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Research Article

Mixed Type Reverse Order Law for the Core Inverse in C^*-Algebras

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Abstract. In this paper, several equivalent conditions related to the reverse order law for the core inverse in C^*-algebras has been determined.

Keywords. Moore-Penrose inverse; Reverse order law; C^*-algebra; Core inverse

Mathematics Subject Classification (2020). 16W10; 15A09; 46L05

1. Introduction

The core inverse for a complex matrix were introduced by Baksalary and Trenkler [1]. Let $\mathscr{A} \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the ring of all $n \times n$ complex matrices. A matrix $X \in M_n(\mathbb{C})$ is called core inverse of A, if it satisfies $AX = P_A$ and $R(X) \subseteq R(A)$, where $R(A)$ denotes the column space of A, and P_A is the orthogonal projector onto $R(A)$, and if such a matrix exists, then it is unique and denoted by $A^{\#}$.

Many author have studied the necessary and sufficient conditions for the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ to hold in setting of matrices, operators, C^*-algebra. This formula cannot trivially be extended to the other generalized inverse of the product ab. Since the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ does not always holds, it is not easy to simplify various expressions that involve the core inverse of a product. In addition to $(ab)^{\#} = b^{\#}a^{\#}, (ab)^{\#}$ may be expressed as $(ab)^{\#} = b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}, (ab)^{\#} = b^*(a^*abb*)^{\#}a^*$, etc. These equalities are called mixed-type

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reverse order laws for the core inverse of a product and some of them are in fact equivalent (see [4], [15], [20]).

The reverse order law $(ab)^{\#} = b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}$ was first studied by Galperin and Waksman [8]. A Hilbert space version of their result was studied by Isumino [10]. Many results investigated the reverse order law $(ab)^{\#} = b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}$ for complex matrices appeared in Tian's papers [20] and [22], where the author used mostly properties of the rank of a complex matrices. In [15], a set of equivalent condition for this reverse order rule for the Moore-Penrose inverse in the setting of C^*-algebra is studied.

Xiong and Qin [24] concerning the following mixed-type reverse order laws for the Moore-Penrose inverse of a product of Hilbert space operators: $(ab)^{\#} = b^{\#}(abb^{\#})^{\#}$, $(ab)^{\#} = (a^{\#}ab)^{\#}a^{\#}$, $(ab)^{\#} = b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}$. They used the technique of block operator matrices. We extend the results from [24] to more general form.

2. Preliminaries

Definition 2.1 ([14]). An element a is Hermitian if $a^* = a$, and a is called an idempotent if $a^2 = a$. A Hermitian idempotent is said to be a projection.

Definition 2.2 ([1]). Let $A \in M_{n\times n}$. A matrix $A^{\#} \in M_{n\times n}$ satisfying: (i) $AA^{\#} = P_A$, and (ii) $R(A^{\#}) \subseteq R(A)$ is called core inverse of A.

Definition 2.3 ([1]). The core inverse of $a \in \mathscr{R}$ is the element $x \in \mathscr{R}$ which satisfies:

$$(1)\ axa = a \quad (2)\ xax = x \quad (3)\ (ax)^* = ax \quad (6)\ xa^2 = a \quad (7)\ ax^2 = x.$$

The element x is unique if it exist and is denoted by $a^{\#}$.

Definition 2.4 ([13]). Let $\mathscr{A}$ be unitial C^*-algebra. The element $a \in \mathscr{A}$ has the core inverse if there exists $x \in \mathscr{A}$ such that

$$axa = a,\ \ x\mathscr{A} = a\mathscr{A} \ \text{ and } \ \mathscr{A}x = \mathscr{A}a^*.$$

The unique core inverse will be denoted by $a^{\#}$.

Definition 2.5 ([14]). Let $\mathscr{R}$ be a unital C^*-algebra. An elements $a \in \mathscr{R}$ is regular if there exists some $x \in \mathscr{R}$ satisfying $axa = a$. The set all regular elements of $\mathscr{R}$ will be denoted by $\mathscr{R}^{\#}$.

Definition 2.6 ([14]). An elements a is said to be normal if $aa^{\#} = a^{\#}a$.

Definition 2.7 ([7]). An elements a is said to be invertible if $ab = ba = e$.

Theorem 2.8 ([14]). *For any $a \in \mathscr{R}^{\#}$, the following is satisfied:*

(i) $(a^{\#})^{\#} = a$;

(ii) $(a^*)^{\#} = (a^{\#})^*$;

(iii) $(a^*a)^{\#} = a^{\#}(a^{\#})^*$;

(iv) $(aa^*)^{\#} = (a^{\#})^*a^{\#}$;

(v) $a^* = a^{\#}aa^* = a^*aa^{\#}$;

(vi) $a^{\#} = (a^*a)^{\#}a^* = a^*(aa^*)^{\#}$;

(vii) $(a^*)^{\#} = a(a^*a)^{\#} = (aa^*)^{\#}a$.

Lemma 2.9. *If $a,b \in R$ such that a is regular, then $b \in a\{1,3,6,7\} \Longleftrightarrow a^*ab = a^*$.*

Proof. Let $b \in a\{1,3,6,7\}$, then we get

$$a^*ab = a^*(ab)^* = (aba)^* = a^*, \quad \text{(by using } (ax)^* = ax).$$

Conversely, the equality $a^*ab = a^*$ implies

$$a = aba = aab = a^2b = a$$
$$b = bab = abb = ab^2 = b$$
$$(ab)^* = b^*a^* = b^*a^*ab = (ab)^*ab \text{ is selfadjoint}$$

and

$$aba = (ab)^*a = (a^*ab)^* = (a^*)^* = a$$
$$ab^2 = abb = (abb)^* = (b)^*(ab)^* = bab = b$$
$$ba^2 = baa = (baa)^* = (a)^*(ba)^* = aba = a$$

Hence $b \in a\{1,3,6,7\}$. □

3. Reverse Order Laws $(a^{\#}ab)^{\#}a^{\#} = (ab)^{\#}, b^{\#}(abb^{\#})^{\#} = (ab)^{\#}$ and $b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} = (ab)^{\#}$ for Core Inverse

In this section, we have given the equivalent conditions related to reverse order laws $(a^{\#}ab)^{\#}a^{\#} = (ab)^{\#}, b^{\#}(abb^{\#})^{\#} = (ab)^{\#}$ and $b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} = (ab)^{\#}$ for core inverse in C^* algebra.

Theorem 3.1. *If $a,b,a^{\#}ab \in \mathscr{R}^{\#}$, then the following statements are equivalent:*

(1) $a^*ab\mathscr{R} \subseteq a^{\#}ab\mathscr{R}$;

(2) $(a^{\#}ab)^{\#}a^{\#} \in (ab)\{3,6,7\}$;

(3) $(a^{\#}ab)^{\#}a^{\#} = (ab)^{\#}$;

(4) $(a^{\#}ab)\{3,6,7\}a\{3,6,7\} \subseteq (ab)\{3,6,7\}$.

Proof. (2) $\Longrightarrow$ (1): Let $x = (a^{\#}ab)^{\#}a^{\#}$, Since $(a^{\#}ab)^{\#}a^{\#} \in (ab)\{3,6,7\}$, then

$$\begin{aligned}(ab)x(ab) &= ab((a^{\#}ab)^{\#}a^{\#})ab \\ &= abb^{\#}(a^{\#}aa^{\#})ab \qquad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\ &= abb^{\#}a^{\#}ab \\ &= ab(ab)^{\#}ab \\ &= ab\end{aligned}$$

$$\begin{aligned}(ab)x &= ab(a^{\#}ab)^{\#}a^{\#} \\ ((ab)x)^* &= (ab(a^{\#}ab)^{\#}a^{\#})^* \\ &= (abb^{\#}a^{\#}aa^{\#})^* \qquad (a^{\#}aa^{\#} = a^{\#})\end{aligned}$$

$$
\begin{aligned}
&= (abb^{\#}a^{\#})^* \\
&= (ab(ab)^{\#})^* \\
&= ab(ab)^{\#} \qquad (3.1)\\
&= abb^{\#}a^{\#} \\
&= abb^{\#}a^{\#}aa^{\#} \\
&= ab(a^{\#}ab)^{\#}a^{\#} \\
x(ab)^2 &= (a^{\#}ab)^{\#}a^{\#}(ab)^2 \\
&= (a^{\#}ab)^{\#}a^{\#}abab \\
&= b^{\#}(a^{\#}aa^{\#})abab \qquad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\
&= b^{\#}a^{\#}(ab)(ab) \\
&= (ab)^{\#}(ab)(ab) \\
&= ab \\
(ab)x^2 &= ab((a^{\#}ab)^{\#}a^{\#})^2 \\
&= ab(a^{\#}ab)^{\#}a^{\#}(a^{\#}ab)^{\#}a^{\#} \\
&= abb^{\#}a^{\#}aa^{\#}(a^{\#}ab)^{\#}a^{\#} \\
&= abb^{\#}a^{\#}(a^{\#}ab)^{\#}a^{\#} \\
&= ab(ab)^{\#}(a^{\#}ab)^{\#}a^{\#} \\
&= ab(ab)^{\#}b^{\#}a^{\#}aa^{\#} \\
&= ab(ab)^{\#}(ab)^{\#}aa^{\#} \\
&= (ab)^{\#}(ab)(ab)^{\#}aa^{\#} \qquad (\text{using definition for core invertible}) \\
&= (ab)^{\#}aa^{\#} \\
&= b^{\#}a^{\#}aa^{\#} \\
&= (a^{\#}ab)^{\#}a^{\#}
\end{aligned}
$$

which gives

$$
\begin{aligned}
a^*ab &= a^*(ab(a^{\#}ab)^{\#}a^{\#})ab \\
&= a^*(a^{\#})^*a^{\#}ab(a^{\#}ab)^{\#}a^*ab \quad (\text{using 3-inverse}) \\
&= (a^{\#}a)^*a^{\#}ab(a^{\#}ab)^{\#}a^*ab \\
&= a^{\#}(aa^{\#}a)b(a^{\#}ab)^{\#}a^*ab \quad (\text{since } aa^{\#}a = a) \\
&= a^{\#}ab(a^{\#}ab)^{\#}a^*ab.
\end{aligned}
$$

Therefore,

$$a^*ab\mathscr{R} = a^{\#}ab(a^{\#}ab)^{\#}a^*ab\mathscr{R} \subseteq a^{\#}ab\mathscr{R}.$$

(1) $\Longrightarrow$ (4): The assumption $a^*ab \subseteq a^{\#}ab\mathscr{R}$ implies that

$$a^*ab = a^{\#}abx, \quad \text{for some } x \in \mathscr{R}.$$

Now, for any $(a^{\#}ab)^{(1,3)} \in (a^{\#}ab)\{1,3\}$ and $a^{(1,3)} \in a\{1,3\}$.

Let $x = (a^{\#}ab)^{\#}$, $a = (a^{\#}ab)$, then now

$$\begin{aligned} a^*ab &= a^{\#}abx \\ &= (a^{\#}ab)(a^{\#}ab)^{(1,3)}(a^{\#}abx) \\ &= (a^{\#}ab)(a^{\#}ab)^{(1,3)}a^*ab. \end{aligned} \tag{3.2}$$

$$\begin{aligned} xa^2 &= (a^{\#}ab)^{\#}(a^{\#}ab)^2 \\ &= (a^{\#}ab)^{\#}(a^{\#}ab)(a^{\#}ab) \\ &= (a^{\#}ab)(a^{\#}ab)^{\#}(a^{\#}ab) \quad \text{(using definition for core invertible)} \\ &= (a^{\#}ab) \\ ax^2 &= (a^{\#}ab)((a^{\#}ab)^{\#})^2 \\ &= (a^{\#}ab)(a^{\#}ab)^{\#}(a^{\#}ab)^{\#} \\ &= (a^{\#}ab)^{\#}(a^{\#}ab)(a^{\#}ab)^{\#} \quad \text{(using definition for core invertible)} \\ &= (a^{\#}ab)^{\#} \end{aligned}$$

Applying the involution to (3.2), we obtain

$$\begin{aligned} (a^*ab)^* &= (a^{\#}ab(a^{\#}ab)^{(1,3)}a^*ab)^* \\ b^*a^*a &= b^*a^*a[(a^{\#}ab)(a^{\#}ab)^{(1,3)}]^* \\ &= b^*a^*aa^{\#}ab(a^{\#}ab)^{(1,3)} \quad (\text{since } aa^{\#}a = a) \\ b^*a^*a &= b^*a^*ab(a^{\#}ab)^{(1,3)}. \end{aligned} \tag{3.3}$$

Post multiplying the equality (3.3) by $a^{(1,3)}$, we get

$$\begin{aligned} b^*a^*aa^{(1,3)} &= b^*a^*ab(a^{\#}ab)^{(1,3)}a^{(1,3)} \\ b^*a^*(aa^{(1,3)})^* &= b^*a^*ab(a^{\#}ab)^{(1,3)}a^{(1,3)} \\ b^*(aa^{(1,3)}a)^* &= b^*a^*ab(a^{\#}ab)^{(1,3)}a^{(1,3)} \\ b^*a^* &= b^*a^*ab(a^{\#}ab)^{(1,3)}a^{(1,3)}. \end{aligned} \tag{3.4}$$

From the equality (3.4) and Lemma 2.9, we deduce that $(a^{\#}ab)^{(1,3)}a^{(1,3)} \in (ab)\{1,3\}$, for any $(a^{\#}ab)^{(1,3)} \in (a^{\#}ab)\{1,3\}$ and $a^{(1,3)} \in a\{1,3\}$. So, $(a^{\#}ab)^{(1,3)}.a^{(1,3)} \subseteq ab\{1,3\}$.

(4) $\Longrightarrow$ (2): Obviously, because $(a^{\#}ab)^{\#} \in (a^{\#}ab)\{1,3\}$ and $a^{\#} \in a\{1,3\}$.

(2) $\Longrightarrow$ (3): It is easy to check this equivalence. □

Theorem 3.2. *If $a, b, a^{\#}abb^{\#} \in \mathscr{R}^{\#}$, then the following statements are equivalent:*

(1) $a^*ab\mathscr{R} \subseteq a^{\#}ab\mathscr{R}$ *and* $bb^*a^*\mathscr{R} \subseteq bb^{\#}a^*\mathscr{R}$*;*

(2) $b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} \in (ab)\{3,6,7\}$*;*

(3) $b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} = (ab)^{\#}$*;*

(4) $b\{3,6,7\}.(a^{\#}abb^{\#})\{3,6,7\}.a\{3,6,7\} \subseteq (ab)\{3,6,7\}$.

Proof. (2) $\Longrightarrow$ (1): Let $x = b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}$.
Since $b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} \in (ab)\{3,6,7\}$ then

$$(ab)x(ab) = ab(b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#})ab$$

$$
\begin{aligned}
&= abb^{\#}bb^{\#}a^{\#}aa^{\#}ab \\
&= abb^{\#}a^{\#}ab \quad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\
&= ab(ab)^{\#}ab \\
&= ab \\
(ab)x &= abb^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} \\
((ab)x)^* &= (abb^{\#}(a^{\#}abb^{\#})^{\#}a^{\#})^* \\
&= (abb^{\#}bb^{\#}a^{\#}aa^{\#})^* \\
&= (abb^{\#}a^{\#})^* \quad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\
&= (ab(ab)^{\#})^* \qquad (3.5) \\
&= ab(ab)^{\#} \\
&= abb^{\#}a^{\#} \\
&= abb^{\#}bb^{\#}a^{\#}aa^{\#} \\
&= abb^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} \\
x(ab)^2 &= b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}(ab)^2 \\
&= b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}abab \\
&= (b^{\#}bb^{\#})(a^{\#}aa^{\#})(ab)(ab) \quad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\
&= b^{\#}a^{\#}(ab)(ab) \\
&= (ab)^{\#}(ab)(ab) \\
&= ab \\
(ab)x^2 &= ab(b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#})^2 \\
&= abb^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#} \\
&= ab(b^{\#}bb^{\#})(a^{\#}aa^{\#})(b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}) \quad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\
&= abb^{\#}a^{\#}(b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}) \\
&= (ab)(ab)^{\#}(b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}) \\
&= b^{\#}(a^{\#}abb^{\#})^{\#}a^{\#}
\end{aligned}
$$

which gives

$$
\begin{aligned}
a^*ab &= a^*(abb^{\#}(a^{\#}abb^{\#})^{\#}a^{\#})ab \\
&= a^*(a^{\#})^*a^{\#}abb^{\#}(a^{\#}abb^{\#})^{\#}a^*ab \quad (\text{using } (3.5)) \\
&= (aa^{\#})^*a^{\#}abb^{\#}(a^{\#}abb^{\#})^{\#}a^*ab \\
&= a^{\#}aa^{\#}abb^{\#}(a^{\#}abb^{\#})^{\#}a^*ab \quad (\text{since } a^{\#}aa^{\#} = a^{\#}) \\
&= a^{\#}abb^{\#}(a^{\#}abb^{\#})^{\#}a^*ab
\end{aligned}
$$

which yields $a^*ab\mathscr{R} \subseteq a^{\#}ab\mathscr{R}$.

(1) $\Longrightarrow$ (4): From $a^*ab\mathscr{R} \subseteq a^{\#}ab\mathscr{R}$, by $b\mathscr{R} = bb^{\#}\mathscr{R}$, we get $a^*abb^{\#}\mathscr{R} \subseteq a^{\#}abb^{\#}$. Thus, $a^*abb^{\#} = a^{\#}abb^{\#}x$, for some $x \in \mathscr{R}$. Then, for any $(a^{\#}abb^{\#})^{(1,3)} \in (a^{\#}abb^{\#})\{1,3\}, a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$.

Let $x=(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}, a=(a^{\textcircled{\#}}abb^{\textcircled{\#}})$. Then

$$\begin{aligned}
a^*abb^{\textcircled{\#}} &= a^{\textcircled{\#}}abb^{\textcircled{\#}}x\\
&= ((a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}(a^{\textcircled{\#}}abb^{\textcircled{\#}}))x\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}(a^{\textcircled{\#}}abb^{\textcircled{\#}}x)\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^*aba^{\textcircled{\#}}. \qquad (3.6)\\
xa^2 &= (a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^2\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}}) \quad \text{(using definition for core invertible)}\\
&= a^{\textcircled{\#}}abb^{\textcircled{\#}}\\
ax^2 &= (a^{\textcircled{\#}}abb^{\textcircled{\#}})((a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}})^2\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}} \quad \text{(using definition for core invertible)}\\
&= (a^{\textcircled{\#}}abb^{\textcircled{\#}})^{\textcircled{\#}}.
\end{aligned}$$

Applying the involution to (3.6), we get

$$\begin{aligned}
(a^*abb^{\textcircled{\#}})^* &= ((a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^*abb^{\textcircled{\#}})^*\\
(bb^{\textcircled{\#}})^*(a^*a)^* &= (bb^{\textcircled{\#}})^*(a^*a)^*((a^{\textcircled{\#}}abb^{\textcircled{\#}})(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)})^*\\
bb^{\textcircled{\#}}a^*a &= bb^{\textcircled{\#}}a^*aa^{\textcircled{\#}}abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)} \quad \text{(since } aa^{\textcircled{\#}}a=a\text{)}\\
&= bb^{\textcircled{\#}}a^*abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}. \qquad (3.7)
\end{aligned}$$

Permultiplying by b^* and postmultiplying by $a^{(1,3)}$ in (3.7), we get

$$\begin{aligned}
b^*bb^{\textcircled{\#}}a^*aa^{(1,3)} &= b^*bb^{\textcircled{\#}}a^*abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^{(1,3)}\\
b^*bb^{(1,3)}a^*aa^{(1,3)} &= b^*bb^{(1,3)}a^*abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^{(1,3)}\\
b^*(bb^{(1,3)})^*a^*(aa^{(1,3)})^* &= b^*(bb^{(1,3)})^*a^*abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^{(1,3)}\\
(bb^{(1,3)}b)^*(aa^{(1,3)}a)^* &= (bb^{(1,3)}b)^*a^*abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^{(1,3)}\\
b^*a^* &= b^*a^*abb^{\textcircled{\#}}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^{(1,3)}. \qquad (3.8)
\end{aligned}$$

By (3.8) and Lemma 2.9, we observe that $b^{(1,3)}(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)}a^{(1,3)} \in (ab)\{1,3\}$, for any $(a^{\textcircled{\#}}abb^{\textcircled{\#}})^{(1,3)} \in (a^{\textcircled{\#}}abb^{\textcircled{\#}})\{1,3\}.a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$. Hence, $b\{1,3\}.(a^{\textcircled{\#}}abb^{\textcircled{\#}})\{1,3\}.a\{1,3\} \subseteq (ab)\{1,3\}$.

(4) $\Longrightarrow$ (2) $\Longrightarrow$ (3): Obviously. □

4. Reverse Order Laws $(a^*ab)^{\textcircled{\#}}a^*=(ab)^{\textcircled{\#}}$, $b^*(abb^*)^{\textcircled{\#}}=(ab)^{\textcircled{\#}}$ and $b^*(a^*abb^*)^{\textcircled{\#}}a^*=(ab)^{\textcircled{\#}}$ for Core Inverse

In this section, we consider necessary and sufficient conditions for reverse order law $(a^*ab)^{\textcircled{\#}}a^* = (ab)^{\textcircled{\#}}$, $b^*(abb^*)^{\textcircled{\#}}=(ab)^{\textcircled{\#}}$ and $b^*(a^*abb^*)^{\textcircled{\#}}a^*=(ab)^{\textcircled{\#}}$ for core inverse in C^* algebra.

Theorem 4.1. *If $a,b,a^*ab \in \mathscr{R}^{\textcircled{\#}}$, then the following statements are equivalent:*

(1) $a^{\textcircled{\#}}ab\mathscr{R} \subseteq a^*ab\mathscr{R}$;

(2) $(a^*ab)^{\textcircled{\#}}a^* \in (ab)\{1,3\}$;

(3) $(a^*ab)^{\textcircled{\#}}a^*\{1,3\} = (ab)^{\textcircled{\#}}$;

(4) $(a^*ab)\{1,3\}.(a^{\textcircled{\#}})^*\{1,3\} \subseteq (ab)\{1,3\}$.

Proof. (2) $\Longrightarrow$ (1): Let $x = (a^*ab)^{\textcircled{\#}}a^*$.

Since $(a^*ab)^{\textcircled{\#}}a^* \in (ab)\{3,6,7\}$, we have

$$\begin{aligned}
(ab)x(ab) &= ab(a^*ab)^{\textcircled{\#}}a^*ab \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}(a^*)^{\textcircled{\#}}a^*ab \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^*ab \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}(aa^{\textcircled{\#}})^*ab \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}aa^{\textcircled{\#}}ab \quad (\text{since } aa^{\textcircled{\#}}a = a) \qquad (4.1)\\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}ab \\
&= ab(ab)^{\textcircled{\#}}ab \\
&= ab \\
(ab)x &= ab(a^*ab)^{\textcircled{\#}}a^* \\
((ab)x)^* &= (ab(a^*ab)^{\textcircled{\#}}a^*)^* \\
&= (abb^{\textcircled{\#}}a^{\textcircled{\#}}(a^*)^{\textcircled{\#}}a^*)^* \\
&= (abb^{\textcircled{\#}}a^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^*)^* \\
&= (abb^{\textcircled{\#}}a^{\textcircled{\#}}(aa^{\textcircled{\#}})^*)^* \\
&= (abb^{\textcircled{\#}}a^{\textcircled{\#}}aa^{\textcircled{\#}})^* \qquad (4.2)\\
&= (abb^{\textcircled{\#}}a^{\textcircled{\#}})^* \\
&= (ab(ab)^{\textcircled{\#}})^* \\
&= ab(ab)^{\textcircled{\#}} \qquad = abb^{\textcircled{\#}}a^{\textcircled{\#}} \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}aa^{\textcircled{\#}} \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}(aa^{\textcircled{\#}})^* \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^* \\
&= abb^{\textcircled{\#}}a^{\textcircled{\#}}(a^*)^{\textcircled{\#}}a^* \\
&= ab(a^*ab)^{\textcircled{\#}}a^* \\
x(ab)^2 &= (a^*ab)^{\textcircled{\#}}a^*(ab)^2 \\
&= (a^*ab)^{\textcircled{\#}}a^*(ab)(ab) \\
&= b^{\textcircled{\#}}a^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^*(ab)(ab) \\
&= b^{\textcircled{\#}}a^{\textcircled{\#}}(aa^{\textcircled{\#}})^*(ab)(ab) \\
&= b^{\textcircled{\#}}(a^{\textcircled{\#}}aa^{\textcircled{\#}})(ab)(ab) \quad (\text{since } a^{\textcircled{\#}}aa^{\textcircled{\#}} = a) \\
&= b^{\textcircled{\#}}a^{\textcircled{\#}}(ab)(ab) \\
&= (ab)^{\textcircled{\#}}(ab)(ab) \\
&= ab \qquad (4.3)\\
(ab)x^2 &= ab((a^*ab)^{\textcircled{\#}}a^*)^2
\end{aligned}$$

$$
\begin{aligned}
&= ab(a^*ab)^{\#}a^*((a^*ab)^{\#}a^*)\\
&= abb^{\#}a^{\#}(a^{\#})^*a^*(a^*ab)^{\#}a^*\\
&= abb^{\#}a^{\#}(aa^{\#})^*(a^*ab)^{\#}a^* \quad (\text{since } (ab)^* = ab) \qquad (4.4)\\
&= abb^{\#}a^{\#}aa^{\#}(a^*ab)^{\#}a^*\\
&= abb^{\#}a^{\#}(a^*ab)^{\#}a^*\\
&= ab(ab)^{\#}(a^*ab)^{\#}a^*\\
&= (a^*ab)^{\#}a^* \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
a^{\#}ab &= a^{\#}(ab(a^*ab)^{\#}a^*)ab\\
&= a^{\#}aa^*ab(a^*ab)^{\#}a^{\#}ab\\
&= a^*ab(a^*ab)^{\#}a^{\#}ab\,.
\end{aligned}
$$

Thus, the condition (3.1) is satisfied.

(1) $\Longrightarrow$ (4): First, by the inclusion $a^{\#}ab\mathscr{R} \subseteq a^*ab\mathscr{R}$, we conclude that $a^{\#}ab = a^*aby$, for some $y \in \mathscr{R}$. Further, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$ and $a' \in (a^{\#})^*\{1,3\}$.

Let $x = (a^*ab)^{\#}, a = (a^*ab)$, we get

$$
\begin{aligned}
a^{\#}ab &= a^*aby\\
&= (a^*ab)(a^*ab)^{(1,3)}(a^*aby)\\
&= a^*ab(a^*ab)^{(1,3)}a^{\#}ab \qquad (4.6)\\
xa^2 &= (a^*ab)^{\#}(a^*ab)^2\\
&= (a^*ab)^{\#}(a^*ab)(a^*ab)\\
&= (a^*ab)(a^*ab)^{\#}(a^*ab)\\
&= (a^*ab)\\
ax^2 &= (a^*ab)((a^*ab)^{\#})^2\\
&= (a^*ab)(a^*ab)^{\#}(a^*ab)^{\#}\\
&= (a^*ab)^{\#}(a^*ab)(a^*ab)^{\#}\\
&= (a^*ab)^{\#}\,.
\end{aligned}
$$

Applying the involution to (4.6), we get

$$
\begin{aligned}
(a^{\#}ab)^* &= ((a^*ab)(a^*ab)^{(1,3)}a^{\#}ab)^*\\
b^*a^*(a^{\#})^* &= b^*a^*(a^{\#})^*((a^*ab)(a^*ab)^{(1,3)})^*\\
b^*(a^{\#}a)^* &= b^*(a^{\#}a)(a^*ab)(a^*ab)^{(1,3)}\\
b^*a^{\#}a &= b^*a^{\#}aa^*ab(a^*ab)^{(1,3)}\\
&= b^*a^*ab(a^*ab)^{(1,3)}\,. \qquad (4.7)
\end{aligned}
$$

Postmultiply both side by a' in (4.7), we get

$$
\begin{aligned}
b^*a^{\#}aa' &= b^*a^*ab(a^*ab)^{(1,3)}a'\\
b^*a^*(a^{\#}a') &= b^*a^*ab(a^*ab)^{(1,3)}a'
\end{aligned}
$$

$$b^*a^*(a^{\#})^*[(a^{\#})^*]^{\#} = b^*a^*ab(a^*ab)^{(1,3)}a'$$
$$b^*a^*(a^{\#})^*(a)^* = b^*a^*ab(a^*ab)^{(1,3)}a'$$
$$b^*(aa^{\#}a)^* = b^*a^*ab(a^*ab)^{(1,3)}a'$$
$$b^*a^* = b^*a^*ab(a^*ab)^{(1,3)}a',$$

which implies, by Lemma 2.9, $(a^*ab)^{(1,3)}a' \in (ab)\{1,3\}$, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$ and $a' \in (a^{\#})^*\{1,3\}$, that is, the condition (4.6) holds.

(4) $\Longrightarrow$ (2): By Theorem 2.8, $a^* = [(a^{\#})^{\#}]^* = [(a^{\#})^*]^{\#} \in (a^{\#})^*\{1,3\}$ and this implication follows.

(2) $\Longrightarrow$ (3): Obviously. □

Theorem 4.2. *If $a,b,a^*abb^* \in \mathscr{R}^{\#}$, then the following statement are equivalent:*

(1) $a^{\#}ab\mathscr{R} \subseteq a^*ab\mathscr{R}$; $bb^{\#}a^*\mathscr{R} \subseteq bb^*a^*\mathscr{R}$;

(2) $b^*(a^*abb^*)^{\#}a^* \in (ab)\{3,6,7\}$;

(3) $b^*(a^*abb^*)^{\#}a^* = (ab)^{\#}$;

(4) $(b^{\#})^*\{1,3\}.(a^*abb^*)\{1,3\}.(a^{\#})^*\{1,3\} \subseteq (ab)\{1,3\}$.

Proof. (2) $\Longrightarrow$ (1): Let $x = b^*(a^*abb^*)^{\#}a^*$.
Since $b^*(a^*abb^*)^{\#}a^* \in (ab)\{3,6,7\}$. Then, now

$$\begin{aligned}
(ab)x(ab) &= ab(b^*(a^*abb^*)^{\#}a^*)ab \\
&= abb^*(b^*)^{\#}b^{\#}a^{\#}(a^*)^{\#}a^*ab \\
&= abb^*(b^{\#})^*b^{\#}a^{\#}(a^{\#})^*a^*ab \\
&= ab(b^{\#}b)^*b^{\#}a^{\#}(aa^{\#})^*ab \\
&= a(bb^{\#}b)b^{\#}a^{\#}(aa^{\#}a)b \\
&= abb^{\#}a^{\#}ab \quad (\text{since } aa^{\#} = a) \\
&= ab(ab)^{\#}ab \\
&= ab
\end{aligned}$$

$$\begin{aligned}
(ab)x &= abb^*(a^*abb^*)^{\#}a^* \\
((ab)x)^* &= (abb^*(a^*abb^*)^{\#}a^*)^* \\
&= (abb^*(b^*)^{\#}b^{\#}a^{\#}(a^*)^{\#}a^*)^* \\
&= (abb^*(b^{\#})^*b^{\#}a^{\#}(a^{\#})^*a^*)^* \\
&= (ab(b^{\#}b)^*b^{\#}a^{\#}(aa^{\#})^*)^* \\
&= (abb^{\#}bb^{\#}a^{\#}aa^{\#})^* \\
&= (abb^{\#}a^{\#})^* \\
&= (ab(ab)^{\#})^* \\
&= ab(ab)^{\#} \\
&= abb^{\#}a^{\#} \\
&= abb^{\#}bb^{\#}a^{\#}aa^{\#}
\end{aligned}$$

$$
\begin{aligned}
&= ab(b^{\circledast}b)^* b^{\circledast} a^{\circledast} (aa^{\circledast})^* \\
&= abb^*(b^{\circledast})^* b^{\circledast} a^{\circledast} (a^{\circledast})^* a^* \\
&= abb^*(b^*)^{\circledast} a^{\circledast} (a^*)^{\circledast} a^* \\
&= abb^*(a^*abb^*)^{\circledast} a^* \\
x(ab)^2 &= b^*(a^*abb^*)^{\circledast} a^*(ab)^2 \\
&= b^*(a^*abb^*)^{\circledast} a^*(ab)(ab) \\
&= b^*(b^{\circledast})^* b^{\circledast} a^{\circledast}(a^{\circledast})^* a^*(ab)(ab) \\
&= (b^{\circledast}b)^* b^{\circledast} a^{\circledast}(aa^{\circledast})^* abab \\
&= b^{\circledast}bb^{\circledast}a^{\circledast}aa^{\circledast}abab \\
&= b^{\circledast}a^{\circledast}(ab)(ab) \\
&= (ab)^{\circledast}(ab)(ab) \\
&= ab \\
(ab)x^2 &= ab(b^*(a^*abb^*)^{\circledast}a^*)^2 \\
&= abb^*(a^*abb^*)^{\circledast}a^*(b^*(a^*abb^*)^{\circledast}a^*) \\
&= abb^*(b^{\circledast})^*b^{\circledast}a^{\circledast}(a^{\circledast})^*a^*b^*(a^*abb^*)^{\circledast}a^{\circledast} \\
&= ab(b^{\circledast}b)^*b^{\circledast}a^{\circledast}(aa^{\circledast})^*b^*(a^*abb^*)^{\circledast}a^* \\
&= abb^{\circledast}bb^{\circledast}a^{\circledast}aa^{\circledast}b^*(a^*abb^*)^{\circledast}a^* \\
&= abb^{\circledast}a^{\circledast}b^*(a^*abb^*)^{\circledast}a^* \\
&= ab(ab)^{\circledast}b^*(a^*abb^*)^{\circledast}a^* \\
&= b^*(a^*abb^*)^{\circledast}a^*
\end{aligned}
$$

which gives

$$
\begin{aligned}
a^{\circledast}ab &= a^{\circledast}(abb^*(a^*abb^*)^{\circledast}a^*)ab \\
&= a^{\circledast}aa^*abb^*(a^*abb^*)^{\circledast}a^{\circledast}ab \\
&= a^*abb^*(a^*abb^*)^{\circledast}a^{\circledast}ab
\end{aligned}
$$

implying

$$a^{\circledast}ab\mathscr{R} \subseteq a^*ab\mathscr{R}.$$

(1) $\Longrightarrow$ (4): If $a^{\circledast}ab\mathscr{R} \subseteq a^*ab\mathscr{R}$, by $b\mathscr{R} = bb^*\mathscr{R}$, we see $a^{\circledast}abb^*\mathscr{R} \subseteq a^*abb^*\mathscr{R}$ and $a^{\circledast}abb^* = a^*abb^*y$, for some $y \in \mathscr{R}$. For any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}, a' \in (a^{\circledast})^*\{1,3\}$ and $b' \in (b^{\circledast})^*\{1,3\}$.

Let $x = (a^*abb^*)^{\circledast}$,

$$
\begin{aligned}
a^{\circledast}abb^* &= a^*abb^*(a^*abb^*)^{(1,3)}(a^*abb^*y) \\
&= a^*abb^*(a^*abb^*)^{(1,3)}a^{\circledast}abb^* \qquad (4.8) \\
xa^2 &= (a^*abb^*)^{\circledast}(a^*abb^*)^2 \\
&= (a^*abb^*)^{\circledast}(a^*abb^*)(a^*abb^*) \\
&= (a^*abb^*)(a^*abb^*)^{\circledast}(a^*abb^*) \\
&= a^*abb^*
\end{aligned}
$$

$$\begin{aligned} ax^2 &= a^*abb^*((a^*abb^*)^{\#})^2 \\ &= a^*abb^*(a^*abb^*)^{\#}(a^*abb^*)^{\#} \\ &= (a^*abb^*)^{\#}(a^*abb^*)(a^*abb^*)^{\#} \\ &= (a^*abb^*)^{\#}. \end{aligned}$$

Applying the involution to (4.8), we get

$$\begin{aligned} (a^{\#}abb^*)^* &= ((a^*abb^*)(a^*abb^*)^{(1,3)}a^{\#}abb^*)^* \\ (bb^*)^*(a^{\#}a)^* &= (bb^*)^*(a^{\#}a)^*((a^*abb^*)(a^*abb^*)^{(1,3)})^* \\ bb^*a^{\#}a &= bb^*a^{\#}aa^*abb^*(a^*abb^*)^{(1,3)} \quad (\text{since } (ax)^* = ax) \\ &= bb^*a^*abb^*(a^*abb^*)^{(1,3)}. \end{aligned} \tag{4.9}$$

Multiplying (4.9) from the left side by $b^{\#}$ and from the right side by a', we get

$$\begin{aligned} b^{\#}bb^*a^{\#}aa' &= b^{\#}bb^*a^*abb^*(a^*abb^*)^{(1,3)}a' \\ (b^{\#}b)^*b^*a^*(a^{\#}a') &= (b^{\#}b)^*b^*a^*abb^*(a^*abb^*)^{(1,3)}a' \\ (b)^*(b^{\#})^*b^*a^*(a^{\#})^*((a^{\#})^*)^{\#} &= (bb^{\#}b)^*a^*abb^*(a^*abb^*)^{(1,3)}a' \\ (bb^{\#}b)^*a^*(aa^{\#})^* &= b^*a^*abb^*(a^*abb^*)^{(1,3)}a' \\ (bb^{\#}b)^*(aa^{\#}a)^* &= b^*a^*abb^*(a^*abb^*)^{(1,3)}a' \\ b^*a^* &= b^*a^*abb'(a^*abb^*)^{(1,3)}a'. \end{aligned}$$

Thus, by Lemma 2.9, $b'(a^*abb^*)^{(1,3)} \in (ab)\{1,3\}$, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$, $a' \in (a^{\#})\{1,3\}$ and $b' \in (b^{\#})^*$, which is equivalent to $(b^{\#})^*\{1,3\}(a^*abb^*)\{1,3\} \in (a^{\#})^*\{1,3\} \subseteq (ab)\{1,3\}$.

(4) $\Longrightarrow$ (2) $\Longrightarrow$ (3): These part can be check easy. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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