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On Cesàro Sequence Space defined by an Orlicz Function

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Abstract. In this paper we provide a suitable generalization of the sequence space ces_M of Orlicz [6] by using a sequence of strictly positive real numbers and study various topological properties and inclusion relations which generalize several known results of Orlicz [6], Shiue [9], Sanhan and Suantai [8], and Leibowitz [3].

1. **Introduction**

Lindenstrauss and Tzafriri [4] used the idea of an Orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty$ for some $\rho > 0$. The space ℓ_M equipped with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}$$

is a BK space [1, p. 300] usually called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \le p < \infty$. We recall [1, 4] that an Orlicz function M is a function from $[0,\infty)$ to $[0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Note that an Orlicz function is always unbounded.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$. It is easy to see [2] that always $K > 2$. A simple example of an Orlicz function which satisfies the Δ_2-condition for all values of u is given by $M(u) = a|u|^\alpha (\alpha > 1)$, since $M(2u) = a2^\alpha |u|^\alpha = 2^\alpha M(u)$. The Orlicz function $M(u) = e^{|u|} - |u| - 1$ does not satisfy the Δ_2-condition.

The Δ_2-condition is equivalent to the inequality $M(lu) \le K(l)M(u)$ which holds for all values of u, where l can be any number greater than unity.

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An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x p(t)dt$$

where p known as the kernel of M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Let w and ℓ^0 denote the spaces of all scalar and real sequences, respectively. For $1 < p < \infty$, Shiue [9] introduced the Cesàro sequence space ces_p by

$$ces_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

and showed that it is a Banach space when equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}.$$

Some geometric properties of the Cesàro sequence space ces_p were studied by many authors. Sanhan and Suantai [8] introduced and studied a generalized Cesàro sequence space $ces(p)$, where $p = (p_n)$ is a bounded sequence of positive real numbers. For any Orlicz function M, Orlicz [6] introduced and studied the Cesàro sequence space ces_M.

In this paper we propose to extend ces_M to a more general space $ces(M, p)$ in the same manner as ℓ_1 was extended to $\ell(p)$ (Simons [10]). We study various algebraic and topological properties of this space. Certain inclusion relations between $ces(M, p)$ spaces have been established. Some information on multipliers for $ces(M, p)$ space has also been given. We also define composite space $ces(M^\nu, p)$ by using composite Orlicz function M^ν.

We now introduce the generalization of Cesàro sequence space using an Orlicz function.

Definition 1.1. Let M be an Orlicz function and $p = (p_n)$ be a bounded sequence of positive real numbers. We define the following sequence space

$$ces(M, p) = \left\{ x \in w : \sum_{n=1}^{\infty} \left[M \left(\frac{\frac{1}{n} \sum_{k=1}^{n} |x_k|}{\rho} \right) \right]^{p_n} < \infty \text{ for some } \rho > 0 \right\}.$$

Some well-known spaces are obtained by specializing M and p.

(i) If $M(x) = x, p_n = p (1 \leq p < \infty)$ for all n, then $ces(M, p) = ces_p$ (Shiue [9]).
(ii) If $M(x) = x$, then $ces(M, p) = ces(p)$ (Sanhan and Suantai [8]).
(iii) If $p_n = 1$ for all n, then $ces(M, p) = ces_M$ (Orlicz [6]) and $ces(M, p) = ces_\Phi$ for an Orlicz function Φ (Petrot and Suantai [7]).

The following inequalities (see, e.g., [5, p. 190]) are needed throughout the paper.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k p_k$, then for any complex a_k and b_k,

$$|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k}), \tag{1.1}$$

where $C = \max(1, 2^{H-1})$. Also for any complex λ,

$$|\lambda|^{p_k} \le \max(1, |\lambda|^H). \tag{1.2}$$

2. Linear Topological Structure of $ces(M,p)$ Space

In this section we establish some algebraic and topological properties of the sequence space defined above.

Theorem 2.1. *For any Orlicz function M, $ces(M,p)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x, y \in ces(M,p)$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some $\rho_3 > 0$ such that

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\alpha x_k + \beta y_k|}{\rho_3} \right) \right]^{p_n} < \infty .$$

Since $x, y \in ces(M,p)$, there exist a positive ρ_1 and ρ_2 such that

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |x_k|}{\rho_1} \right) \right]^{p_n} < \infty$$

and

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |y_k|}{\rho_2} \right) \right]^{p_n} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\alpha x_k + \beta y_k|}{\rho_3} \right) \right]^{p_n} \\
&\quad \le \sum_{n=1}^{\infty} \frac{1}{2^{p_n}} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |x_k|}{\rho_1} \right) + M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |y_k|}{\rho_2} \right) \right]^{p_n} \\
&\quad \le \max(1, 2^{H-1}) \left(\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |x_k|}{\rho_1} \right) \right]^{p_n} + \sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |y_k|}{\rho_2} \right) \right]^{p_n} \right)
\end{aligned}$$

so that $\alpha x + \beta y \in ces(M,p)$. This proves that $ces(M,p)$ is a linear space over $\mathbb{C}$. □

Theorem 2.2. *$ces(M,p)$ is a topological linear space, paranormed by*

$$g(x)=\inf\left\{\rho^{p_n/G}:\left(\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n}\right)^{\frac{1}{G}}\leq 1\right\}, \tag{2.1}$$

where $H=\sup p_n<\infty$ and $G=\max(1,H)$.

The proof follows by using standard techniques and the fact that every paranormed space is a topological linear space [11, p. 37].

Proposition 2.3 ([1, p. 300])**.** *We have, for x in ℓ_M, the inequality*

$$\sum_{i\geq 1}M\left(\frac{|x_i|}{\|x\|_{(M)}}\right)\leq 1,$$

where $\|x\|_{(M)}=\inf\left\{k>0:\sum_{i\geq 1}M\left(\frac{|x_i|}{k}\right)\leq 1\right\}$.

Theorem 2.4. *Let $1\leq p_n<\infty$, then $ces(M,p)$ is a Fréchet space paranormed by* (2.1).

Proof. In view of Theorem 2.2, it suffices to prove the completeness of $ces(M,p)$. Let $(x^{(s)})$ be a Cauchy sequence in $ces(M,p)$. Let r and x_0 be fixed. Then for each $\frac{\epsilon}{rx_0}>0$ there exists a positive integer N such that

$$g(x^{(s)}-x^{(t)})<\frac{\epsilon}{rx_0},\quad \text{for all } s,t\geq N.$$

Using (2.1) and Proposition 2.3, we get

$$\left(\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)}-x_k^{(t)}|}{g(x^{(s)}-x^{(t)})}\right)\right]^{p_n}\right)^{1/G}\leq 1.$$

Thus

$$\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)}-x_k^{(t)}|}{g(x^{(s)}-x^{(t)})}\right)\right]^{p_n}\leq 1.$$

Since $1\leq p_n<\infty$, it follows that $M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)}-x_k^{(t)}|}{g(x^{(s)}-x^{(t)})}\right)\leq 1$, for each $n\geq 1$.

We choose $r>0$ such that $\left(\frac{x_0}{2}\right)rp\left(\frac{x_0}{2}\right)\geq 1$, where p is the kernel associated with M. Hence,

$$M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)}-x_k^{(t)}|}{g(x^{(s)}-x^{(t)})}\right)\leq\left(\frac{x_0}{2}\right)rp\left(\frac{x_0}{2}\right)$$

for each $n \in \mathbb{N}$. Using the integral representation of Orlicz function M, we get

$$\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)} - x_k^{(t)}| \le \frac{rx_0}{2}g(x^{(s)} - x^{(t)}) < \frac{\epsilon}{2}, \quad \text{for all } s,t \ge N.$$

Hence for each fixed k, $(x_k^{(s)})$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, as $s \to \infty$, $x_k^{(s)} \to x_k$, say, for each k. For given $\epsilon > 0$, choose an integer $n_0 > 1$ such that $g(x^{(s)} - x^{(t)}) < \epsilon$ for all $s,t \ge n_0$ and a $\rho > 0$, such that $g(x^{(s)} - x^{(t)}) < \rho < \epsilon$. Since

$$\left(\sum_{n=1}^{m}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)} - x_k^{(t)}|}{\rho}\right)\right]^{p_n}\right)^{1/G} \le 1, \quad \text{for all } \ s,t \ge n_0.$$

Now, using continuity of M and taking $t \to \infty$ in the above inequality, we get

$$\left(\sum_{n=1}^{m}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k^{(s)} - x_k|}{\rho}\right)\right]^{p_n}\right)^{1/G} \le 1, \quad \text{for all } s \ge n_0.$$

Letting $m \to \infty$, we get $g(x^{(s)} - x) < \rho < \epsilon$ for all $s \ge n_0$. Thus $(x^{(s)})$ converges to x in the paranorm of $ces(M,p)$. Since $(x^{(s)}) \in ces(M,p)$ and M is continuous, it follows that $x \in ces(M,p)$. □

3. **Inclusion between** $ces(M,p)$ **Spaces**

We now investigate some inclusion relations between $ces(M,p)$ spaces.

Theorem 3.1. *If $p = (p_n)$ and $q = (q_n)$ are bounded sequences of positive real numbers with $0 < p_n \le q_n < \infty$ for each n, then for any Orlicz function M, $ces(M,p) \subseteq ces(M,q)$.*

Proof. Let $x \in ces(M,p)$. Then there exists some $\rho > 0$ such that

$$\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n} < \infty .$$

This implies that $M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right) \le 1$ for sufficiently large values of n, say $n \ge n_0$ for some fixed $n_0 \in \mathbb{N}$. Since M is non-decreasing and $p_n \le q_n$, we have

$$\sum_{n \ge n_0}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{q_n} \le \sum_{n \ge n_0}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n} < \infty.$$

This shows that $x \in ces(M,q)$ and completes the proof. □

Theorem 3.2. *If $r = (r_n)$ and $t = (t_n)$ are bounded sequences of positive real numbers with $0 < r_n,\ t_n < \infty$ and if $p_n = \min(r_n, t_n)$, $q_n = \max(r_n, t_n)$, then for any Orlicz function M, $ces(M,p) = ces(M,r) \cap ces(M,t)$ and $ces(M,q) = G$, where G is the subspace of w generated by $ces(M,r) \cup ces(M,t)$.*

Proof. It follows from Theorem 3.1 that $ces(M,p) \subseteq ces(M,r) \cap ces(M,t)$ and that $G \subseteq ces(M,q)$.

For any complex λ, $|\lambda|^{p_n} \leq \max(|\lambda|^{r_n}, |\lambda|^{t_n})$; thus $ces(M,r) \cap ces(M,t) \subseteq ces(M,p)$.

Let $A = \{n : r_n \geq t_n\}$ and $B = \{n : r_n < t_n\}$. If $x \in ces(M,q)$, we write

$$y_n = x_n \ (n \in A) \quad \text{and} \quad y_n = 0 \ (n \in B);$$

$$z_n = 0 \ (n \in A) \quad \text{and} \quad z_n = x_n \ (n \in B).$$

Then since $x \in ces(M,q)$, there exists some $\rho > 0$ such that

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |x_k|}{\rho} \right) \right]^{q_n} < \infty .$$

Now

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |y_k|}{\rho} \right) \right]^{r_n} = \sum_{n\in A} + \sum_{n\in B} = \sum_{n\in A} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |x_k|}{\rho} \right) \right]^{q_n} < \infty$$

and so $y \in ces(M,r) \subseteq G$. Similarly, $z \in ces(M,t) \subseteq G$. Thus, $x = y + z \in G$. We have proved that $ces(M,q) \subseteq G$, which gives the required result. □

Corollary 3.3. *The three conditions $ces(M,r) \subseteq ces(M,t)$, $ces(M,p) = ces(M,r)$ and $ces(M,t) = ces(M,q)$ are equivalent.*

Corollary 3.4. *$ces(M,r) = ces(M,t)$ if and only if $ces(M,p) = ces(M,q)$.*

Finally some information on multipliers for $ces(M,p)$ is given below. For any set E of sequences the space of multipliers of E, denoted by $S(E)$, is given by

$$S(E) = \{a \in w : ax \in E \text{ for all } x \in E\}.$$

Theorem 3.5. *For an Orlicz function M which satisfies the Δ_2-condition, we have $\ell_\infty \subset S(ces(M,p))$.*

Proof. Let $a = (a_k) \in \ell_\infty$, $T = \sup_k |a_k|$ and $x = (x_k) \in ces(M,p)$. Then

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |x_k|}{\rho} \right) \right]^{p_n} < \infty \quad \text{for some } \rho > 0 .$$

Since M satisfies the Δ_2-condition, there exists a constant K such that

$$\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |a_k x_k|}{\rho} \right) \right]^{p_n} \leq \sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |a_k|\,|x_k|}{\rho} \right) \right]^{p_n}$$

$$\leq \sum_{n=1}^{\infty}\left[M\left(\frac{(1+[T])\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n}$$

$$\leq (K(1+[T]))^H\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n} < \infty,$$

where $[T]$ denotes the integer part of T. Hence $a \in S(ces(M,p))$. □

4. Composite Space $ces(M^v,p)$ using Composite Orlicz Function M^v

Taking Orlicz function M^v instead of M in the space $ces(M,p)$, we can define the composite space $ces(M^v,p)$ as follows.

Definition 4.1. For a fixed natural number v, we define

$$ces(M^v,p) = \left\{x \in w : \sum_{n=1}^{\infty}\left[M^v\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n} < \infty \text{ for some } \rho > 0\right\}.$$

Theorem 4.2. *For any Orlicz function M and $v \in \mathbb{N}$,*

(i) *$ces(M^v,p) \subseteq ces(p)$ if there exists a constant $\alpha \geq 1$ such that $M(t) \geq \alpha t$ for all $t \geq 0$.*

(ii) *Suppose there exists a constant β, $0 < \beta \leq 1$ such that $M(t) \leq \beta t$ for all $t \geq 0$ and let $m, v \in \mathbb{N}$ be such that $m < v$, then $ces(p) \subseteq ces(M^m,p) \subseteq ces(M^v,p)$.*

Proof. (i) Since $M(t) \geq \alpha t$ for all $t \geq 0$ and M is non-decreasing and convex, we have $M^v(t) \geq \alpha^v t$ for each $v \in \mathbb{N}$. Let $x \in ces(M^v,p)$. Using (1.2), we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p_n} \leq \max(1,\rho^H)\max(1,\alpha^{-vH})\sum_{n=1}^{\infty}\left[M^v\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n}$$

and hence $x \in ces(p)$.

(ii) Since $M(t) \leq \beta t$ for all $t \geq 0$ and M is non-decreasing and convex, we have $M^m(t) \leq \beta^m t$ for each $m \in \mathbb{N}$. The first inclusion is easily proved by using (1.2). To prove the second inclusion, suppose that $v - m = r$ and let $x \in ces(M^m,p)$. Again, using (1.2), we have

$$\sum_{n=1}^{\infty}\left[M^v\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n} \leq \max(1,\beta^{rH})\sum_{n=1}^{\infty}\left[M^m\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)\right]^{p_n}$$

and hence $x \in ces(M^v,p)$. □

Example 4.3. The examples of functions satisfying the conditions given in Theorem 4.2(i),(ii) are $M_1(t) = e^t - 1 \geq t$ and $M_2(t) = \frac{t^2}{1+t} \leq t$ for all $t \geq 0$, respectively.

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