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Factorization of Polynomials with Analytic Coefficients

Wayne Lawton

Abstract We study monic univariate polynomials whose coefficients are analytic functions of a real variable and whose roots lie in a specified analytic curve. These include characteristic polynomials of unitary and hermitian matrices whose entries are analytic functions. We use a result of Newton to prove that every polynomial in such a class is a product of degree one polynomials in the class.

1. Introduction

$\mathbb{R}$ and $\mathbb{C}$ are the real and complex numbers and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Functions defined by their Taylor series are called analytic. For $r > 0$, $A(\mathbb{D}_r)$ is the ring of analytic functions on the open disc $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $A((-r,r))$ is the ring of analytic functions on the open interval $(-r,r)$. We let $\mathbb{C}[z] \subset \mathscr{C}_0^\omega \subset \mathbb{C}[[z]]$ denote the rings of polynomials, power series with complex coefficients that are absolutely convergent in $\mathbb{D}_r$ for some $r > 0$, and formal power series. We identify $\mathscr{C}_0^\omega$ with the rings of germs of functions in $\cup_{r>0} A((-r,r))$ and of functions in $\cup_{r>0} A(\mathbb{D}_r)$.

$\mathscr{C}_0^\omega[z]$ is the ring of polynomials with coefficients in $\mathscr{C}_0^\omega$. Let $P(z) \in \mathscr{C}_0^\omega[z]$ be a monic polynomial of degree $d \geq 1$. Then there exist $r > 0$ and $a_0, \dots, a_{d-1} \in A(\mathbb{D}_r)$ such that $P(z) = z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \in \mathscr{C}_0^\omega[z]$. For $w \in \mathbb{D}_r$ we define $P_w(z) \in \mathbb{C}[z]$ by $P_w(z) = z^d + a_{d-1}(w) z^{d-1} + \cdots + a_1(w) z + a_0(w)$. If $\gamma \subset \mathbb{C}$ we say that $P(z)$ *has roots in* γ if there exists $s \in (0,r]$ such that for every $t \in (-s,s)$ all roots of $P_t(z)$ are in γ. We say that $P(z)$ is *completely reducible* if factors into monic polynomials in $\mathscr{C}_0^\omega[z]$ having degree one, or equivalently, if there exist $u \in (0,s]$ and $\lambda_1, \dots, \lambda_d \in A(\mathbb{D}_u)$ such that for every $w \in \mathbb{D}_u$, $\lambda_1(w), \dots, \lambda_d(w)$ are the roots (with multiplicity) of $P_w(z)$. The polynomial $z^2 - t^2$ is completely reducible but the polynomial $z^2 - t$ is not. In Section 3 we prove:

Theorem 1.1. *Every monic polynomial $P(z) \in \mathscr{C}_0^\omega[z]$ that has roots in an analytic curve $\gamma \subset \mathbb{C}$ is completely reducible.*

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2. Preliminary Results

$\gamma \subset \mathbb{C}$ is an analytic curve if it is a real analytic submanifold of dimension 1. This means that for every point $p \in \gamma$ there exist $\epsilon > 0$, an open neighborhood U of p in γ, and an analytic diffeomorphism $f : (-\epsilon, \epsilon) \to U$ with $f(0) = p$. Then $f'(0) \neq 0$. For $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$ we define $\Re z = x$ and $\Im z = y$.

Lemma 2.1. *If $\gamma \subset \mathbb{C}$ is an analytic curve and $p \in \gamma$, then there exist $c \in \mathbb{C}\backslash\{0\}$, $\delta > 0$, an open neighborhood V of p in γ, and an analytic function $h : (-\delta, \delta) \to \mathbb{R}$ such that $h(0) = 0$, $h'(0) = 0$, and*

$$\Im\left[\frac{z-p}{c}\right] = h\left(\Re\left[\frac{z-p}{c}\right]\right), \quad z \in V. \tag{2.1}$$

Proof. Since γ is an analytic curve and $p \in \gamma$, there exist $\epsilon > 0$, an open neighborhood U of p in γ, and an analytic diffeomorphism $f : (-\epsilon, \epsilon) \to U$ such that $f(0) = p$. Let $c = f'(0)$. Then $c \neq 0$. Construct $g : (-\epsilon, \epsilon) \to \mathbb{C}$ by $g(t) = (f(t) - p)/c$ and $\psi = \Re g$. Since $\psi(0) = 0$ and $\psi'(0) = 1$, the implicit function theorem for real analytic functions ([2, Theorem 1.4.3]) implies that there exist $\delta > 0$ and an analytic function $\phi : (-\delta, \delta) \to (-\epsilon, \epsilon)$ such that $\phi(0) = 0$, $\phi'(0) = 1$, and $\psi(\phi(t)) = t$, $t \in (-\delta, \delta)$. Construct $h : (-\delta, \delta) \to \mathbb{R}$ by $h(t) = \Im g(\phi(t))$. Therefore $h(0) = \Im g(\phi(0)) = \Im g(0) = \Im 0 = 0$ and $h'(0) = \Im(g'(0)\phi'(0)) = \Im 1 = 0$. Let $V = f(\phi((-\delta, \delta)))$. Then V is an open neighborhood of p in γ, and for every $z \in V$ there exists $t \in (-\delta, \delta)$ with $z = f(\phi(t))$. Therefore

$$\frac{z-p}{c} = \frac{f(\phi(t)) - p}{c} = g(\phi(t)).$$

Equation (2.1) follows since $\Im g(\phi(t)) = h(t) = h(\psi(\phi(t)) = h(\Re g(\phi(t)))$. □

Lemma 2.2. *If $P(z)$ is a monic polynomial that is irreducible in $\mathscr{C}_0^\omega[z]$ and has degree $d \geq 2$ then there exist $r > 0$ and $\eta \in A(\mathbb{D}_r)$ such that*

$$P_{w^d}(z) = \prod_{k=0}^{d-1} [z - \eta(e^{2\pi i k/d} w)], \quad w \in \mathbb{D}_r\,. \tag{2.2}$$

Proof. Abhyankar ([1, Newton's Theorem and Supplements 1 and 2 on page 89]) proves a version of this result for polynomials with coefficients in the ring of formal power series $\mathbb{C}[[w]]$ and says that it was proved by Newton in 1660 [5]. The version in Lemma 2.2 for coefficients in $\mathscr{C}_0^\omega$ follows from Weierstrass' M-test. □

Lemma 2.3. *If η in Equation (2.2) has the Taylor expansion $\eta(w) = \sum_{n=0}^\infty \eta_n w^n$, then there exists $L \geq 1$ such that $\eta_L \neq 0$ and d does not divide L.*

Proof. Otherwise there exists $\mu \in A(\mathbb{D}_{r^d})$ such that $\eta(w) = \mu(w^d)$, $w \in \mathbb{D}_r$. Then Equation (2.2) implies that $P_{w^d}(z) = (z - \mu(w^d))^d$, $w \in \mathbb{D}_r$. Since the function $w \to w^d$ maps $\mathbb{D}_r$ onto $\mathbb{D}_{r^d}$, $P_w(z) = (z - \mu(w))^d$, $w \in \mathbb{D}_{r^d}$, so $P(z)$ is not irreducible in $\mathscr{C}_0^\omega[z]$. This contradiction completes the proof. □

3. **Proof of Theorem 1.1**

Assume to the contrary that there exist an analytic curve $\gamma \subset \mathbb{C}$ and a monic polynomial $P(z) \in \mathscr{C}_0^\omega[z]$ of degree $d \geq 2$ that has roots in γ and is not completely reducible. We may assume that $P(z)$ is irreducible in $\mathscr{C}_0^\omega[z]$ so Lemma 2.2 implies there exist $r > 0$ and $\eta \in A(\mathbb{D}_r)$ that satisfy Equation (2.2). Since the roots of $P(z)$ are in γ, there exists $s \in (0, r]$ such that $\eta(w) \in \gamma$ whenever $w^d \in \mathbb{R}$ and $w \in \mathbb{D}_s$. Let $p = \eta(0)$. Lemma 2.1 implies that there exist $c \in \mathbb{C}\backslash\{0\}$, $\delta > 0$, an open neighborhood V of p in γ, and an analytic function $h : (-\delta, \delta) \to \mathbb{R}$ such that $h(0) = 0$, $h'(0) = 0$, and Equation (2.1) holds. Since η is continuous there exists $u \in (0, s]$ such that $\eta(w) \in V$ whenever $w^d \in \mathbb{R}$ and $w \in \mathbb{D}_u$. Construct $\lambda = (\eta - p)/c$ with Taylor series $\sum_{n=0}^{\infty} \lambda_n w^n$. Then $\lambda_0 = 0$ and Lemma 2.3 implies that there exists a smallest integer $L \geq 1$ such that $\lambda_L \neq 0$ and d does not divide L. Choose $k \in \{0, 1, 2, \ldots, d-1\}$ such that $\Im(e^{\pi i k L/d}\lambda_L) \neq 0$ and construct $\zeta(t) = \Im \lambda(e^{\pi i k/d} t)$, $t \in (-u, u)$ with Taylor series $\sum_{n=0}^{\infty} \zeta_n t^n$. Then $\zeta_L = \Im(e^{\pi i k L/d}\lambda_L) \neq 0$. If $t \in (-u, u)$ then $\eta(e^{\pi i k L/d} t) \in V$ so Equation (2.1) gives $\zeta(t) = h(\Re \lambda(e^{\pi i k L/d} t))$. The facts that $1 \leq m < L$ implies that d divides m or $\lambda_m = 0$, $\lambda_0 = 0$, $h'(0) = 0$, and d does not divide L, imply that $\zeta_L = 0$.

This contradiction completes the proof.

Remark 3.1. In ([3, Corollary 1]) we proved that a monic $P(z) \in \mathscr{C}_0^\omega[z]$ of degree 2 that has roots in $\mathbb{T}$ is completely reducible and used results in [4] to prove that the eigenvalues of certain unitary matrices (arising in quantum physics) with analytic entries are global analytic functions on $\mathbb{T}$ if the characteristic polynomials of the matrices are completely reducible. Theorem 1.1 ensures this condition holds.

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Wayne Lawton, *Department of Mathematics, Mahidol University, Bangkok 10400, Thailand; School of Mathematics and Statistics, University of Western Australia, Perth, Australia.*
E-mail: scwlw@mahidol.ac.th

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