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Research Article

On Bicomplex Jacobsthal Numbers

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Abstract. In this article, bicomplex Jacobsthal and bicomplex Jacobsthal-Lucas numbers are introduced and some properties related to them are investigated.

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Mathematics Subject Classification (2020). 11B37; 05A15; 11B39

1. Introduction

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then the set **BC** of bicomplex numbers is defined as follows:

$$\mathbf{BC} = \{z_1 + z_2\mathrm{j} \mid z_1, z_2 \in \mathbf{C} \text{ and } \mathrm{j}^2 = -1\}, \tag{1.1}$$

where i and j are commuting imaginary units, i.e., ij = ji and **C** is the set of complex numbers with the imaginary unit i. So the set of bicomplex numbers can be expressed by the basis $\{1, \mathrm{i}, \mathrm{j}, \mathrm{ij}\}$ as

$$\mathbf{BC} = \{x_1 + \mathrm{i}y_1 + \mathrm{j}x_2 + \mathrm{ij}y_2 \mid x_1, x_2, y_1, y_2 \in \mathbf{R}, \mathrm{ij} = \mathrm{ji}, \mathrm{j}^2 = -1 = \mathrm{i}^2\}. \tag{1.2}$$

Now, the addition and the multiplication of bicomplex numbers are defined in a natural way: given $b_1 = z_1 + \mathrm{j}z_2$ and $b_2 = z_1' + \mathrm{j}z_2'$ in **BC**, then

$$b_1 + b_2 = (z_1 + z_1') + \mathrm{j}(z_2 + z_2'), \tag{1.3}$$

$$b_1 \cdot b_2 = (z_1 \cdot z_1' - z_2 \cdot z_2') + \mathrm{j}(z_1 \cdot z_2' + z_2 \cdot z_1'). \tag{1.4}$$

The multiplication of a bicomplex number $b = x_1 + y_1\mathrm{i} + x_2\mathrm{j} + y_2\mathrm{ij}$ by a real scalar k is defined as

$$kb = kx_1 + ky_1\mathrm{i} + kx_2\mathrm{j} + ky_2\mathrm{ij}. \tag{1.5}$$

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Thus, **BC** is a real vector space according to addition and scalar multiplication.

From the Propositions 1 and 3 in [12], $(\mathbf{BC}, +, \cdot)$ is a commutative ring. **BC** also has zero divisors and non-trivial idempotent elements.

Given a bicomplex number $b = z_1 + z_2\mathrm{j}$, its (bicomplex) conjugate is defined by $\overline{b} = z_1 - z_2\mathrm{j}$.

From, Proposition 3 in [12], $v_1 = \frac{1+\mathrm{ij}}{2}$ and $v_2 = \frac{1-\mathrm{ij}}{2}$ are zero divisors. Both v_1 and v_2 are linearly independent and also satisfy the following rules:

$$v_1 + v_2 = 1, \quad v_1 - v_2 = \mathrm{ij}, \quad v_1 \cdot v_2 = 0, \quad v_1 = {v_1}^2, \quad v_2 = {v_2}^2. \tag{1.6}$$

For $t \in \mathbf{BC}$ can be described in terms of the elements v_1 and v_2;

$$t = q_1 + \mathrm{j}q_2 = \mu_1 v_1 + \mu_2 v_2 = (q_1 - \mathrm{i}q_2)v_1 + (q_1 + \mathrm{i}q_2)v_2. \tag{1.7}$$

The eq. (1.7) is described as the idempotent representation of element t. Indeed, μ_1 and μ_2 are defined as idempotent coefficients (see, for example [13]). Now, for $q_1, q_2 \in \mathbf{BC}$, q_1 and q_2 can be stated as

$$q_1 = \mu_1 v_1 + \mu_2 v_2, \quad q_2 = \xi_1 v_1 + \xi_2 v_2 \tag{1.8}$$

and the algebraic operations in this set are described as

$$q_1 + q_2 = (\mu_1 + \xi_1)v_1 + (\mu_2 + \xi_2)v_2, \tag{1.9}$$

$$q_1 \cdot q_2 = (\mu_1 \cdot \xi_1)v_1 + (\mu_2 \cdot \xi_2)v_2, \tag{1.10}$$

$${q_1}^m = {\mu_1}^m v_1 + {\mu_2}^m v_2. \tag{1.11}$$

In [1, 3, 6, 9, 12–14] authors have studied bicomplex numbers. Some of them are as follows; Halici [6] studied bicomplex numbers with coefficients from Fibonacci sequence and gave some identities. In [14], Nurkan and Guven introduced bicomplex Fibonacci and bicomplex Lucas numbers and they computed d'Ocagne, Cassini and Catalan identities of them. In [1], Aydin defined bicomplex Pell and bicomplex Pell-Lucas numbers, she investigated some algebraic properties of bicomplex Pell and bicomplex Pell-Lucas numbers. Babadag [3], introduced a new generation of dual bicomplex Fibonacci numbers, author gave some formulas, facts and properties about dual bicomplex Fibonacci numbers.

2. Bicomplex Jacobsthal and Jacobsthal-Lucas Numbers

The Jacobsthal sequence $\{J_u\}$ is defined by the following recursive relation, for $u \ge 0$

$$J_{u+2} = J_{u+1} + 2J_u \tag{2.1}$$

with initial values $J_0 = 0, \quad J_1 = 1$.

The Jacobsthal-Lucas sequence $\{j_u\}$ is defined by the following recursive relation, for $u \ge 0$

$$j_{u+2} = j_{u+1} + 2j_u, \tag{2.2}$$

where $j_0 = 2$, $j_1 = 1$.

The Binet formulas for these sequences are given by

$$J_u = \frac{\lambda^u - \gamma^u}{\lambda - \gamma}, \quad j_u = \lambda^u + \gamma^u, \tag{2.3}$$

where λ, γ are roots of the equation $x^2 - x - 2$ associated to the recurrence relations (2.1) and (2.2). For more knowledge about Jacobsthal numbers, one can see [2, 4, 8, 10, 11, 16, 18].

From the aid of the Jacobsthal and Jacobsthal-Lucas numbers, we next define two bicomplex sequence with coefficients are from Jacobsthal and Jacobsthal-Lucas sequences, so, let us denote them $\mathbf{BC}_J$ and $\mathbf{BC}_j$, respectively. For $m \geq 0$

$$\mathbf{BC}_J = \{\mathcal{J}_m + \mathcal{J}_{m+2}\mathrm{j} \mid \mathcal{J}_m = J_m + \mathrm{i}J_{m+1}, \mathrm{j}^2 = -1\} \tag{2.4}$$

and

$$\mathbf{BC}_j = \{g_m + g_{m+2}\mathrm{j} \mid g_m = j_m + \mathrm{i}j_{m+1}, \mathrm{j}^2 = -1\}. \tag{2.5}$$

Let us write bicomplex Jacobsthal and bicomplex Jacobsthal-Lucas sequences as follows:

$$\mathbf{BC}_J = \{N_0, N_1, N_2, \dots, N_m, \dots\}, \quad \mathbf{BC}_j = \{K_0, K_1, K_2, \dots, K_m, \dots\}. \tag{2.6}$$

Now, any element N_m in $\mathbf{BC}_J$ can be written in terms of the elements v_1 and v_2:

$$N_m = \mathcal{J}_m + \mathrm{j}\mathcal{J}_{m+2} = \mu_m v_1 + \xi_m v_2 = (\mathcal{J}_m - \mathrm{i}\mathcal{J}_{m+2})v_1 + (\mathcal{J}_m + \mathrm{i}\mathcal{J}_{m+2})v_2. \tag{2.7}$$

Hence, it can be written as

$$N_m = \mu_m v_1 + \xi_m v_2, \tag{2.8}$$

where

$$\mu_m = \mathcal{J}_m - \mathrm{i}\mathcal{J}_{m+2}, \quad \xi_m = \mathcal{J}_m + \mathrm{i}\mathcal{J}_{m+2}. \tag{2.9}$$

Considering eq. (2.8), the idempotent representation of the element N_m is unique in $\mathbf{BC}_J$.

Using the idempotent representation of N_m and N_k and the algebraic properties of v_1 and v_2, the following results are obtained:

$$N_m + N_k = (\mu_m + \mu_k)v_1 + (\xi_m + \xi_k)v_2, \tag{2.10}$$

$$N_m \cdot N_k = (\mu_m \cdot \mu_k)v_1 + (\xi_m \cdot \xi_k)v_2, \tag{2.11}$$

$$N_m^p = \mu_m^p v_1 + \xi_m^p v_2. \tag{2.12}$$

Lemma 2.1. *For $m \geq 2$, $\mathbf{BC}_J$ has the following recursive relation*

$$N_{m+2} = N_{m+1} + 2N_m, \tag{2.13}$$

where $N_0 = \mu_0 v_1 + \xi_0 v_2$ and $N_1 = \mu_1 v_1 + \xi_1 v_2$.

Proof. From eqs. (2.8) and (2.9),

$$N_m = \mathcal{J}_m + \mathcal{J}_{m+2}\mathrm{j} = (\mathcal{J}_m - \mathrm{i}\mathcal{J}_{m+2})v_1 + (\mathcal{J}_m + \mathrm{i}\mathcal{J}_{m+2})v_2.$$

The coefficients μ_m and ξ_m in (2.8) give us the following recurrence relation:

$$\mu_{m+2} = \mu_{m+1} + 2\mu_m, \quad \xi_{m+2} = \xi_{m+1} + 2\xi_m. \tag{2.14}$$

$$\begin{aligned} N_{m+1} + 2N_m &= (\mu_{m+1} + 2\mu_m)v_1 + (\xi_{m+1} + 2\xi_m)v_2 \\ &= \mu_{m+2}v_1 + \xi_{m+2}v_2 \\ &= N_{m+2}. \end{aligned}$$

□

Using the properties of Jacobsthal and Jacobsthal-Lucas numbers, the following proposition is obtained (for details, see [8]).

Proposition 2.2. *For $N_m \in \mathbf{BC}_J$ and $K_m \in \mathbf{BC}_j$,*

$$N_{m+1} + 2N_{m-1} = K_m, \tag{2.15}$$

$$N_m + K_m = 2N_{m+1}, \tag{2.16}$$

$$K_{m+1} + 2K_{m-1} = 9N_m, \tag{2.17}$$

$$3N_m + K_m = 2^{m+1}(1+2i)(1+4j), \tag{2.18}$$

$$K_{m+1} + K_m = 3(N_{m+1} + N_m) = 3.2^n(1+2i)(1+4j), \tag{2.19}$$

$$K_{m+1} - K_m = 2^n(1+2i)(1+4j) + 2(-1)^{n+1}(1-i)(1+j), \tag{2.20}$$

$$K_{m+1} - 2K_m = 3(2N_m - N_{m+1}) = 3(-1)^{m+1}(1-i)(1+j), \tag{2.21}$$

$$2K_{m+1} + K_{m-1} = 3(2N_{m+1} + N_{m-1}) + 6(-1)^{m+1}(1-i)(1+j), \tag{2.22}$$

$$K_{m+r} + K_{m-r} = 2^{m-r}(2^{2r}+1)(1+2i)(1+4j) + 2(-1)^{m-r}(1-i)(1+j), \tag{2.23}$$

$$K_{m+r} - K_{m-r} = 3(N_{m+r} - N_{m-r}) = 2^{m-r}(2^{2r}-1)(1+2i)(1+4j), \tag{2.24}$$

$$K_m = 3N_m + 2(-1)^m(1-i)(1+j). \tag{2.25}$$

Proof. The proof of Proposition 2.2 can be done from the definitions of J_m, j_m, $\mathbf{BC}_J$, $\mathbf{BC}_j$ and interrelationships in [8]. □

Theorem 2.3 (Binet Formula)**.** *The Binet formula for* $N_p \in \mathbf{BC}_J$ *is,*

$$N_p = \frac{1}{3}\{(E_1\lambda^p + D_1\gamma^p)v_1 + (E_2\lambda^p + D_2\gamma^p)v_2\}, \tag{2.26}$$

where the values λ *and* γ *are the roots of the characteristic equation in* (2.13)*,* $E_1 = \mu_1 - \mu_0\gamma$*,* $D_1 = -\mu_1 + \mu_0\lambda$*,* $E_2 = \xi_1 - \xi_0\gamma$ *and* $D_2 = \xi_0\lambda - \xi_1$*.*

Proof. From (2.14), $\xi_{p+2} = \xi_{p+1} + 2\xi_p$ and $\mu_{p+2} = \mu_{p+1} + 2\mu_p$. Now both formulas have the same relation as $N_{p+2} = N_{p+1} + 2N_p$, also

$$N_{p+2} = \mu_{p+2}v_1 + \xi_{p+2}v_2. \tag{2.27}$$

Thus, the Binet formula for μ_p can be written as

$$\mu_p = \frac{1}{3}(\lambda^p E_1 + \gamma^p D_1) \tag{2.28}$$

here $D_1 = 2i$, $E_1 = 9-2i$, $\mu_0 = 3$ and $\mu_1 = 6-2i$.

Similarly, the Binet formula for ξ_p can be given as

$$\xi_p = \frac{1}{3}(E_2\lambda^p + D_2\gamma^p) \tag{2.29}$$

where $\xi_0 = 2i-3$, $\xi_1 = 4i-4$, $E_2 = 6i-7$ and $D_2 = -2$.

Hence, by using eqs. (2.28), (2.27) and (2.29), we obtain

$$N_p = \frac{1}{3}\{(D_1\gamma^p + E_1\lambda^p)v_1 + (D_2\gamma^p + E_2\lambda^p)v_2\}.$$

the proof is completed. □

Now, $\gamma = -1$ and $\lambda = 2$ are the roots of the equation $w^2 - w - 2 = 0$ which related to eq. (2.13). Based on these; the following results are written:

$$\lambda - \gamma = 3, \lambda + \gamma = 1, \quad \lambda\gamma = -2 \quad . \tag{2.30}$$

Theorem 2.4 (Cassini's Identity)**.** *For* $\mathrm{m} \geq 1$*,*

$$N_{m-1}N_{m+1} - N_m^2 = (-2)^{m-1}(i+3)(3-5j). \tag{2.31}$$

Proof. If the left side of (2.31) is taken and the Binet formulas are used for N_{m}, $N_{\mathrm{m}-1}$ and $N_{\mathrm{m}+1}$ here, the following expression is obtained

$$\begin{aligned} N_{\mathrm{m}-1}N_{\mathrm{m}+1}-N_{\mathrm{m}}^{2} &= \Big\{\frac{1}{3}[(E_1\lambda^{\mathrm{m}-1}+D_1\gamma^{\mathrm{m}-1})v_1+(E_2\lambda^{\mathrm{m}-1}+D_2\gamma^{\mathrm{m}-1})v_2] \\ &\qquad \times\frac{1}{3}[(E_1\lambda^{\mathrm{m}+1}+D_1\gamma^{\mathrm{m}+1})v_1+(E_2\lambda^{\mathrm{m}+1}+D_2\gamma^{\mathrm{m}+1})v_2]\Big\} \\ &\qquad -\frac{1}{9}[(E_1\lambda^{\mathrm{m}}+D_1\gamma^{\mathrm{m}})v_1+(E_2\lambda^{\mathrm{m}}+D_2\gamma^{\mathrm{m}})v_2]^2 \\ &= \frac{1}{9}(\lambda\gamma)^{m-1}(\lambda-\gamma)^2\{E_1D_1v_1+E_2D_2v_2\}. \end{aligned}$$

Since

$$E_1D_1=18i+4 \text{ and } E_2D_2=-12i+14, \tag{2.32}$$

the following expression is easily seen

$$N_{\mathrm{m}-1}N_{\mathrm{m}+1}-N_{\mathrm{m}}^{2}=(-2)^{\mathrm{m}-1}(i+3)(3-5j). \qquad \square$$

Theorem 2.5 (Catalan's Identity)**.** *For every nonnegative integer number* m *and* t *such that* $\mathrm{t}\geq\mathrm{m}$*,*

$$N_{\mathrm{t}+\mathrm{m}}N_{\mathrm{t}-\mathrm{m}}-N_{\mathrm{t}}^{2}=(-2)^{\mathrm{t}-\mathrm{m}}J_{\mathrm{m}}^{2}\{(i+3)(3-5j)\}. \tag{2.33}$$

Proof. Using the Binet formulas of $N_{\mathrm{t}+\mathrm{m}}$, $N_{\mathrm{t}-\mathrm{m}}$ and N_{t} in the left side of (2.33), the following expression is written

$$\begin{aligned} N_{\mathrm{t}+\mathrm{m}}N_{\mathrm{t}-\mathrm{m}}-N_{\mathrm{t}}^{2} &= \Big\{\frac{1}{3}[(E_1\lambda^{\mathrm{t}+\mathrm{m}}+D_1\gamma^{\mathrm{t}+\mathrm{m}})v_1+(E_2\lambda^{\mathrm{t}+\mathrm{m}}+D_2\gamma^{\mathrm{t}+\mathrm{m}})v_2] \\ &\qquad \times\frac{1}{3}[(E_1\lambda^{\mathrm{t}-\mathrm{m}}+D_1\gamma^{\mathrm{t}-\mathrm{m}})v_1+(E_2\lambda^{\mathrm{t}-\mathrm{m}}+D_2\gamma^{\mathrm{t}-\mathrm{m}})v_2] \\ &\qquad -\frac{1}{9}[(E_1\lambda^{\mathrm{t}}+D_1\gamma^{\mathrm{t}})v_1+(E_2\lambda^{\mathrm{t}}+D_2\gamma^{\mathrm{t}})v_2]^2\Big\} \\ &= \frac{1}{9}\{E_1D_1[\lambda^{\mathrm{t}+\mathrm{m}}\gamma^{\mathrm{t}-\mathrm{m}}+\lambda^{\mathrm{t}-\mathrm{m}}\gamma^{\mathrm{t}+\mathrm{m}}-2(\lambda\gamma)^{\mathrm{t}}]v_1 \\ &\qquad +E_2D_2[\lambda^{\mathrm{t}+\mathrm{m}}\gamma^{\mathrm{t}-\mathrm{m}}+\lambda^{\mathrm{t}-\mathrm{m}}\gamma^{\mathrm{t}+\mathrm{m}}-2(\lambda\gamma)^{\mathrm{t}}]v_2\} \\ &= \frac{1}{9}(\lambda\gamma)^{\mathrm{t}-\mathrm{m}}(\lambda^{\mathrm{m}}-\gamma^{\mathrm{m}})^2[E_1D_1v_1+E_2D_2v_2]. \end{aligned}$$

By eqs. (2.32), (2.30) and (2.3), we obtain

$$N_{\mathrm{t}+\mathrm{m}}N_{\mathrm{t}-\mathrm{m}}-N_{\mathrm{t}}^{2}=(-2)^{\mathrm{t}-\mathrm{m}}J_{\mathrm{m}}^{2}\{(i+3)(3-5j)\}. \qquad \square$$

Theorem 2.6 (d'Ocagne's Identity)**.** *For integers t and u which are positive and different from each other*

$$N_tN_{u+1}-N_uN_{t+1}=(-2)^tJ_{u-t}(i+3)(3-5j), \quad u\geq t. \tag{2.34}$$

Proof. Using the Binet formula for N_t and N_u on the left-hand side of eq. (2.34), the following can be written

$$N_tN_{u+1}-N_uN_{t+1}=\Big\{\frac{1}{3}[(E_1\lambda^t+D_1\gamma^t)v_1+(E_2\lambda^t+D_2\gamma^t)v_2]$$

$$\times \frac{1}{3}[(E_1\lambda^{u+1}+D_1\gamma^{u+1})v_1+(E_2\lambda^{u+1}+D_2\gamma^{u+1})v_2]\Big\}$$
$$-\Big\{\frac{1}{3}[(E_1\lambda^{u}+D_1\gamma^{u})v_1+(E_2\lambda^{u}+D_2\gamma^{u})v_2]$$
$$\times \frac{1}{3}[(E_1\lambda^{t+1}+D_1\gamma^{t+1})v_1+(E_2\lambda^{t+1}+D_2\gamma^{t+1})v_2]\Big\}$$
$$=\frac{1}{9}\{E_1D_1(\lambda\gamma)^t(\lambda-\gamma)(\lambda^{u-t}-\gamma^{u-t})v_1+E_2D_2(\lambda\gamma)^t(\lambda-\gamma)(\lambda^{u-t}-\gamma^{u-t})v_2\}.$$

From (2.32), (2.30) and (2.3)

$$N_tN_{u+1}-N_uN_{t+1}=(-2)^tJ_{u-t}(i+3)(3-5j).$$ □

Theorem 2.7. *The sum of the first m terms of the sequence* $\mathbf{BC}_J$ *is given by*

$$\sum_{l=1}^{m} N_l=\frac{N_{m+2}-N_2}{2}. \tag{2.35}$$

Proof. The proof is obtained from eqs. (2.26) and (2.30). □

Theorem 2.8. *The generating function of the bicomplex Jacobsthal numbers is*

$$\sum_{l=0}^{\infty} N_lt^l=\frac{N_0+t(N_1-N_0)}{1-t-2t^2}. \tag{2.36}$$

Proof. Let $A(t)=\sum_{l=0}^{\infty}N_lt^l$. In this case, with (2.13), the following equation is obtained

$$A(t)-tA(t)-2t^2A(t)=N_0+t(N_1-N_0).$$

Thus, $A(t)=\frac{N_0+t(N_1-N_0)}{1-t-2t^2}$. □

Proposition 2.9. *For* $N_m\in\mathbf{BC}_J$*,*

$$N_m+(\overline{N_m})_i+(\overline{N_m})_j+(\overline{N_m})_{ij}=4J_m.$$

Proof. The proof can be obtained from (2.4). □

Matrix representations of bicomplex Jacobsthal numbers are studied below:

Let $\begin{bmatrix} N_{s+1} & N_s \\ N_s & N_{s-1}\end{bmatrix}$ be a matrix with entries bicomplex Jacobsthal numbers.

Theorem 2.10. *Let* $s\geq 1$ *be an integer. Then*

$$\begin{bmatrix} N_{s+1} & N_s \\ N_s & N_{s-1}\end{bmatrix}=\begin{bmatrix} N_2 & N_1 \\ N_1 & N_0\end{bmatrix}\cdot\begin{bmatrix} 1 & 1 \\ 2 & 0\end{bmatrix}^{s-1}.$$

Proof. The proof will be done by induction method. If $s=1$, then the result is obvious.

Assume that

$$\begin{bmatrix} N_{s+1} & N_s \\ N_s & N_{s-1}\end{bmatrix}=\begin{bmatrix} N_2 & N_1 \\ N_1 & N_0\end{bmatrix}\cdot\begin{bmatrix} 1 & 1 \\ 2 & 0\end{bmatrix}^{s-1}.$$

We prove that

$$\begin{bmatrix} N_{s+2} & N_{s+1} \\ N_{s+1} & N_s\end{bmatrix}=\begin{bmatrix} N_2 & N_1 \\ N_1 & N_0\end{bmatrix}\cdot\begin{bmatrix} 1 & 1 \\ 2 & 0\end{bmatrix}^{s}.$$

From eq. (2.13),

$$\begin{aligned}\begin{bmatrix} N_2 & N_1 \\ N_1 & N_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{s} &= \left(\begin{bmatrix} N_2 & N_1 \\ N_1 & N_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{s-1} \right) \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} N_{s+1} & N_s \\ N_s & N_{s-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} N_{s+1} + 2N_s & N_{s+1} \\ N_s + 2N_{s-1} & N_s \end{bmatrix} \\ &= \begin{bmatrix} N_{s+2} & N_{s+1} \\ N_{s+1} & N_s \end{bmatrix}.\end{aligned}$$

So it is true for $s+1$. Thus, the proof is complete. □

3. Conclusion

We examine the bicomplex Jacobsthal numbers. In Section 2, we have given the Binet formula, the generating function, the Catalan identity, the Cassini identity, the d'Ocagne identity and some results. Using these results, some properties of other bicomplex numbers can be calculated in the future.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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