth grade, we obtain

$$
\begin{aligned}
f^{-1}(W_C)((h+t)k-hk) &= W_C(f[(h+t)k-hk]) \\
&= W_C([f(h)+f(t)]f(k)-f(h)f(k)) \\
&\geq \min(W_C(f(t))) \\
&= \min(f^{-1}(W_C)(t)).
\end{aligned}
$$

For non-membership grade,

$$
\begin{aligned}
f^{-1}(V_C)((h+t)k-hk) &= V_C(f[(h+t)k-hk]) \\
&= V_C([f(h)+f(t)]f(k)-f(h)f(k)) \\
&\leq \max(V_C(f(t))) \\
&= \max(f^{-1}(V_C)(t)).
\end{aligned}
$$

Therefore, $f^{-1}(C)$ is a Pythagorean fuzzy ideal of N. □

Theorem 6.4. *Let N_1 and N_2 be two near rings and f be a homomorphism of N_1 and N_2. If C is a Pythagorean fuzzy ideal of N_1, then $f(C)$ is a Pythagorean fuzzy ideal of N_2.*

Proof. Let $k_1,k_2,k_3 \in N_2$ and $h_1,h_2,h_1 \in N_1$. Then, the following are observed.
For truth grade, we can write

$$\begin{aligned} f(W_C(k_1-k_2)) &= \sup_{h_1,h_2\in f^{-1}(N_2)} W_C(h_1-h_2) \\ &\geq \sup_{h_1,h_2\in f^{-1}(N_2)} \min(W_C(h_1),W_C(h_2)) \\ &= \min(\sup_{h_1\in f^{-1}(N_2)} W_C(h_1), \sup_{h_2\in f^{-1}(N_2)} W_C(h_2)) \\ &= \min(f(W_C(k_1)), f(W_C(k_2))). \end{aligned}$$

Also,

$$\begin{aligned} f(W_C(k_1+k_2-k_1)) &= \sup_{h_1\in h_2\in f^{-1}(N_2)} W_C(h_1+h_2-h_1) \\ &\geq \sup_{h_1\in f^{-1}(N_2)} W_C(h_1) \\ &= f(W_C(k_1)). \end{aligned}$$

Furthermore, we write

$$\begin{aligned} f(W_c(k_1k_2)) &= \sup_{h_1,h_2\in f^{-1}(N_2)} W_C(h_1h_2) \\ &\geq \sup_{h_2\in f^{-1}(N_2)} W_C(h_2) \\ &= f(W_C(k_2)). \end{aligned}$$

Finally, we get

$$\begin{aligned} f(W_C)((k_1+k_3)k_2-k_1k_2) &= \sup_{h_1,h_2,h_3\in f^{-1}(N_2)} W_C((h_1+h_3)h_2-h_1h_2) \\ &\geq \sup_{h_1,h_2,h_3\in f^{-1}(N_2)} W_C(h_3) \\ &= f(W_C(k_3)). \end{aligned}$$

For non-membership grade, we deduce that

$$\begin{aligned} &f(V_C(k_1-k_2)) \leq \max(f(V_C(k_1)), f(V_C(k_2))) \\ &f(V_C(k_1+k_2-k_1)) \leq f(V_C(k_2)) \\ &f(V_C(k_1k_2)) \leq f(V_C(k_2)) \\ &f(V_C((k_1+k_3)k_2+k_1k_2)) \leq f(V_C(k_3)). \end{aligned}$$

Hence, $f(C)$ is a Pythagorean fuzzy ideal of N_2. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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