Communications in Mathematics and Applications
Vol. 12, No. 4, pp. 803–813, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i4.1538

Research Article

Fixed Points for the *G*-Contraction on *E*-*b*-Metric Spaces with a Graph

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Received: March 16, 2021 **Revised:** July 12, 2021 **Accepted:** July 20, 2021

Abstract. In this paper, we study fixed points for the G-contraction on E-b-Metric spaces endowed with a graph. The work in this paper should be seen as a generalization of [7].

Keywords. Fixed points; E-b-metric space; Graph; Banach G-contraction

Mathematics Subject Classification (2020). 47H09; 47H10

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1. Introduction

Banach contraction principle proved by Polish Mathematician Stefan Banach in his thesis is a very easy tool to find the fixed point of a mapping in the theory of metric spaces. According to this principle, a mapping which is contractive in nature and defined on a complete metric space to itself has a unique fixed point. Kannan [9] also studied some results on fixed points. Afterward, most of the Mathematicians worked on it and generalize this principle in different metric spaces. Bakhtin [1] came out with the definition of b-metric space in 1989. Cevik and Altun [2] introduced the definition of vector metric space in 2009. Petre [4] developed the definition of the vector b-metric space in 2014. From the literature survey of the last ten years, it is observed that fixed point theory was combined with Graph theory.

Fixed point theory with a graph theory was firstly studied by Kirk and Espinola [8] in 2006. After that Jachymski [7] in 2008, studied the Banach contraction principle and generalized

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it to mappings on a metric space endowed with a graph. Samreen et al. [10] proved some fixed point theorems in b-metric space endowed with graph. In this paper, we generalize most of the results of Jachymski [7].

2. Preliminaries and Definitions

Now, we will give some basic definitions and results in the field of fixed point theory and graph theory before going to main results.

Definition 2.1 ([2]). A partially ordered linear space is a quadruple $(E,+,\cdot,\leq_E)$ where $(E,+,\cdot)$ is a linear space over the field R of Real numbers and $\leq_E$ is a partial ordering on E such that

(i) $u \leq_E v \Rightarrow \lambda u \leq_E \lambda v$, where λ is non-negative real number.

(ii) $u \leq_E v \Rightarrow u+w \leq_E v+w$, for every $u,v,w \in E$.

Definition 2.2 ([2]). A Riesz space, or vector lattice, is a partially ordered linear space $(E,+,\cdot,\leq_E)$ such that $(E,\leq_E)$is a lattice.

Notation 2.3 ([2]). If $\{u_n\}$ is a decreasing sequence in Riesz space whose $\inf\{u_n\} = u$, then we say sequence $\{u_n\}$ is directed downwards and we use the symbol $u_n \downarrow u$.

Definition 2.4 ([6]). A Riesz space E is said to be Archimedean if it is true for every $u \in E_+$ that the decreasing sequence $\{\frac{1}{n}u\}$ satisfies $\frac{1}{n}u \downarrow 0$, where $E_+ = \{u \in E : u \geq_E 0\}$.

Lemma 2.5 ([9]). *If $u \leq_E ku$ in a Riesz space E, where $u \in E_+$ and $k \in [0,1)$ then $u = 0$.*

Example 2.6 ([6]). R^2 is an Archimedean Riesz space with coordinate wise ordering but with lexicographical ordering, it is non-Archimedean Riesz space.

Definition 2.7 ([6]). Suppose $Z \neq \phi$ and E is a Riesz space. A function $d : Z \times Z \to E_+$ is said to be E-b-metric if, for any $u_1,u_2,u_3 \in Z$ and $s \geq 1$ any real number, the following conditions are satisfied:

(i) $d(u_1,u_2) \leq_E s[d(u_1,u_3)+d(u_3,u_2)]$.

(ii) $d(u_1,u_2) = 0$ iff $u_1 = u_2$.

The triplet (Z,d,E) is said to be E-b-metric space or vector-b-metric space.

Definition 2.8 ([6]). Suppose $Z = C[-2,2] = E$ and $d : C[-2,2] \times C[-2,2] \to E_+$ be defined as

$$d(f,g) = |f-g|^p, \quad p > 1.$$

Then (Z,d,E) is E-b-metric space with $s = 2^{p-1} > 1$. Since the function $|u|^p$ $(p > 1)$is convex, we have

$$\left(\left|\frac{1}{2}u+\frac{1}{2}v\right|\right)^p \leq \frac{1}{2}|u|^p + \frac{1}{2}|v|^p$$

so that

$$(|u+v|)^p \leq 2^{p-1}(|u|^p + |v|^p).$$

Therefore

$$\begin{aligned} d(f_1,f_3) &= (|f_1-f_3|)^p \\ &= (|f_1-f_2+f_2-f_3|)^p \\ &\le 2^{p-1}[|f_1-f_2|^p+|f_2-f_3|^p] \\ &= 2^{p-1}[d(f_1,f_2)+d(f_2,f_3)]. \end{aligned}$$

Thus condition (i) of Definition 2.7 holds with $s=2^{p-1}>1$.

Example 2.9 ([6]). Suppose $Z=L^p[0,1]$ where $p\in(0,1)$ and $E=R^2$ and $d:L^p[0,1]\times L^p[0,1]\to R^2_+$ such that $d(f,g)=(\alpha\|f-g\|_p,\beta\|f-g\|_p)$ where $\alpha,\beta\ge 0$ and $\alpha+\beta>0$. (Z,d,R^2) is an E-b-metric space with $s=2^{1/p}>1$.

We also know that $L^p[0,1]$ is Riesz space with partial order $f\ge g$ iff $f(u)\ge g(u)$ for all $u\in[0,1]$.

Example 2.10. Let $Z=\{0,1,2\}$, $E=R^2$ and $d:Z\times Z\to R^2$ be defined as

$$d(1,2)=d(2,1)=(2,2),$$
$$d(0,1)=d(1,0)=(2,2),$$
$$d(0,2)=d(2,0)=(8,8).$$

Since $d(2,0)=(8,8)\le d(2,1)+d(1,0)$. So (Z,d,E) is not a metric space but it is E-b-metric space with $s\ge 2$.

Definition 2.11 ([2]). A sequence $\langle u_n\rangle$ in (Z,d,E) is said to be E-converge to some $u\in Z$, written as $u_n\xrightarrow{d,E}u$, if there exists a sequence $\langle a_n\rangle$ in E such that $a_n\downarrow 0$ and $d(u_n,u)\le_E a_n$ for all n.

Definition 2.12 ([2]). A sequence $\langle u_n\rangle$ in (Z,d,E) is said to be E-Cauchy if there exists a sequence $\langle a_n\rangle$ in E such that $a_n\downarrow 0$ and $d(u_n,u_{n+p})\le_E a_n$ for all n and p.

Definition 2.13 ([7]). Two sequences $\{u_n\}_{n\in N}$ and $\{v_n\}_{n\in N}$ in Riesz space E are said to be E-Cauchy Equivalent if both sequences are E-Cauchy sequence and there is a sequence $\{a_n\}$ in E such that $a_n\downarrow 0$ and $d(u_n,v_n)\le_E a_n$ for all n.

Definition 2.14 ([7]). Let T be a mapping on an E-b-metric space (Z,d,E) to itself. T is said to be a Picard operator (P.O.) if T has a unique fixed point u^* and there exists a sequence a_n in E such that $a_n\downarrow 0$ and $d(T^nu,u^*)\le a_n$ for all n.

Definition 2.15 ([7]). Let T be a mapping on E-b-metric space (Z,d,E) to itself. T is said to be a weakly Picard operator if for every $u\in Z$, sequence T^nu E-converges to fixed point of T and this fixed point need not be unique.

2.1 Preliminaries of Graph Theory

Notation 2.16. We use the symbol $E'(G)$ which means set of edges of graph G.

Suppose (Z,d,E) is an E-b-metric space and $\Delta = \{(u,u) : u \in Z\}$. A graph G is a pair (V,E') where $E' = E'(G)$, the set of its edges such that $\Delta \subset E'(G)$ and $V = V(G)$ is a set of vertices coinciding with Z. Assume the graph has no parallel edges.

The graph G^{-1} is obtained by reversing the direction of edges where it's set of edges and vertices are defined as follows:

$$V(G^{-1}) = V(G) \text{ and } E'(G^{-1}) = \{(u,v) \in Z \times Z : (v,u) \in E'(G)\}.$$

Consider the graph $\tilde{G}$ consisting all the edges of G and G^{-1} and all vertices of G i.e.

$$E'(\tilde{G}) = E'(G) \cup E'(G^{-1}) \text{ and } V(\tilde{G}) = V(G).$$

Definition 2.17. A subgraph $G'(U,F)$ of a graph $G(V,E)$ is a graph in which $U \subseteq V$ and F contains all those edges of E which are associated with vertices of U.

Definition 2.18. Suppose G is a graph. A path in graph G from vertex u to vertex v of length $n\{n \in \mathrm{N} \cup \{0\}\}$ is a sequence $(u_i)_{i=0}^{n}$ of $n+1$ distinct vertices such that $u_0 = u$, $u_n = v$ and $(u_i, u_{i+1}) \in E'(G)$ for $i = 1,2,\dots,n$.

Definition 2.19. Suppose G is a graph. We say that graph G is connected if there is a path between any two vertices of G and it is weakly connected if $\tilde{G}$ is connected.

Notion 2.20. Consider any vertex u in a graph G. We denote the component of G containing u by G_u which is a subgraph of G. The subgraph G_u contained all those vertices and edges of G which involves in any path in G starting at u. We can observe vertices of G_u as an equivalence class $[u]_G$ defined on $V(G)$ by the relation R which is defined as vertex u is related to vertex v if there is a path from u to v. By this relation R, we can see easily, $V(G_u) = [u]_G$. It is very easy exercise to check the subgraph G_u is connected.

Definition 2.21 ([7])**.** Let (Z,d) be a metric space with graph G. A mapping $f : Z \to Z$ is said to be a Banach G-contraction if

(i) for all $u,v \in Z$

$$(u,v) \in E'(G) \Rightarrow (fu,fv) \in E'(G).$$

(ii) there exists $\alpha \in [0,1)$, for all $u,v \in Z$

$$(u,v) \in E'(G) \Rightarrow d(fu,fv) \le \alpha d(u,v).$$

Remark. We need not to define Banach-G contraction in E-b-metric space with graph separately because in condition (ii) of Definition 2.21 less than or equal to ($\le$) symbol can be replaced by general partial order relation ($\le_E$).

Definition 2.22 ([7])**.** Let (Z,d) be a metric space with graph G. The function $f : Z \to Z$ is said to be orbitally G-continuous if for all $u,v \in Z$ and any sequence $\{k_n\}$, $n \in \mathrm{N}$ of positive integers, $f^{k_n}u \longrightarrow v$ and $(f^{k_n}u, f^{k_{n+1}}u) \in E'(G)$ imply that $f(f^{k_n}u) \longrightarrow fv$ for $n \in \mathrm{N}$.

Proposition 2.23. *Let (Z,d,E) is an E-b metric space with constant $s \geq 1$ and graph G. If $f : Z \to Z$ is a G-contraction with $\alpha \in [0,1)$ then f is both a $\tilde{G}$-contraction and G^{-1}-contraction.*

Proof. Since f is G-contraction. Therefore, for all $u,v \in Z$,

$$(u,v) \in E'(G) \Rightarrow (fu,fv) \in E'(G)$$

and $\alpha \in [0,1)$, for all $u,v \in Z$.

$$(u,v) \in E'(G) \Rightarrow d(fu,fv) \leq_E \alpha d(u,v).$$

Now

$$(u,v) \in E'(G) \Rightarrow (v,u) \in E'(G^{-1}).$$

Thus if

$$(fu,fv) \in E'(G) \Rightarrow (fv,fu) \in E'(G^{-1}).$$

Since d is E-b-metric.

We have

$$(v,u) \in E'(G^{-1}) \Rightarrow d(fv,fu) \leq_E \alpha d(v,u).$$

Thus f is G^{-1}-contraction in E-b-metric space Z.
Similarly, we can show f is $\tilde{G}$-contraction. □

3. Main Results

Let $\mathrm{Fix} f = \{u \in Z : f(u) = u\}$. Throughout this section, G is considered as a directed graph.

Lemma 3.1. *Let (Z,d,E) be an E-b-metric space with constant $s \geq 1$ and graph G. Let $f : Z \to Z$ be a G-contraction with constant α. Therefore, given $u \in Z$ and $v \in [u]_{\tilde{G}}$, there exists an element $C(u,v) \in E_+$ satisfying*

$$d(f^n u, f^n v) \leq_E \alpha^n C(u,v), \quad \textit{for all } n \in \mathrm{N}.$$

Proof. Suppose $u \in Z$ and $v \in [u]_{\tilde{G}}$. Then, there is a finite sequence $(u_i)_{i=0}^N$ in $\tilde{G}$ from u to v i.e. $u_0 = u$, $u_N = v$ and $(u_{i-1},u_i) \in E'(\tilde{G})$ for $i = 1,2,\dots,N$. By Proposition 2.23, f is a $\tilde{G}$-contraction.

$$\Rightarrow \quad (f^n u_{i-1}, f^n u_i) \in E'(\tilde{G}) \text{ and } d(fu_{i-1},fu_i) \leq_E \alpha d(u_{i-1},u_i).$$

An easy induction shows

$$d(f^2 u_{i-1}, f^2 u_i) \leq_E \alpha d(fu_{i-1},fu_i) \leq_E \alpha^2 d(u_{i-1},u_i),$$

$$d(f^n u_{i-1}, f^n u_i) \leq_E \alpha^n d(u_{i-1},u_i), \quad \text{for all } n \in N \text{ and } i = 1,2,\dots,N.$$

Now, we use the triangular inequality of E-b-metric, we get

$$\begin{aligned} d(f^n u, f^n v) &\leq_E s[d(f^n u, f^n u_1) + d(f^n u_1, f^n v)] \\ &= sd(f^n u, f^n u_1) + sd(f^n u_1, f^n v). \end{aligned}$$

By repeated use of Triangular inequality

$$\Rightarrow \quad d(f^n u, f^n v) \leq_E sd(f^n u, f^n u_1) + s^2 d(f^n u_1, f^n u_2) + s^3 d(f^n u_2, f^n u_3) + \cdots$$

$$+s^{N-1}d(f^n u_{N-2}, f^n u_{N-1}) + s^N d(f^n u_N, f^n v)$$
$$\leq_E \alpha^n \left(\sum_{i=1}^{N} s^i d(u_{i-1}, u_i)\right).$$

So, it is suffices to set

$$C(u,v) = \sum_{i=1}^{N} s^i d(u_{i-1}, u_i). \qquad \square$$

Theorem 3.2. *If (Z,d,E) is an Archimedean, E-b-metric space with constant $s \geq 1$ and directed weakly connected graph G then for any G-contraction $f : Z \to Z$ with constant α and $\alpha s \in [0,1)$, given $u,v \in Z$, $(f^n u)_n$ and $(f^n v)_n$ are E-Cauchy equivalent.*

Proof. Suppose $u \in Z$, then $[u]_{\tilde{G}} = Z$ so $fu \in [u]_{\tilde{G}}$.
Now by Lemma 3.1, we get $d(f^n u, f^{n+1}u) \leq_E \alpha^n C(u, fu)$.
Since$v = fu$, we obtain

$$d(f^n u, f^{n+1}u) \leq_E \alpha^n C(u, fu), \quad n \geq 1, \text{ for all } n \in N. \tag{3.1}$$

Now, we will claim that $\{f^n u\}_n$is an E-Cauchy sequence. For this consider $m > n$, using the triangular inequality, we get

$$d(f^n u, f^m u) \leq_E sd(f^n u, f^{n+1}u) + s^2 d(f^{n+1}u, f^{n+2}u) + s^3 d(f^{n+2}u, f^{n+3}u) + \dots$$
$$+ s^{m-n} d(f^{m-1}u, f^m u)$$

From the use of (3.1)

$$\leq_E [s\alpha^n + s^2\alpha^{n+1} + \dots + s^{m-n}\alpha^{m-1}]C(u, fu)$$
$$\leq_E s\alpha^n [1 + s\alpha + s^2\alpha^2 + \dots + s^{m-n-1}\alpha^{m-n-1}]C(u, fu)$$
$$\leq_E \frac{s\alpha^n}{1 - s\alpha} C(u, fu).$$

Since $\alpha s \in [0,1)$, $s \geq 1$ and $\alpha \in [0,1)$

$$\Rightarrow \quad d(f^n u, f^m u) \leq_E \frac{\alpha^n s}{1 - s\alpha} C(u, fu) \downarrow 0$$

$\Rightarrow$ $\{f^n u\}$ is E-Cauchy sequence.

Since $v \in [u]_{\tilde{G}}$ and use of above lemma yields $d(f^n u, f^n v) \leq \alpha^n C(u,v)$. Hence $\{f^n u\}$ and $\{f^n v\}$ are E-Cauchy equivalent. $\square$

Theorem 3.3. *Let (Z,d,E) be an E-b-metric space endowed with constant $s \geq 1$ and a directed graph G. If $f : Z \to Z$ is G-contraction with constant α and $\alpha s \in [0,1)$, given $u,v \in Z$, $\{f^n u\}$ and $\{f^n v\}$ are E-Cauchy equivalent, then Card*$(\mathrm{Fix} f) \leq 1$.

Proof. Let f be a G-contraction and we assume that $u,v \in \mathrm{Fix} f$.
$\Rightarrow$ $fu = u$ and $fv = v$
$\Rightarrow$ $f^n u = u$ and $f^n v = v$

Since $\{f^n u\}$ and $\{f^n v\}$ are E-Cauchy equivalent, so by definition of E-Cauchy equivalent there exists $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(f^n u, f^n v) \le a_n$ for all n.

$$\Rightarrow \quad d(u,v) \le a_n \downarrow 0, \quad \text{for all } n$$

$$\Rightarrow \quad u = v.$$

□

Corollary 3.4. *Let (Z,d,E) be Archimedean, complete E-b-metric space endowed with constant $s \ge 1$ and a graph G. If G is weakly connected, then for any G-contraction $f: Z \to Z$ with constant α and $\alpha s \in [0,1)$, there exists $u^* \in Z$ such that $f^n u \xrightarrow{d,E} u^*$ for all $u \in Z$ and $(f^n u, u^*) \in E'(\tilde{G})$.*

Proof. Result of this corollary implies from Theorem 3.2 and Theorem 3.3. □

Proposition 3.5. *Suppose that $f:(Z,d,E) \to (Z,d,E)$ is a G-contraction with constant α and $\alpha s \in [0,1)$ such that for some $u_0 \in Z$, $f(u_0) \in [u_0]_{\tilde{G}}$. Then $[u_0]_{\tilde{G}}$ is f-invariant and $f/[u_0]_{\tilde{G}}$ is a $\tilde{G}_{u_0}$-contraction. Furthermore, if $u,v \in [u_0]_{\tilde{G}}$, then $(f^n u)_{n\in N}$ and $(f^n v)_{n\in N}$ are E-Cauchy equivalent.*

Proof. Let $u \in [u_0]_{\tilde{G}}$. Then there is a path $(u_i)_{i=0}^N$ in $\tilde{G}$ from u_0 to u, i.e. $u_N = u$ and $(u_{i-1}, u_i) \in E'(\tilde{G})$ for $i = 1,2,\dots,N$.
Because f is G-contraction, so f is also $\tilde{G}$-contraction which yields $(fu_{i-1}, fu_i) \in E'(\tilde{G})$ for $i = 1,2,\dots,N$ i.e. $(fu_i)_{i=0}^N$ is path in $\tilde{G}$ from fu_0 to fu.
Thus $f(u) \in [f(u_0)]_{\tilde{G}}$.
Since by hypothesis, $fu_0 \in [u_0]_{\tilde{G}}$ i.e. $[f(u_0)]_{\tilde{G}} = [u_0]_{\tilde{G}}$.

$$\Rightarrow \quad f(u) = [u_0]_{\tilde{G}}.$$

Thus $[u_0]_{\tilde{G}}$ is f-invariant.

Now, let $(u,v) \in E'(\tilde{G}_{u_o})$. This means that there is path $(u_i)_{i=0}^N$ in $\tilde{G}$ from u_0 to v such that $u_{N-1} = u$.
Since $f(u_0) \in [u_0]_{\tilde{G}}$, let $(v_i)_{i=0}^M$ be a path in $\tilde{G}$ from u_0 to fu_0. Repeating the argument from the last part of the proof, we infer $(v_0, v_1, \dots, v_M, fu_1, \dots, fu_N)$ is a path in $\tilde{G}$ from u_0 to fv, in particular $(fu_{N-1}, fu_N) \in E'(\tilde{G}_{u_0})$ i.e. $(fu, fv) \in E'(\tilde{G}_{u_0})$. Moreover, since $E'(\tilde{G}_{u_0}) \subseteq E'(\tilde{G})$ and f is $\tilde{G}$-contraction. We conclude that contraction condition for $\tilde{G}_{u_0}$ holds. Thus $f[u_0]_{\tilde{G}}$ is a $\tilde{G}_{u_0}$-E-contraction. Last statement of theorem is directly implied from Theorem 3.2. □

Theorem 3.6. *Let (Z,d,E) be an Archimedean, complete E-b-metric space endowed with graph with constant $s \ge 1$. Suppose G satisfying the following property*
P. *If $(u_n)_{n\in N}$ is any sequence in Z where $u_n \xrightarrow{d,E} u$ and there is edge between consecutive terms of sequence i.e. $(u_n, u_{n+1}) \in E'(G)$ for any $n \in \mathrm{N}$ then there is a subsequence $(u_{k_n})_{n\in N}$ with $(u_{k_n}, u) \in E'(G)$.*
Let $Z_f = \{u \in Z : (u, fu) \in E'(G)\}$ and $f: Z \to Z$ be a G-contraction with constant α and $\alpha s \in [0,1)$. Then

(1) *For all $u \in Z_f$, $f/[u]_{\tilde{G}}$ is a Picard operator.*

(2) *If G is weakly connected graph and $Z_f \neq \emptyset$, then f is Picard operator on Z.*

(3) *If* $Z' = \cup\{[u]_{\tilde{G}} : x \in Z_f\}$ *then* f/Z' *is Weakly Picard operator on* Z.

(4) *If* $f \subseteq E'(G)$ *i.e. there is an edge between every element of* Z *and its image, then* f *is Weakly Picard operator on* Z.

(5) $Card(\text{Fix} f) = Card\{[u]_{\tilde{G}} : u \in Z_f\}$.

(6) $\text{Fix} f \neq \emptyset \Leftrightarrow Z_f \neq \emptyset$.

(7) f *has a unique fixed point iff* $\exists\ u_0 \in Z_f$ *such that* $Z_f \subseteq [u_0]_{\tilde{G}}$.

Proof. (1) Firstly, we will claim that there exists $u^* \in [u]_{\tilde{G}}$ such that $f^n u \xrightarrow{d,E} u^*$ and u^* is fixed point of f i.e. $fu^* = u^*$.

Let $u \in Z_f$, then $fu \in [u]_{\tilde{G}}$. Therefore, by Proposition 3.5 if $v \in [u]_{\tilde{G}}$, then $\{f^n v\}_n$ and $\{f^n u\}_n$ are E-Cauchy equivalent. Because Z is E-Complete, so there exists u^* such that

$$f^n u \xrightarrow{d,E} u^* \tag{3.2}$$

i.e. $\{f^n v\}_n$ E-converges to some element of Z say u^* and also $\{f^n u\}_n$ E-converges to $u^* \in Z$.

Because $(u, fu) \in E'(G)$ so, by using the definition of G-contraction, we obtain

$$(f^n u, f^{n+1} u) \in E'(G), \quad \text{for all}\, n \in N. \tag{3.3}$$

Since Z has the property P, therefore there exists a subsequence $f^{k_n} u$ such that

$$(f^{k_n} u, u^*) \in E'(G), \quad \text{for all}\, n.$$

From (3.3), we obtain a path $(u, fu, f^2 u, \dots, f^{k_n} u, u^*)$ from u to u^* in graph G (also in $\tilde{G}$).

Thus $u^* \in [u]_{\tilde{G}}$.

Now, we will claim that u^* is fixed point of f.

Since f is G-contraction, so we have $d(f^{k_n+1} u, fu^*) \leq_E \alpha d(f^{k_n} u, u^*)$.

Also, we know that limit of E-convergent sequence and its subsequence is same, so $f^{k_n} u \xrightarrow{d,E} u^*$ and also $f^{k_n+1} u \xrightarrow{d,E} u^*$. Therefore, there exists a sequence $\{a_n\} \downarrow 0$ in E such that

$$d(f^{k_n+1} u, fu^*) \leq_E a_{k_n+1} \downarrow 0, \quad \text{for all } n$$

and

$$d(f^{k_n} u, fu^*) \leq_E a_{k_n} \downarrow 0, \quad \text{for all } n.$$

Therefore

$$\begin{aligned} d(u^*, fu^*) &\leq_E s[d(u^*, f^{k_n} u) + d(f^{k_n} u, fu^*)] \\ &\leq_E s[d(u^*, f^{k_n} u) + sd(f^{k_n} u, f^{k_n+1} u) + sd(f^{k_n+1} u, fu^*)] \\ &\leq_E s[a_{k_n} + s\alpha d(f^{k_n-1} u, f^{k_n} u) + s\alpha a_{k_n}] \\ &\leq_E \left[s a_{k_n} + \frac{\alpha^{k_n} s^3}{1 - s\alpha} C(u, fu) + s^2 \alpha a_{k_n} \right] \downarrow 0 \end{aligned}$$

where $d(f^{k_n} u, f^{k_n+1} u) \leq_E \frac{\alpha^{k_n} s^2}{1-s\alpha} C(u, fu) \downarrow 0$ and $C(u, fu) \in E$

$$d(u^*, fu^*) \leq 0 \quad \Rightarrow \quad fu^* = u^*.$$

Thus $f/[u]_{\tilde{G}}$ is a Picard operator.

(2) If we assume G a weakly connected graph, then $[u]_{\tilde{G}} = Z \Rightarrow f$ is Picard operator.

(3) Let $Z' = \cup\{[u]_{\tilde{G}} : u \in Z_f\}$. Now, we prove that f/Z' is Weakly Picard operator.
$f : [u]_{\tilde{G}} \to [u]_{\tilde{G}}$ is a Picard operator by (1)
For all $u \in Z_f$, $(u, fu) \in E'(G)$.
Consider arbitrary element $v \in Z' = \cup[u]_{\tilde{G}}$. This imply that there exists $u \in Z_f$ such that $v \in [u]_{\tilde{G}}$. Thus $v \in Z' \Rightarrow v \in [u]_{\tilde{G}}$. Because $f/[u]_{\tilde{G}}$ is Picard operator, so f/Z' is Weakly Picard operator.

(4) We have $Z_f = \{u \in Z : (u, fu) \in E'(G)\}$. If $f \subseteq E'(G)$ which means there exists an edge between the element and its image under f, then $Z_f = Z$.

$$Z' = \cup\{[u]_{\tilde{G}} : u \in Z_f\} = \cup\{[u]_{\tilde{G}} : u \in Z\} = Z.$$

From (3), f/Z' is Weakly Picard operator.

$\Rightarrow \quad f/Z$ is Weakly Picard operator.

(5) Consider a mapping ψ from $\mathrm{Fix} f$ to $\{[u]_{\tilde{G}} : u \in Z_f\}$ i.e.

$$\psi : \mathrm{Fix} f \to A = \{[u]_{\tilde{G}} : u \in Z_f\}.$$

Defined $u \to \psi(u) = [u]_{\tilde{G}}$.

We will prove thar ψ is one-one (injection) and onto (surjection).

We have $\psi(\mathrm{Fix} f) \subset A$.

Surjection. Let $[u]_{\tilde{G}} \in A$, thus $u \in Z_f$. From (1), $f/[u]_{\tilde{G}}$ is a Picard operator i.e. $f^n u \xrightarrow{d,E} u^*$ such that u^* is a fixed point of f i.e. $fu^* = u^* \in \mathrm{Fix} f$.

Then $u^* \in [u]_{\tilde{G}} \cap \mathrm{Fix} f \quad \Rightarrow \quad \psi(u^*) = [u]_{\tilde{G}} \in A$.

Thus for any $[u]_{\tilde{G}} \in A$ there exists $u^* \in \mathrm{Fix} f$ such that $\psi(u^*) = [u]_{\tilde{G}}$.

Injection. Suppose $u_1, u_2 \in \mathrm{Fix} f$ such that $\psi(u_1) = \psi(u_2)$, then $[u_1]_{\tilde{G}} = [u_2]_{\tilde{G}}$

$\Rightarrow \quad u_2 \in [u_1]_{\tilde{G}}$.

By using (1) we get $f[u]_{\tilde{G}}$ is Picard operator, then

$$f^n u_2 \xrightarrow{d,E} u_1 \quad \Rightarrow \quad [u_1]_{\tilde{G}} \cap \mathrm{Fix} f = \{u_1\}.$$

Since $u_2 \in \mathrm{Fix} f$ and $u_1 \in Fixf$, then $u_1 = u_2$, which shows ψ is a injection.

$\Rightarrow \quad \mathrm{Card}(\mathrm{Fix} f) = \mathrm{Card} A$.

(6) Condition (6) can be obtained from (5) as a consequence.

(7) f has a unique fixed point iff $\exists\ u_0 \in Z_f$ such that $Z_f \subseteq [u_0]_{\tilde{G}}$.

Let $u_0 \in Z_f$ such that $Z_f \subseteq [u_0]_{\tilde{G}}$. Then from (1) part and (5) part, f has a unique fixed point because $\mathrm{Card}(\mathrm{Fix} f) = \mathrm{Card}\{[u]_{\tilde{G}} : u \in Z_f\}$.

For the converse part, Assume that there is only one fixed point of f i.e. $u_0 \in Z$ such that $f(u_0) = u_0$.

$\Rightarrow \quad u_0 \in Z_f$ because $(u_0, fu_0) \in E'(G)$.

As we assume earlier that $\Delta \subseteq E'(G)$.

Now, we show that $Z_f \subseteq [u_0]_{\tilde{G}}$.

For this let $v \in Z_f$.

$\Rightarrow \quad (v, fv) \in E'(G)$.

But from (1) part if $v \in Z_f$, then f is P.O. on $[v]_{\tilde{G}}$.

$\Rightarrow \quad f(v) = v$

$\Rightarrow \quad u_0 \in [v]_{\tilde{G}}$

$\Rightarrow \quad v \in [u_0]_{\tilde{G}}$

$\Rightarrow \quad Z_f \subseteq [u_0]_{\tilde{G}}$ □

Theorem 3.7. *Assume (Z, d, E) is an Archimedean, complete E-b metric space with graph G and $f : Z \to Z$ is a G-contraction with constant α and $\alpha s \in [0,1)$. Also, assume f is orbitally G-continuous mapping.*

Let $Z_f = \{u \in Z, (u, fu) \in E'(G)\}$. Under above hypothesis we can obtain following conclusions.

(1) *If $u \in Z_f$ and $v \in [u]_{\tilde{G}}$ then the sequence $(f^n v)_{n \in N}$ E-converges to fixed point of mapping f and limit of sequence $f^n v$ is independent of v.*

(2) *If graph G is weakly connected and Z_f is non empty then f is Picard operator.*

(3) *If $f \subseteq E(G)$ i.e. for all $u \in Z$ we have $\{(u, fu) : u \in Z\} \subseteq E'(G)$ then f is Weakly Picard operator.*

Proof. (1) Let $u \in Z_f$ and $v \in [u]_{\tilde{G}}$. Then by Proposition 3.5, $\{f^n u\}_{u \in N}$ and $\{f^n v\}_{v \in N}$ E-converges to same point u^* (say). Since $u \in Z_f$ that implies $(u, f(u)) \in E'(G)$. By using the definition of G_E-contraction, we have $(fu, f^2 u) \in E'(G)$ and $(f^n u, f^{n+1} u) \in E'(G)$ for $n \in N$.

Since f is orbitally G-continuous. Therefore

$$f(f^n u) \xrightarrow{d,E} f(u^*)$$

i.e.

$$d(f^{n+1} u, fu^*) \leq_E a_n \downarrow 0, \quad \text{for some } \{a_n\} \subseteq E.$$

Now

$$d(fu^*, u^*) \leq_E sd(fu^*, f^{n+1} u^*) + sd(f^{n+1} u, u^*) \leq_E s(a_n + b_n) \downarrow 0$$

$\Rightarrow \quad f(u^*) = u^*$

(2) Suppose Z_f is non empty and G is weakly connected.

Therefore for arbitrary $u \in Z_f$, $[u]_{\tilde{G}} = Z$, part (1) imply that f is Picard operator. Since

$f \subseteq E(G)$

$\Rightarrow \quad Z_f \neq \phi$

$\Rightarrow \quad$ For each $u \in Z_f$, f is P.O. on $[u]_{\tilde{G}}$ which imply that f is Weakly Picard operator on Z. □

4. Conclusions

We generalized the results of Jachymski [7] by considering E-b-metric space with graph in place of metric space with graph.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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