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Research Article

Transit Index of Subdivision Graphs

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Abstract. The concept of transit of a vertex and transit index of a graph was defined by the authors in their previous work. The transit of a vertex v is “the sum of the lengths of all shortest path with v as an internal vertex” and the transit index of a graph G is the sum of the transit of all the vertices of it. In this paper, we investigate transit index of sub-division graphs.

Keywords. Transit index; Majorized shortest path; Transit decomposition; Subdivision graph

Mathematics Subject Classification (2020). 05C10; 05C12

1. Introduction

It is well known that the chemical behaviour of a compound is dependent upon the structure of its molecules. *Quantitative Structure Activity Relationship* (QSAR) studies and *Quantitative Structure Property Relationship* (QSPR) studies are active areas of chemical research that focus on the nature of this dependency. A topological index is a numeric quantity that is mathematically derived from the structural graph of a molecule. The first reported use of a topological index in chemistry was by Wiener in his study of paraffin boiling points. In [5], transit index of a graph was introduced by the authors and its correlation with one of the physiochemical property-MON of octane isomers was established.

In this paper, we discuss transit index of subdivision of a tree, a graph with no odd cycles and of certain graph classes.

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Throughout G denotes a simple, connected, undirected graph with vertex set V and edge set E, for undefined terms we refer [1].

2. Preliminaries

Definition 2.1 ([5]). Let v be a vertex of G. Then the transit of v denoted by $T(v)$ is "the sum of the lengths of all shortest path with v as an internal vertex" and the transit index of G denoted by $TI(G)$ is

$$TI(G) = \sum_{v \in V} T(v).$$

Lemma 2.2 ([5]). *For a vertex v of G, $T(v) = 0$ iff $\langle N[v] \rangle$ is a clique, or $T(v) = 0$ iff v is a simplicial vertex of G.*

Theorem 2.3 ([5]). *For a path P_n, transit index will be* $\frac{n(n+1)(n^2-3n+2)}{12}$.

Theorem 2.4 ([6]). *Let C_n be a cycle with n even. Then*

(i) $TI(C_n) = \frac{n^2(n^2-4)}{24}$,

(ii) $TI(C_{n+1}) = \frac{n(n^2-4)(n+1)}{24}$,

Definition 2.5. A path M through v is called a majorized shortest path through v, abbreviated as *msp*(v), if it satisfies the following conditions:

(i) M is a shortest path in G with v as an internal vertex.

(ii) There exist no path M' such that, M' is a shortest path in G with v as an internal vertex and M as a sub-path of it.

We denote the collection of all $msp(v)$ by $\mathscr{M}_v$ and $\bigcup_{v \in V} \mathscr{M}_v$ by $\mathscr{M}_G$.

Definition 2.6. A decomposition of a graph G into a collection of sub graphs $\tau = \{T_1, T_2, \dots, T_r\}$, where each T_i is either a chord-less cycle in G or a majorized shortest path of G such that $TI(G) = \sum_i TI(T_i) - \sum_{i \neq j} TI(T_i \cap T_j) + \sum_{i \neq j \neq k} TI(T_i \cap T_j \cap T_k) - \dots$ is called a transit decomposition of G. We denote a transit decomposition of minimum cardinality by $\tau_{\min}$.

Definition 2.7 ([6]). Two vertices v_1 and v_2 of a graph are called transit identical if the shortest paths passing through them are same in number and length.

Definition 2.8 ([3]). The edge subdivision operation for an edge $\{u, v\} \in E$ is the deletion of $\{u, v\}$ from G and the addition of two edges $\{u, w\}$ and $\{w, v\}$ along with the new vertex w.

Definition 2.9 ([3]). A graph which has been derived from G by a sequence of edge subdivision operations is called a subdivision of G.

3. Subdivision of Trees

Theorem 3.1. *Let G be a tree. Let S denote the graph got by subdividing every edge of G. Then $\mathscr{M}_S$ is got by subdividing paths of $\mathscr{M}_G$.*

Proof. Let $M : v_1v_2\dots v\dots v_{k-1}v_k \in \mathscr{M}_G$. Let $M' : v_1u_1v_2u_2\dots v\dots v_{k-1}uk-1v_k$ be the subdivision of M.

Claim 1: M' is a shortest path connecting v_1 to v_k in S.

If possible let M' be not a shortest path connecting v_1 to v_k in S. Then there exist some path $N' : v_1n_1n_2\dots n_sv_k$, where $s+1 < 2k-2$. Clearly, $n_1, n_3, \dots, n_s$ are subdivision vertices. Hence the path $N : v_1n_2n_4\dots n_{s-1}v_k$ is a path in G connecting v_1 to v_k of length $\frac{s-3}{2}+2$.

But $\frac{s-3}{2}+2 \le k-1$, a contradiction to the fact that M is a shortest path connecting v_1 to v_k. Hence the claim.

Claim 2: There exist no path M'' in S such that M'' is a shortest path with v as an internal vertex and M' as a subpath of it.

Suppose on the contrary, let M'' be a shortest path in S with v as an internal vertex and M' as a subpath of it. Then M'' connects two pendant vertices of S. Let $M'' = z_1u_1\dots M'\dots u_{s-1}z_s$. Then the path $M'' - \{u_1, u_2, \dots, u_{s-1}\}$ is a path in G with M as a subpath and v as an internal vertex. This is a contraction. Hence the claim.

These two claims prove the theorem. □

4. Sub Division of a Graph

Theorem 4.1. *Let G be a graph with no odd cycles. Let $\tau_{\min}$ be a transit decomposition of G of minimum cardinality. τ' denotes the collection of all sub division of paths/cycles in $\tau_{\min}$. Then τ' is a transit decomposition of $S(G)$, the sub division graph of G.*

Proof. Let $\tau_{\min} = \{T_1, T_2, \dots, T_r\}$ and $\tau' = \{T'_1, T'_2, \dots, T'_r\}$. Since $\tau_{\min}$ is of minimum cardinality, every cycle of G belongs to $\tau_{\min}$. Also, note that every path in $S(G)$ have subdivision vertices in alternate position.

Claim: If $M' : v_1, v_2, \dots, v_k$ is a shortest path in in $S(G)$, then M' is a subpath of some $T'_i \in \tau'$. Here three cases arise. In each case we will show the claim is true.

Case 1: Both v_1 and v_k are in G. The path got by deleting the subdivision vertices from M' will be a path connecting v_1 to v_k in G and will be a shortest path. Hence it will be part of some path/cycle, say T_i in $\tau_{\min}$. Clearly, M' will be part of $T'_i \in \tau'$.

Case 2: Either of v_1 or v_k is in G. Without loss of generality let us assume v_1 is in G and v_k is a subdivision vertex. Clearly, the path $v_1, v_3, \dots, v_{k-1}$ is a shortest path in G. Suppose $w \neq v_{k-1}$ is a neighbour of v_k. Since G has no odd cycles it is clear that v_{k-3} is not a neighbour of w. We claim that the path $M : v_1, v_3, \dots, v_{k-1}, w$ is a shortest path in G, which will prove the theorem for *Case* 2.

On the contrary let us assume that M is not the shortest path from v_1 to w. Then it is evident that some (atleast v_{k-3}, v_{k-1} and w) or all of the vertices in M are part of a cycle. Let us assume that the vertices $v, \dots, v_{k-1}$, w are part of a cycle. Then the paths $v \to v_{k-1}$ and $v \to w$ are of same length, a contradiction to the fact that G has no odd cycles. Hence our claim.

Case 3: Both v_1 and v_k are subdivision vertices. Hence $d(v_1) = d(v_k) = 2$. Let $u \neq v_2$ and $w \neq v_{k-1}$ be the neighbours of v_1 and v_k respectively. If $uv_2v_4 \dots w$ is a shortest path of G, we are done. If the edges uv_2 and $v_{k-1}w$ are not part of a cycle, there is nothing to prove. Since G contains cycles, some or all of the vertices of the path $uv_2v_4 \dots w$ may lie on same or different cycles. Hence there can be more than one path connecting u to w. Due to our assumption that M' is the shortest path and due to the fact that G contains only even cycles, the path $uv_2v_4 \dots w$ is a shortest path.
Hence the proof. □

Tadpole Graph

The tadpole graph $T_{m,n}$ is a special type of graph consisting of a cycle on $m (\geq 3)$ vertices and a path on n vertices, connected by a bridge, say e.

Corollary 4.2. *Let G denote the tadpole graph $T_{2m,n}$. Then the transit decomposition for $S(G), \tau_{S(G)} = \{T'_1, T'_2, T'_3\}$, where $T'_1, T'_2 \simeq P_{2m+2n+1}$ and $T'_3 \simeq C_{4m}$.*

Proof. The graph G contains no odd cycles. Hence if $\tau_{\min}$ is a transit decomposition of G with minimum cardinality, the transit decomposition for $S(G), \tau_{S(G)}$ is got by subdividing every edge of paths/cycles in $\tau_{\min}$.
Since C_{2m} is a cycle in G, $C_{2m} \in \tau_{\min}$. Let $e = uv$ be the bridge in G, with u as a vertex of the cycle C_{2m}. Let u' be vertex diametrically opposite to u. The paths connecting the pendant vertex of G to u' are of length $n+m$ and form majorized paths of G. Hence $\tau_{\min} = \{T_1, T_2, T_3\}$ where $T_1 \simeq C_{2m}$ and $T_1, T_2 \simeq P_{n+m+1}$. Thus, by Theorem 4.1, the result follows. □

Remark 4.3. Consider the tadpole graph $T_{2m+1,n}$. This graph has an odd cycle and hence the Theorem 4.1 does not hold good here. To find the transit index of its subdivision graph we form the transit decomposition, $\tau_{S(G)}$. It is evident that here $S(G) = T_{4m+2,2n}$. Therefore, $\tau_{S(G)} = \{T_1, T_2, T_3\}$, where $T_1 \simeq C_{4m+2}$, $T_2 \simeq T_3 \simeq P_{2m+2n+2}$. Also, note that $T_1 \cap T_2 \simeq T_1 \cap T_3 \simeq P_{2m+2}$ and $T_2 \cap T_3 \simeq P_{2n+1}$.

Proposition 4.4. *Let G be not a cycle and let τ be a transit decomposition of G. If τ' denotes the collection of all subdivision of elements of τ, τ' will be a transit decomposition of $S(G)$, the sub division graph of G, only if every edge of G is part of some majorized path in τ.*

Proof. Suppose on the contrary, let $e = uv$ be not a part of any majorized path in τ. Clearly, e belongs to some cycle, say C. Let the sub division of e be uwv. Let w' be any vertex of G that is not in C. Then the shortest path connecting w to w' in $S(G)$ will not be a sub path of any element in τ' which proves τ' is not a transit decomposition of $S(G)$ □

5. Transit Index of Subdivision Graphs

Theorem 5.1. *Let G be the graph got by subdividing every edge of the path P_n. Then $TI(G) = TI(P_n) + \frac{n(n-1)(15n^2-31n+14)}{12}$.*

Proof. Since G is got by subdividing every edge of P_n, $G \simeq P_{2n-1}$.
Hence by Theorem 2.3, we get $TI(G) - TI(P_n) = \frac{n(n-1)(15n^2-31n+14)}{12}$. □

Theorem 5.2. *Let G be the graph got by subdividing every edge of a cycle. Then*

$$TI(G) = \begin{cases} TI(C_n) + \frac{n(n-1)(5n^2+6n+1)}{8}, & n \text{ odd}, \\ TI(C_n) + \frac{n^2(5n^2-4)}{8}, & n \text{ even}. \end{cases}$$

Proof. For a cycle C_n its subdivision graph is the cycle C_{2n}. Now, using Theorem 2.4, the result follows. □

Theorem 5.3. *Let G be the graph got by the subdivision of a single edge $e = uv_1$ of the star graph $K_{1,n-1}$. Then $TI(G) = n^2 + 3n - 8 = TI(K_{1,n-1}) + 6n - 10$.*

Proof. In the graph G every vertex other then the central vertex u and the newly added vertex v have transit zero.
The shortest paths through u are the one's connecting v to other $(n-2)$ vertices of star and the ones connecting v_1 to the $(n-2)$ vertices of star, i.e.

$$T(u) = (n-1)(n-2) + 3(n-2) = TI(K_{1,n-1}) + 3(n-2).$$

The shortest path through v are those connecting v_1 to other $(n-2)$ vertices of star and connecting v_1 to u, i.e.

$$T(v) = 3(n-2) + 2.$$

Hence $TI(G) = n^2 + 3n - 8 = TI(K_{1,n-1}) + 6n - 10$. □

Theorem 5.4. *Let G be the graph got by the subdivision of every edge of the star graph $K_{1,n-1}$. Then $TI(G) = (n-1)(13n-24) = TI(K_{1,n-1}) + 2(n-1)(6n-11)$.*

Proof. In G let the pendant vertices be $v_1, v_2, \dots, v_{n-1}$, newly added vertices be $u_1, u_2, \dots, u_{n-1}$ and the center vertex be u.

$$T(v_i) = 0, \quad \text{for all } i.$$

The shortest paths through u are:

(a) connecting v_i to v_j of length 4,

(b) connecting v_i to u_j of length 3,

(c) connecting u_i to u_j of length 2.

Therefore, $T(u) = (n-1)(6n-12)$.

The shortest paths through u_i are:

(a) connecting v_i to v_j of length 4,

(b) connecting v_i to u_j of length 3,

(c) connecting v_i to u of length 2.

Therefore, $T(u_i) = 7n - 12$.

This gives

$$TI(G) = (n-1)(n-2) + 2(n-1)(6n-11)$$
$$= TI(K_{1,n-1}) + 2(n-1)(6n-11).$$ □

Theorem 5.5. *Let G be the bistar got by joining the apex vertex of two stars $K_{1,n}$ by an edge. If $S(G)$ denotes its subdivision graph, $TI(S(G)) = TI(G) + 124n^2 + 4n + 2$.*

Proof. Consider Figure 1, all vertices other than the vertices of the type u,v,w have transit zeroİt can be easily verified that $T(u) = 32n^2$, $T(v) = 32n^2 + 2n + 2$ and $T(w) = 18n + 2$. There are two vertices of type u and $2n$ vertices of the type w. This shows that $TI(S(G)) = 132n^2 + 6n + 2$.

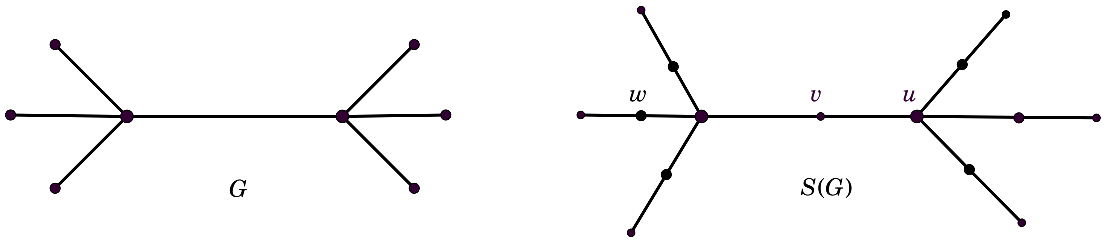

Figure 1. Bistar and its subdivision graph

Also, $TI(G)$ can be computed from the figure as $8n^2 + 2n$. Hence the result. □

Theorem 5.6. *Let G be the graph got by the subdividing a single edge $e = uv$ of the complete graph K_n. Then $TI(G) = 6n - 10$.*

Proof. Let the new vertex be w. After the subdivision the distance between u and v becomes 2. Also, the diameter of the graph is now 2.

All the $n-2$ vertices of K_n other than u and v are adjacent to each other and at a distance 2 from w. Hence $T(u) = T(v) = 2(n-2)$.

The only shortest path through w is the one connecting u to v. Hence $T(w) = 2$.

For the remaining $(n-2)$ vertices the shortest path through it is the one connecting u to v, of length 2. Therefore, $TI(G) = 6n - 10$. □

Theorem 5.7. *Let G be the graph got by sub dividing every edge of the complete graph K_n. Then $TI(G) = n(11n^2 - 40n + 37)$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of K_n, and $u_1, u_2, \dots, u_m$ be the sub division vertices, where $m = C(n,2)$. In G, $d(v_i) = n-1$ and $d(u_i) = 2$. Note that v_i's are transit identical and so are u_i's. We will calculate $T(v_i)$ and $T(u_i)$ separately and hence compute $TI(G)$.

1. *Computing* $T(v_i)$

Let us fix v_1 (refer Figure 2). It is adjacent to $n-1$ vertices of the type u_i. The shortest path connecting these $n-1$ vertices are of length 2 and passes through v_1. Hence contribute $C(n-1,2)\times 2=(n-1)(n-2)$ to the transit of v_1. The $(n-1)$ vertices of the type u_i adjacent to v_1 travels through v_1 to reach other $(n-2)$ vertices of type v_i, each of length 3. Hence add $(n-2)(n-1)\times 3$ to $T(v_1)$.

For u_1 there are $m-(2n-3)$ vertices of the type u_i at a distance 4 from it. For each such vertex u_i there are two paths passing through v_1 of length 4. Hence contribute $4\times 2\times(m-2n+3)$ to the transit of v.

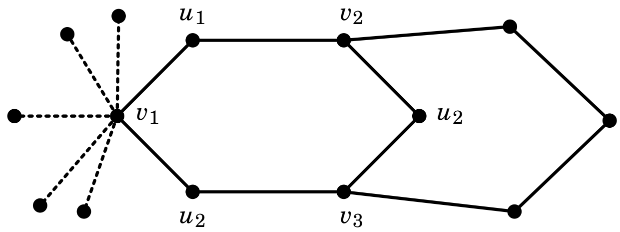

Figure 2. v_1 and adjacent vertices

Therefore

$$\begin{aligned}T(v)&=(n-1)(n-2)+(n-2)(n-1)\times 3+4\times 2\times(m-2n+3)\\&=8n^2-32(n-1).\end{aligned}$$

2. *Computing* $T(u_i)$

Now consider u_1 (refer Figure 2).

The path connecting v_1 to v_2 of length 2 passes through u_1.

All the $(n-2)$ vertices of the type u_i adjacent to v_2 passes through u_1 to reach v_1. All these paths are of length 3. Hence add $2\times 3\times(n-2)$ to transit of u_1.

Therefore

$$T(u_1)=6n-10.$$

Therefore

$$\begin{aligned}TI(G)&=\sum_i T(v_i)+\sum_i T(u_i)\\&=n(11n^2-40n+37).\end{aligned}$$

□

6. Conclusion

In computational graph theory, the operations on graphs played an important role. Subdivision is an important aspect in graph theory which allows one to calculate properties of some complicated graphs in terms of some easier graphs. In this paper we have found transit index of subdivision graphs for certain important graphs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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