**Communications in Mathematics and Applications**
Volume 4 (2013), Number 1, pp. 21–28

http://www.rgnpublications.com

# Application of Chebyshev Polynomials to the Approximate Solution of Singular Integral Equations of the First Kind with Cauchy Kernel on the Real Half-line

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**Abstract** In this paper, exact solution of the characteristic equation with *Cauchy* kernel on the real half-line is presented. Next, the *Chebyshev* polynomials of the second kind, $U_n(x)$, and fourth kind, $W_n(x)$, are used to derive numerical solutions of *Cauchy*-type singular integral equations of the first kind on the real half-line. The collocation points are chosen as the zeros of the *Chebyshev* polynomials of the first kind, $T_{n+2}(x)$, and third kind, $V_{n+1}(x)$. Moreover, estimations of errors of the approximated solutions are presented. The numerical results are given to show the accuracy of the methods presented.

## 1. Introduction

Let us consider the equation

$$\frac{1}{\pi}\int_0^{+\infty}\frac{\varphi(\sigma)}{\sigma-x}d\sigma+\frac{1}{\pi}\int_0^{+\infty}k(x,\sigma)\varphi(\sigma)d\sigma=f(x),\quad x>0, \tag{1.1}$$

where $k(x,\sigma)$ and $f(x)$ are given real-valued H lder continuous functions and $\varphi(x)$ is an unknown function. The theory of equations of the form (1.1) and their approximate solutions for the case in which the integration line is a closed or open curve of finite length can be found in many references [1, 2, 4, 5, 9].

We apply the transformation of the form (see [7, 8])

$$\frac{1}{\sigma-x}=\frac{x+1}{\sigma+1}\frac{1}{\sigma-x}+\frac{1}{\sigma+1}, \tag{1.2}$$

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2010 *Mathematics Subject Classification*. 65R20.
*Key words and phrases*. Singular integral equation; Cauchy kernel; Approximate solution; Chebyshev polynomials; Collocation points.

and rewrite (1.1) in the form

$$\frac{1}{\pi}\int_0^{+\infty}\frac{(x+1)\varphi(\sigma)}{(\sigma+1)(\sigma-x)}d\sigma+\frac{1}{\pi}\int_0^{+\infty}\frac{\varphi(\sigma)}{\sigma+1}d\sigma$$
$$+\frac{1}{\pi}\int_0^{+\infty}k(x,\sigma)\varphi(\sigma)d\sigma=f(x),\quad x>0\,. \tag{1.3}$$

We assume that the behavior of the function $k(x,\sigma)$ as $\sigma\to+\infty$ is described by the relation

$$k(x,\sigma)=\frac{k_0(x,\sigma)}{(\sigma+1)^{\alpha}},\quad \alpha>1\,,$$

where $k_0(x,\sigma)$ is a H  lder continuous function. By setting

$$x=\frac{1+t}{1-t},\quad \sigma=\frac{1+\tau}{1-\tau}\,, \tag{1.4}$$

we reduce (1.3) to the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-t}d\tau-\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-1}d\tau+\frac{1}{\pi}\int_{-1}^{1}k^*(t,\tau)\psi(\tau)d\tau=g(t),\quad t\in(-1,1)\,, \tag{1.5}$$

where

$$\psi(\tau)=\varphi\left(\frac{1+\tau}{1-\tau}\right),\ \ g(t)=f\left(\frac{1+t}{1-t}\right),\ \ k^*(t,\tau)=\left(\frac{2}{(1-\tau)^2}\right)k\left(\frac{1+t}{1-t},\frac{1+\tau}{1-\tau}\right).$$

We set $k^*(t,\tau)\equiv 0$ in (1.5) and first analyze the equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-t}d\tau-\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-1}d\tau=g(t),\quad t\in(-1,1),\ g(1)=0, \tag{1.6}$$

in two cases.

**Case (I):** If the solution $\psi(t)$ is sought in the class of H  lder continuous functions on $(-1,1)$, bounded at the point $t=1$ and unbounded at the point $t=-1$, then, in view of [5], we have

$$\psi(t)=-\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\sqrt{\frac{1+\tau}{1-\tau}}\frac{(g(\tau)+\gamma)}{\tau-t}d\tau\,, \tag{1.7}$$

where

$$\gamma=\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-1}d\tau\,. \tag{1.8}$$

Using the following relation (see [3]):

$$p.v\int_a^b\left(\frac{b-\tau}{\tau-a}\right)^{\nu}\frac{d\tau}{\tau-\xi}=(\pi\cot\pi\nu)\left(\frac{b-\xi}{\xi-a}\right)^{\nu}-\pi\csc(\pi\nu)\,, \tag{1.9}$$

we can rewrite (1.7) in the form

$$\psi(t) = -\sqrt{\frac{1-t}{1+t}}\left(\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\tau}{1-\tau}}\frac{g(\tau)d\tau}{\tau - t} + \gamma\right), \tag{1.10}$$

where $\gamma$ is an arbitrary constant.

The constant $\gamma$ is uniquely determined if (1.6) is supplemented by the condition

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau - 1}d\tau = \gamma^*, \tag{1.11}$$

equivalent to (1.8). Substituting function (1.10) for $\psi(t)$ into this relation, we obtain $\gamma = \gamma^*$.

**Case (II):** If the solution $\psi(t)$ is sought in the class of H lder continuous functions on $(-1,1)$, bounded at the points $t = \pm 1$, then

$$\psi(t) = -\frac{1}{\pi}\sqrt{1-t^2}\int_{-1}^{1}\frac{(g(\tau)+\gamma)}{\sqrt{1-\tau^2}}\frac{d\tau}{\tau - t}, \tag{1.12}$$

provided that

$$\int_{-1}^{1}\frac{g(\tau)+\gamma}{\sqrt{1-\tau^2}}d\tau = 0, \tag{1.13}$$

(see [5]). Using the following relations

$$\int_{-1}^{1}\frac{T_n(\tau)}{\sqrt{1-\tau^2}}\frac{d\tau}{\tau - t} = \begin{cases}\pi U_{n-1}(t), & n > 0,\\ 0, & n = 0,\end{cases} \tag{1.14}$$

and

$$\int_{-1}^{1}\frac{T_0(t)}{\sqrt{1-t^2}}dt = \pi, \tag{1.15}$$

we can rewrite (1.12) and (1.13) in the form

$$\psi(t) = -\frac{1}{\pi}\sqrt{1-t^2}\int_{-1}^{1}\frac{g(\tau)}{\sqrt{1-\tau^2}}\frac{d\tau}{\tau - t}, \tag{1.16}$$

provided that

$$\gamma = -\frac{1}{\pi}\int_{-1}^{1}\frac{g(\tau)}{\sqrt{1-\tau^2}}d\tau. \tag{1.17}$$

Therefore, in the original variables $x$, $\sigma$, the solution of (1.1) where $k(x,\sigma) \equiv 0$ is expressed in the following forms:

**Case (I):** If the solution $\varphi(x)$ is sought in the class of functions that are H lder continuous on $[\varepsilon, +\infty), \varepsilon > 0$, vanish at infinity, i.e. $\lim_{x\to\infty}\varphi(x) = 0$, and can have an integrable singularity in the neighborhood of $x = 0$, then

$$\varphi(x) = -\frac{1}{\sqrt{x}}\left(\frac{1}{\pi}\int_{0}^{+\infty}\frac{\sqrt{\sigma}(x+1)}{\sigma + 1}\frac{f(\sigma)}{\sigma - x}d\sigma + \gamma\right), \tag{1.18}$$

where $\gamma$ is an arbitrary constant. Additionally, if the solution $\varphi(x)$ satisfies the condition

$$-\frac{1}{\pi}\int_0^{+\infty}\frac{\varphi(\sigma)}{\sigma+1}d\sigma=\gamma^*, \tag{1.19}$$

where $\gamma^*$ is an arbitrary number, then the unique solution of (1.1) is given by the formula (1.18) with $\gamma=\gamma^*$.

**Case (II):** If the solution $\varphi(x)$ is sought in the class of bounded H lder functions on $(0,+\infty)$ vanishing at infinity, then

$$\varphi(x)=-\frac{1}{\pi}\sqrt{x}\int_0^{+\infty}\frac{f(\sigma)}{\sqrt{\sigma}}\frac{d\sigma}{(\sigma-x)}, \tag{1.20}$$

provided that

$$-\frac{1}{\pi}\int_0^{+\infty}\frac{\varphi(\sigma)}{\sigma+1}d\sigma=-\frac{1}{\pi}\int_0^{+\infty}\frac{f(\sigma)}{\sqrt{\sigma}(\sigma+1)}d\sigma. \tag{1.21}$$

## 2. Approximate Solutions of the Complete Equation

In this section, we will derive an approximate solution of (1.5) in two cases.

**Case (I):** An approximate solution in the case that the solution of (1.5) is bounded at the point $t=1$ and unbounded at the point $t=-1$ is expressed of the form

$$\psi_n(\tau)=\sqrt{\frac{1-\tau}{1+\tau}}\sum_{j=0}^{n}\beta_j W_j(\tau), \tag{2.1}$$

where $W_j$ is the *Chebyshev* polynomial of the fourth kind which is defined by the following recurrence relation

$$\begin{aligned} W_0(\tau) &= 1, \quad W_1(\tau)=2\tau+1,\\ W_n(\tau) &= 2\tau W_{n-1}(\tau)-W_{n-2}(\tau), \qquad n\geq 2. \end{aligned} \tag{2.2}$$

We rewrite (1.5) in the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau-t}d\tau+\frac{1}{\pi}\int_{-1}^{1}k^*(t,\tau)\psi(\tau)d\tau=g(t)+\gamma^*, \quad t\in(-1,1), \tag{2.3}$$

where $\gamma^*$ is determined of (1.11). If we substitute (2.1) in (2.3) and use the relation (see [6])

$$\int_{-1}^{1}\sqrt{\frac{1-\tau}{1+\tau}}\frac{W_j(\tau)}{\tau-t}d\tau=-\pi V_j(t), \tag{2.4}$$

we get

$$\sum_{j=0}^{n}\beta_j\left(-V_j(t)+\frac{1}{\pi}Q_j^*(t)\right)=g(t)+\gamma^*, \tag{2.5}$$

where

$$Q_j^*(t) = \int_{-1}^{1} k^*(t,\tau)\sqrt{\frac{1-\tau}{1+\tau}}W_j(\tau)d\tau\,, \tag{2.6}$$

and $V_j$ is the *Chebyshev* polynomial of the third kind which is defined by the following recurrence relation

$$\begin{aligned} V_0(\tau) &= 1, \quad V_1(\tau) = 2\tau - 1, \\ V_n(\tau) &= 2\tau V_{n-1}(\tau) - V_{n-2}(\tau), \quad n \geq 2. \end{aligned} \tag{2.7}$$

Using the zeros of $V_{n+1}(\tau)$,

$$t_i = \cos\left(\frac{(2i-1)\pi}{(2i+3)}\right), \quad i = 1,2,\ldots,n+1\,, \tag{2.8}$$

as the collocation points, we obtain the coefficients $\{\beta_j\}_0^n$ by solving the following system of linear equations

$$\sum_{j=0}^{n}\beta_j\Big(-V_j(t_i) + \frac{1}{\pi}Q_j^*(t_i)\Big) = g(t_i) + \gamma^*, \quad i = 1,2,\ldots,n+1. \tag{2.9}$$

In the special case that $k^*(t,\tau) \equiv 0$, the approximate solution (1.6) is

$$\sum_{j=0}^{n} -\beta_j V_j(t_i) = g(t_i) + \gamma^*, \quad i = 1,2,\ldots,n+1. \tag{2.10}$$

**Case (II):** An approximate solution in the case that the solution of (1.5) is bounded at the points $t = \pm 1$ is expressed of the form

$$\psi_n(\tau) = \sqrt{1-\tau^2}\sum_{j=0}^{n}\alpha_j U_j(\tau)\,, \tag{2.11}$$

where $U_j$ is the *Chebyshev* polynomial of the second kind which is defined by the following recurrence relation

$$\begin{aligned} &U_0(\tau) = 1, \quad U_1(\tau) = 2\tau, \\ &U_n(\tau) = 2\tau U_{n-1}(\tau) - U_{n-2}(\tau), \quad n \geq 2. \end{aligned} \tag{2.12}$$

We rewrite (1.5) in the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(\tau)}{\tau - t}d\tau + \frac{1}{\pi}\int_{-1}^{1}k^*(t,\tau)\psi(\tau)d\tau = g(t) + \gamma, \quad t \in (-1,1), \tag{2.13}$$

where $\gamma$ is determined of (1.17). If we substitute (2.11) in (2.13) and use the relation (see [6])

$$\int_{-1}^{1}\frac{\sqrt{1-\tau^2}U_j(\tau)}{\tau - t}d\tau = -\pi T_{j+1}(t)\,, \tag{2.14}$$

then we will obtain

$$\sum_{j=0}^{n}\alpha_j\Big(-T_{j+1}(t) + \frac{1}{\pi}Q_j(t)\Big) = g(t) + \gamma\,, \tag{2.15}$$

where

$$Q_j(t) = \int_{-1}^{1} k^*(t,\tau)\sqrt{1-\tau^2}U_j(\tau)d\tau\,. \tag{2.16}$$

Let $t_k$ be the zeros of $T_{n+2}(t)$, i.e.

$$t_k = \cos\left(\frac{(2k-1)\pi}{(2n+4)}\right), \quad k = 1,2,\dots,n+2. \tag{2.17}$$

Substituting the collocation points (2.17) in (2.15), we obtain the coefficients $\{\alpha_j\}_0^n$ by solving the following system of linear equations

$$\sum_{j=0}^{n}\alpha_j\left(-T_{j+1}(t_i) + \frac{1}{\pi}Q_j(t_i)\right) = g(t_i) + \gamma, \quad i = 1,2,\dots,n+1.$$

In the special case that $k^*(t,\tau) \equiv 0$, the approximate solution (1.6) is

$$\sum_{j=0}^{n} -\alpha_j T_{j+1}(t_i) = g(t_i) + \gamma, \quad i = 1,2,\dots,n+1.$$

## 3. **Error Estimation**

Now, we give an error estimation for the approximate solutions of (1.5). Let $\psi_n(t)$ be approximate solution and $e_n(t) = \psi_n(t) - \psi(t)$, be the error function associated with $\psi_n(t)$, where $\psi(t)$ is the exact solution of (1.5). Since $\psi_n(t)$ is an approximate solution, it satisfies in

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi_n(\tau)}{\tau - t}d\tau - \frac{1}{\pi}\int_{-1}^{1}\frac{\psi_n(\tau)}{\tau - 1}d\tau + \frac{1}{\pi}\int_{-1}^{1}k^*(t,\tau)\psi_n(\tau)d\tau = g(t) + H_n(t), \tag{3.1}$$

where $H_n(t)$ is a perturbation term and it is obtained from

$$H_n(t) = \frac{1}{\pi}\int_{-1}^{1}\frac{\psi_n(\tau)}{\tau - t}d\tau - \frac{1}{\pi}\int_{-1}^{1}\frac{\psi_n(\tau)}{\tau - 1}d\tau + \frac{1}{\pi}\int_{-1}^{1}k^*(t,\tau)\psi_n(\tau)d\tau - g(t). \tag{3.2}$$

Subtracting (1.5) from (3.2), yields the equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{e_n(\tau)}{\tau - t}d\tau - \frac{1}{\pi}\int_{-1}^{1}\frac{e_n(\tau)}{\tau - 1}d\tau + \frac{1}{\pi}\int_{-1}^{1}k^*(t,\tau)e_n(\tau)d\tau = H_n(t) \tag{3.3}$$

for the error function $e_n(t)$. To find an approximation $\hat{e}_n(t)$ to $e_n(t)$, we can solve (3.3) by the same ways as we did for (1.5). In this case, only the function $g(t)$ will be replaced by the perturbation term $H_n(t)$. Note that the integrals in above equations are considered as the *Cauchy* principal value integrals.

## 4. Numerical Example

In this section, we give a numerical example to clarify accuracy of the presented method. The results of example are reported in Tables 1 and 2. Moreover, we can compare numerical results for $e_n(t) = |\psi_n(t) - \psi(t)|$ and $|\hat{e}_n(t)|$ in Tables 1 and 2. In the case (I), we consider $\gamma^* = \frac{1}{\pi}\int_{-1}^{1} \frac{g(\tau)}{\sqrt{1-\tau^2}} d\tau$.

**Example.**

$$\frac{1}{\pi}\int_0^{+\infty} \frac{\varphi(\sigma)}{\sigma - x} d\sigma = \frac{1}{2x+3}, \quad x > 0. \tag{4.1}$$

**Table 1.** Numerical results in the case (I)

| $x$ | $t$ | $\psi_n(t)$ | $\psi(t)$ | $\lvert e_n(t)\rvert$ | $\lvert \hat{e}_n(t)\rvert$ |
|---|---|---|---|---|---|
| 0.11 | −0.8 | −1.0165553508894 | −1.0165553508893 | 0.9e-13 | 0.12e-12 |
| 0.25 | −0.6 | −0.61737130758516 | −0.61737130758512 | 0.4e-13 | 0.39e-13 |
| 0.33 | −0.5 | −0.50710962108493 | −0.50710962108493 | 0 | 0.34e-13 |
| 0.538 | −0.3 | −0.35318599623723 | −0.35318599623724 | 0.1e-13 | 0 |
| 0.81 | −0.1 | −0.24644645519425 | −0.24644645519422 | 0.3e-13 | 0.22e-13 |
| 1.5 | 0.2 | −0.13299316185546 | −0.13299316185548 | 0.2e-13 | 0 |
| 1.857 | 0.3 | −0.10358852081208 | −0.10358852081207 | 0.1e-13 | 0.14e-13 |
| 5.66 | 0.7 | −0.018570177408511 | −0.018570177408511 | 0 | 0.25e-14 |
| 9 | 0.8 | −0.0056932440108893 | −0.005693244010887 | 0.2e-14 | 0.66e-15 |
| 19 | 0.9 | 0.0026083711357127 | 0.002608371135713 | 0 | 0.91e-15 |

**Table 2.** Numerical results in the case (II)

| $x$ | $t$ | $\psi_n(t)$ | $\psi(t)$ | $\lvert e_n(t)\rvert$ | $\lvert \hat{e}_n(t)\rvert$ |
|---|---|---|---|---|---|
| 0.11 | −0.8 | 0.084465163544256 | 0.084465163544250 | 0.6e-14 | 0 |
| 0.25 | −0.6 | 0.11664236870396 | 0.11664236870397 | 0.1e-13 | 0 |
| 0.33 | −0.5 | 0.12856486930665 | 0.12856486930664 | 0.1e-13 | 0.106e-11 |
| 0.538 | −0.3 | 0.14696001818300 | 0.14696001818299 | 0.1e-13 | 0.442e-12 |
| 0.81 | −0.1 | 0.15929487067914 | 0.15929487067915 | 0.1e-13 | 0.353e-12 |
| 1 | 0 | 0.16329931618555 | 0.16329931618555 | 0 | 0.306e-12 |
| 1.5 | 0.2 | 0.16666666666667 | 0.16666666666667 | 0 | 0 |
| 1.857 | 0.3 | 0.16572087156806 | 0.16572087156806 | 0 | 0 |
| 5.66 | 0.7 | 0.13560353243826 | 0.13560353243826 | 0 | 0.393e-13 |
| 9 | 0.8 | 0.11664236870396 | 0.11664236870397 | 0.1e-13 | 0.209e-13 |
| 19 | 0.9 | 0.086805514244158 | 0.086805514244163 | 0.5e-14 | 0.597e-14 |

## References

[1] N.I. Muskhelishvilli, *Singular Integral Equations*, Nauka, Moscow, 1968 (in Russian).

[2] F.D. Gakhov, *Boundary Value Problems*, Nauka, Moscow, 1977 (in Russian).

[3] R. Estrada and R.P. Kanwal, *Singular Integral Equations*, Birkh user, Boston, 2000.

[4] I.K. Lifanov, L.N. Poltavskii and G.M. Vainikko, *Hypersingular Integral Equations and Their Applications*, Chapman and Hall/CRC, 2004.

[5] I.K. Lifanov, *Singular Integral Equation and Discrete Vortices*, VSO, The Netherlands, 1996.

[6] J.C. Mason and D.C. Handscomb, *Chebyshev Polynomials*, CRC Press LLC, 2003.

[7] D. Pylak, Application of Jacobi polynomials to the approximate solution of a singular integral equation with Cauchy kernel on the real half-line, *Computational Methods in Applied Mathematics* **6**(3) (2006), 326–335.

[8] D. Pylak, R. Smarzewski and M.A. Sheshko, A singular integral equation with a Cauchy kernel on the real half-line, *Differential Equations* **41**(12) (2005), 1696–1708.

[9] M. Kashfi and S. Shahmorad, Approximate solution of a singular integral Cauchy-Kernel equation of the first kind, *Computational Methods in Applied Mathematics* **10**(4) (2010), 359–367.

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*Received* March 9, 2012
*Accepted* June 26, 2012