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Research Article

Optimal System and Exact Solutions of Monge-Ampere Equation

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Abstract. A solution that remains unchanged when transformed under Lie group of point symmetries of the differential equation is an invariant solution of the differential equation. Optimal system of Lie group of point symmetry generators provide all possible invariant solutions of differential equation. Here, using optimal system of Lie point symmetry generators group invariant solutions are obtained. Using these solutions, exact solutions of non-homogeneous Monge-Ampere equation have been presented here.

Keywords. Lie symmetries; Adjoint representations; Commutator relation; Conjugacy classes; Invariant equations

Mathematics Subject Classification (2020). 76M60; 22E70; 35A30; 58J70

1. Introduction

The Monge-Ampere equation

$$u_{xx}u_{yy} - u_{xy}^2 + f(x,y) = 0, \tag{1.1}$$

is a semi-linear non-homogeneous partial differential equation with $f(x,y)$ as non-homogeneous part of the equation. The name "Monge-Ampere equation" has been derived from its early formulation in two different directions. One by the French mathematician, civil engineer

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and the inventor of descriptive geometry Gaspard Monge (1746-1818) [16] while the second by the French physicist Andre Marie Ampere (1775-1836) [2]. In 1781, Gaspard Monge originally formulated and analyzed the problem of optimal transportation, initiating a profound mathematical theory, which connects the different areas of differential geometry, nonlinear partial differential equations, linear programming and probability theory. It was later studied by Minkowski (1864-1909) [14, 15], Lewy (1904-1988) [12], Bernstein (1918-1990) [1] and many others. During the last century the development of Monge-Ampere equation was closely related to geometric problems. It also arise in meteorology and fluid mechanics. In fluid mechanics it is coupled with transport equation, like semi-geostrophic equation. Due to its applications and beautiful theory, Monge-Ampere type equations are important and get lot of attention and are studied extensively [17].

In general, finding solutions of non-linear partial differential equations is not an easy task. Mathematicians have developed different techniques for the solutions of such equations. For finding the solutions, Sophus Lie developed a very useful technique that can be applied to all types of differential equations. His technique is based on the group of transformations that a differential equation may have. Each group of transformations corresponds to a family of group invariant solutions, which are solutions that remain unchanged when transformed under a Lie group of point transformations of the differential equation. By considering group of transformations of a differential equation infinite number of such groups can be obtained which lead to infinite number of group invariant solutions. One can divide these invariant solutions into equivalence classes. A set consisting of exactly one generator from each class of generators is called an optimal system of generators, i.e. a list of group invariant solutions from which every other solution can be derived [19].

In literature, there are many techniques available for obtaining optimal systems and a lot of excellent work has been done by experts e.g. [3, 5–11, 13, 18, 20–23]. Here, we use Peter J. Olver's technique [19] to derive optimal system for different cases of Monge-Ampere equation by assuming different particular values of the non-homogeneous part $f(x, y)$. This system helps us to reduce the semi-linear non-homogeneous Monge-Ampere equation (1.1) into ordinary differential equations. Solutions of these reduced equations give new set of group invariant solutions. Then, by using set of transformations obtained from optimal system we get exact solutions of nonlinear partial differential equation.

In the following sections we find (a) the optimal system by (i) calculating the commutator table for symmetry generators of given differential equation; (ii) constructing adjoint representation table, by conjunction of adjoint map with already calculated commutator relation table; and (iii) construct the conjugacy classes. (b) Using these optimal algebras, equation (1.1) is reduced to ordinary differential equation whose solutions then lead to the solutions of equation (1.1).

2. Lie symmetries and Commutator Relation Table

In equation (1.1) we have two independent variables x and y while u the dependent variable. For such situations we consider following one parameter, ε, group of transformations

$$x^* = x + \varepsilon\xi^1(x,y,u) + o(\varepsilon^2),$$
$$y^* = y + \varepsilon\xi^2(x,y,u) + o(\varepsilon^2),$$
$$u^* = u + \varepsilon\eta(x,y,u) + o(\varepsilon^2).$$

The corresponding second order prolonged symmetry generator is

$$\mathbf{V} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta_x\frac{\partial}{\partial u_{,x}} + \eta_y\frac{\partial}{\partial u_{,y}} + \eta_{xx}\frac{\partial}{\partial u_{,xx}} + \eta_{xy}\frac{\partial}{\partial u_{,xy}} + \eta_{yy}\frac{\partial}{\partial u_{,yy}}. \tag{2.1}$$

By applying generator (2.1) to the equation (1.1), we obtain a system of over determined linear partial differential equations in ξ^1, ξ^2 and η. Solution of these give symmetry generators of equation (1.1). In this paper we are considering two different cases of the non-homogeneous part of equation (1.1), *Case* I: $f(x,y) = e^x$ and *Case* II: $f(x,y) = e^x\phi(y)$.

2.1 Case I: $f(x,y) = e^x$

First consider particular value for non-homogeneous part of non-homogeneous Monge-Ampere equation (1.1) to be e^x. For this case symmetry generators are:

$$\mathbf{V}_1 = \frac{\partial}{\partial u},\ \ \mathbf{V}_2 = \frac{\partial}{\partial y},\ \ \mathbf{V}_3 = x\frac{\partial}{\partial y},\ \ \mathbf{V}_4 = x\frac{\partial}{\partial u},\ \ \mathbf{V}_5 = y\frac{\partial}{\partial u},\ \ \mathbf{V}_6 = \frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u},\ \ \mathbf{V}_7 = y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.$$

Symmetry generators $\mathbf{V}_1$ and $\mathbf{V}_2$ represent translations, $\mathbf{V}_4$ and $\mathbf{V}_5$ represent Galilean transformations whereas $\mathbf{V}_6$ represents Galilean transformation translation in x direction and scaling in u direction and $\mathbf{V}_7$ represents scaling in x and u directions. Commutator relations of these generators are given in Table 1.

Table 1

,	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_7}$
$\mathbf{V_1}$	0	0	0	0	0	$\mathbf{V}_1$	$\mathbf{V}_7$
$\mathbf{V_2}$	0	0	0	0	$\mathbf{V}_1$	0	$\mathbf{V}_2$
$\mathbf{V_3}$	0	0	0	0	$\mathbf{V}_4$	$-\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V_4}$	0	0	0	0	0	$\mathbf{V}_4-\mathbf{V}_1$	$\mathbf{V}_4$
$\mathbf{V_5}$	0	0	$-\mathbf{V}_4$	0	0	$\mathbf{V}_5$	0
$\mathbf{V_6}$	$-\mathbf{V}_4$	0	$\mathbf{V}_2$	$\mathbf{V}_1-\mathbf{V}_4$	$-\mathbf{V}_5$	0	0
$\mathbf{V_7}$	$-\mathbf{V}_1$	$-\mathbf{V}_2$	$-\mathbf{V}_3$	$-\mathbf{V}_4$	0	0	0

2.1.1 Construction of Adjoint Representation Table

To compute adjoint representation, we use the Lie series in conjunction with commutator relation Table 1. The adjoint action is given by the Lie series as

$$Ad(\exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j = \mathbf{V}_j - \varepsilon[\mathbf{V}_i,\mathbf{V}_j] + \frac{\varepsilon^2}{2!}[\mathbf{V}_i,[\mathbf{V}_i,\mathbf{V}_j]] - \cdots,$$

where $[\mathbf{V}_i, \mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action one can construct an adjoint representation Table 2:

Table 2

Ad	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_7}$
$\mathbf{V_1}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6-\varepsilon\mathbf{V}_1$	$\mathbf{V}_7-\varepsilon\mathbf{V}_1$
$\mathbf{V_2}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5-\varepsilon\mathbf{V}_1$	$\mathbf{V}_6$	$\mathbf{V}_7-\varepsilon\mathbf{V}_2$
$\mathbf{V_3}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5-\varepsilon\mathbf{V}_4$	$\mathbf{V}_6+\varepsilon\mathbf{V}_2$	$\mathbf{V}_7-\varepsilon\mathbf{V}_3$
$\mathbf{V_4}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6-\varepsilon(\mathbf{V}_4-\mathbf{V}_1)$	$\mathbf{V}_7-\varepsilon\mathbf{V}_4$
$\mathbf{V_5}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$-\mathbf{V}_4$	$\mathbf{V}_3+\varepsilon\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6-\varepsilon\mathbf{V}_5$	$\mathbf{V}_7$
$\mathbf{V_6}$	$e^{\varepsilon}\mathbf{V}_1$	$e^{-\varepsilon}\mathbf{V}_2$	$\mathbf{V}_3-\varepsilon\mathbf{V}_2$	$e^{\varepsilon}(\mathbf{V}_4-\varepsilon\mathbf{V}_1)$	$e^{\varepsilon}\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$
$\mathbf{V_7}$	$e^{\varepsilon}\mathbf{V}_1$	$e^{\varepsilon}\mathbf{V}_2$	$e^{\varepsilon}\mathbf{V}_3$	$e^{\varepsilon}\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$

2.1.2 Formation of Optimal System

Optimal system constitutes the set of conjugacy classes of group of transformations. Adjoint action gives the conjugacy classes of group of transformations which are written in columns of adjoint representation Table 2. Our aim is to find the set of one dimensional sub algebras, that cover all conjugacy classes. Following Olver's technique we assume a general vector $\mathbf{V}$ as the combination of all symmetry generators. Then by observing columns of adjoint representation Table 2, we try to vanish coefficients of as much symmetry generators as possible by using appropriate adjoint action on general vector $\mathbf{V}$. In this case there are seven symmetry generators. So, non zero general vector is

$$\mathbf{V}=a_1\mathbf{V}_1+a_2\mathbf{V}_2+a_3\mathbf{V}_3+a_4\mathbf{V}_4+a_5\mathbf{V}_5+a_6\mathbf{V}_6+a_7\mathbf{V}_7\,. \tag{2.2}$$

Use judicious application of adjoint map to make maximum possible constants a's to vanish. Assume that $a_7 \neq 0$ and also for convenience $a_7 = 1$. Then the general non zero vector (2.2) becomes

$$\mathbf{V}=a_1\mathbf{V}_1+a_2\mathbf{V}_2+a_3\mathbf{V}_3+a_4\mathbf{V}_4+a_5\mathbf{V}_5+a_6\mathbf{V}_6+\mathbf{V}_7\,.$$

Referring adjoint representation Table 2, if we act on $\mathbf{V}$ by $Ad(\exp(a_4\mathbf{V}_4))$, then coefficient of $\mathbf{V}_4$ vanishes. Name the resulting vector as $\mathbf{V}'$

$$\mathbf{V}'=a_1\mathbf{V}_1+a_2\mathbf{V}_2+a_3\mathbf{V}_3+a_5\mathbf{V}_5+a_6\mathbf{V}_6+\mathbf{V}_7\,.$$

Similarly, if we act on $\mathbf{V}'$ by adjoint map $Ad(\exp(a_3\mathbf{V}_3))$, then the coefficient of $\mathbf{V}_3$ vanishes. Call the resulting vector as $\mathbf{V}''$

$$\mathbf{V}''=a_1\mathbf{V}_1+a_2\mathbf{V}_2+a_5\mathbf{V}_5+a_6\mathbf{V}_6+\mathbf{V}_7\,.$$

Working on same lines we find that, if we act on $\mathbf{V}''$ by $Ad(\exp(a_2\mathbf{V}_2))$, then the coefficient of $\mathbf{V}_2$ vanishes from the general vector $\mathbf{V}''$, which is represented in $\mathbf{V}'''$

$$\mathbf{V}'''=a_1\mathbf{V}_1+a_5\mathbf{V}_5+a_6\mathbf{V}_6+\mathbf{V}_7.$$

Also, if we act on $\mathbf{V}'''$ by $Ad(\exp(a_1\mathbf{V}_1))$, then coefficient of $\mathbf{V}_1$ vanishes and we get the vector free from the coefficients a_4, a_3, a_2 and a_1, that is

$$\mathbf{V}^{iv} = a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7 .$$

Referring adjoint representation Table 2, coefficient of symmetry generator $\mathbf{V}_5$ can be eliminated from above vector if we act on above general non-zero vector by $Ad(\exp(a_5)\mathbf{V}_5)$. After that, we cannot eliminate any other coefficient of symmetry generators. Therefore, two symmetry generators which can be included in optimal algebra of non-homogeneous Monge-Ampere equation are:

(i) $a_6\mathbf{V}_6 + \mathbf{V}_7$.

(ii) $\mathbf{V}_7$.

Continuing in the same way, we find the optimal system of one-dimensional sub algebras of non-homogeneous Monge-Ampere equation (1.1) with $f(x, y) = e^x$ as

$$\begin{aligned}&\mathbf{V}_7 + a_6\mathbf{V}_6, \quad \mathbf{V}_5 + a_3\mathbf{V}_3 + a_2\mathbf{V}_2, \quad \mathbf{V}_4 + a_2\mathbf{V}_2, \quad \mathbf{V}_6 + a_3\mathbf{V}_3, \quad \mathbf{V}_3 + a_1\mathbf{V}_1,\\ &\mathbf{V}_2 + a_1\mathbf{V}_1, \quad \mathbf{V}_7, \quad \mathbf{V}_6, \quad \mathbf{V}_5, \quad \mathbf{V}_4, \quad \mathbf{V}_3, \quad \mathbf{V}_1 .\end{aligned}$$

2.1.3 Reduction

It is not very easy to reduce Monge-Ampere equation (1.1) into ordinary differential equation as it is semi-linear partial differential equation and admits translations and Galilean transformations. Here we are going to show reduction by those optimal sub algebras which reduces equation (1.1) into ordinary differential equations and give solutions. Remaining sub algebras yield either the trivial solution or reduces the order of semi-linear non-homogeneous Monge-Ampere equation.

(i) For symmetry generator $\mathbf{V}_7$, we have $x = \xi$, $u = yU(\xi)$. Substituting these transformations in equation (1.1), one gets the following ordinary differential equation

$$U'^2 - e^{\xi} = 0, \quad \text{where } U' = \frac{dU}{d\xi}. \tag{2.3}$$

Therefore,

$$U(\xi) = c \pm e^{\frac{\xi}{2}}$$

or

$$u(x, y) = y(c \pm e^{\frac{x}{2}}), \tag{2.4}$$

is a solution of semi-linear non-homogeneous Monge-Ampere equation (1.1).

(ii) The combination of $\mathbf{V}_3$ and $\mathbf{V}_6$, yields $\xi = y - \frac{x^2}{2}$, $U = ue^{-\frac{x}{2}}$. Using these transformations equation (1.1) can be transformed into the following ordinary differential equation

$$U''(U - 4U') - U'^2 + 4 = 0, \quad \text{where } U' = \frac{dU}{d\xi}. \tag{2.5}$$

Solution of this equation yields

$$u = (a \pm 2(y - \frac{x^2}{2}))e^{\frac{x}{2}}.$$

(iii) One can write combination of $\mathbf{V}_6$ and $\mathbf{V}_7$ as

$$\mathbf{V} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{3u}{2}\frac{\partial}{\partial u},$$

to get, $\xi = ye^{-x}$, $u = Uy^{\frac{3}{2}}$. These transformations reduce (1.1) into the following ordinary differential equation

$$\frac{3}{4}\xi UU' - \frac{13}{4}\xi^2 U'^2 - \xi^3 U'U'' + \frac{3}{4}\xi^2 UU'' + \frac{1}{\xi} = 0, \text{where } U' = \frac{dU}{d\xi} \tag{2.6}$$

whose solution is $U = 2\xi^{-\frac{1}{2}}$ and the corresponding solution of Monge Ampere equation given by eq. (1.1) is $u(x,y) = ye^{\frac{x}{2}}$.

(iv) The symmetry generator $\mathbf{V}_6$, yields $x = \xi$, $y = \eta$, $u = Ue^{\xi}$. Hence, the reduced ordinary differential equation and its solution are

$$UU'' - (U')^2 + 4 = 0, \tag{2.7}$$

and $U = a \pm 2\xi$. Thus, we have $u = (a \pm 2y)e^{\frac{x}{2}}$, a solution of semi-liner non-homogeneous Monge-Ampere equation (1.1).

2.2 Case II: $f(x,y) = e^x\phi(y)$

Now consider another case of family of non homogeneous Monge-Ampere equation (1.1) with particular value of non homogeneous part as $e^{ax}\phi(y)$.

2.2.1 Lie Symmetries, Commutator Relation and Adjoint Representation Tables

Adopting the well developed method discussed in the previous section for finding symmetry generators of non homogeneous Monge-Ampere equation with $e^x\phi(y)$ as non homogeneous part, we get

$$\mathbf{V}_1 = \frac{\partial}{\partial u}, \quad \mathbf{V}_2 = x\frac{\partial}{\partial u}, \quad \mathbf{V}_3 = y\frac{\partial}{\partial u}, \quad \mathbf{V}_4 = \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u}.$$

Symmetry generator $\mathbf{V}_1$ representing translation, $\mathbf{V}_2$, $\mathbf{V}_3$ representing Galilean transformation while $\mathbf{V}_4$ representing translation in x direction and scaling in u direction. Commutator relations of these four symmetry generators are given in Table 3. Using Table 3 one can obtain the adjoint representation Table 4.

Table 3

,	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$
$\mathbf{V_1}$	0	0	0	$\frac{1}{2}\mathbf{V}_1$
$\mathbf{V_2}$	0	0	0	$\frac{1}{2}\mathbf{V}_2 - \mathbf{V}_1$
$\mathbf{V_3}$	0	0	0	$\frac{1}{2}\mathbf{V}_3$
$\mathbf{V_4}$	$-\frac{1}{2}\mathbf{V_1}$	$\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2$	$-\frac{1}{2}\mathbf{V}_3$	0

Table 4

Ad	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$
$\mathbf{V_1}$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}-\frac{\varepsilon}{2}\mathbf{V_1}$
$\mathbf{V_2}$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}-\frac{\varepsilon}{2}\mathbf{V_2}+\varepsilon\mathbf{V_1}$
$\mathbf{V_3}$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}-\frac{\varepsilon}{2}\mathbf{V_3}$
$\mathbf{V_4}$	$\mathbf{V_1}\mathbf{e}^{\frac{\varepsilon}{2}}$	$\mathbf{V_2}\mathbf{e}^{\frac{\varepsilon}{2}}-\varepsilon\mathbf{V_1}\mathbf{e}^{\frac{\varepsilon}{2}}$	$\mathbf{V_3}\mathbf{e}^{\frac{\varepsilon}{2}}$	$\mathbf{V_4}$

2.2.2 Optimal System

Following the procedure of the previous case we obtain the optimal system of one dimensional sub algebras of (1.1) as

$$\mathbf{V}_4, \quad a_1\mathbf{V}_1+a_2\mathbf{V}_2+\mathbf{V}_3, \quad a_1\mathbf{V}_1+\mathbf{V}_2, \quad \mathbf{V}_2-a_1\mathbf{V}_1, \quad \mathbf{V}_2, \quad \mathbf{V}_1.$$

This is same as the classification of real three and four dimensional Lie algebras done by Patera and Winternitz in [4].

2.2.3 Reduction and Solution

For this case there is only one optimal algebra, $\mathbf{V}_4$, that converts Monge-Ampere equation into the following ordinary differential equation

$$UU''-U'^2+4\phi(\xi)=0, \tag{2.8}$$

using the transformations $y=\xi$, $U=ue^{-\frac{x}{2}}$. Here $\phi(\xi)$ is any arbitrary function, if we consider particular value,

$$\phi(\xi)=\xi e^{\xi}-2e^{\xi}+1,$$

then the solution of this reduced equation is $U=2(e^{\xi}-\xi)$. This solution leads to $u=2(e^{y}-y)e^{\frac{x}{2}}$, be the solution of the original partial differential equation.

By using any other optimal algebra we did not get any transformation that can reduce equation (1.1) to ordinary differential equation. But we are able to just reduce the order of semi linear non homogeneous Monge-Ampere equation or we get its trivial solution.

3. Conclusion

In this paper, solutions of semi-linear non-homogeneous Monge-Ampere equation (1.1) using its optimal sub algebra are obtained. We consider two particular cases by considering e^x and $e^x\phi(y)$ as non-homogeneous parts in Case I and Case II, respectively. All solutions satisfies original partial differential equation (1.1) with respective conditions. Equation (1.1) involves three basic symmetries (symmetries depending on homogeneous part only) $\frac{\partial}{\partial u}$, $x\frac{\partial}{\partial u}$, and $y\frac{\partial}{\partial u}$. Symmetries of this form basically define translation and Galilean translation. Reduction to ordinary differential equation by using such symmetries is not possible here. Because of this we are unable to find solutions from all sub algebras of optimal system. Fortunately, we get some optimal sub algebra which reduces equation (1.1) to an ordinary differential equation with its respective transformations that leads to the new solutions obtained in this paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] N. I. Akhiezer and I. G. Petrovskii, S. N. Bernshtein's contribution to the theory of partial differential equations, *Russian Mathematical Surveys* **16**(2) (1961), 1 – 15, DOI: 10.1070/RM1961v016n02ABEH004101.

[2] A. M. Ampère, Mémoire contenant l'application de la théorie exposée dans le XVII° cahier du Journal de l'École Polytechnique, à l'intégration des équations différentielles partielles du premier et du second ordre, *Journal de l'École Polytechnique* **11**(18) (1820), 1 – 188, URL: http://www.ampere.cnrs.fr/bibliographies/pdf/1820-P248.pdf.

[3] D. Baleanu, M. Inc, A. Yusuf and A. I. Aliyu, Optimal system, nonlinear self-adjointness and conservation laws for generalized shallow water wave equation, *Open Physics* **16** (2018), 364 – 370, DOI: 10.1515/phys-2018-0049.

[4] J. Beckers, J. Patera and P. Winternitz, Subalgebras of real three and four dimensional algebras, *Journal of Mathematical Physics* **18** (1977), 72, DOI: 10.1063/1.523441.

[5] M. L. Gandarias, M. Torrisi and A. Valenti, Symmetry classification and optimal systems of a non-Linear wave equation, *International Journal of Non-Linear Mechanics* **39** (2004), 389 – 398, DOI: 10.1016/S0020-7462(02)00195-6.

[6] E. Hizel, N. C. Turgay and B. Guldogan, Symmetry analysis of three dimensional time independent Schrodinger-Newton equation, *Applied Mathematical Sciences* **2** (2008), 341 – 351, URL: http://www.m-hikari.com/ams/ams-password-2008/ams-password5-8-2008/hizelAMS5-8-2008.pdf.

[7] X.-R. Hu and Y. Chen, Two dimensional symmetry reduction of $(2+1)$-dimensional nonlinear Klien-Gordan Equation, *Applied Mathematics and Computation* **215** (2009), 1141 – 1145, DOI: 10.1016/j.amc.2009.06.049.

[8] B. Jamil, T. Feroze and M. Safdar, Optimal systems and their group-invariant solutions to geodesic equations, *International Journal of Geometric Methods in Modern Physics* **16**(09) (2019), 1950135, DOI: 10.1142/S0219887819501354.

[9] C. M. Khalique and A. Biswas, Analysis of nonlinear Klein-Gordon equation using Lie symmetry analysis, *Applied Mathematics Letter* **23** (2010), 1397 – 1400, DOI: 10.1016/j.aml.2010.07.006.

[10] F. A. Kiraz, A note on one dimensional optimal system of generalized Boussineq equation, *Applied Mathematical Sciences* **2** (2008), 1541 – 1548, URL: http://www.m-hikari.com/ams/ams-password-2008/ams-password29-32-2008/kirazAMS29-32-2008.pdf.

[11] H. Kötz, A technique to classify the similarity solutions of nonlinear partial (integro-)differential equations. I. Optimal systems of solvable Lie subalgebras, *Zeitschrift für Naturforschung A* **47a** (1992), 1161 – 1173, DOI: 10.1515/zna-1992-1114.

[12] H. Lewy, On differential geometry in the large. I. Minkowski's problem, *Transactions of the American Mathematical Society* **43** (1938), 258 – 270, DOI: 10.1090/S0002-9947-1938-1501942-3.

[13] H. Liu, J. Li and L. Liu, Lie symmetry analysis, optimal system and exact solutions to the fifth-order KdV types equation, *Mathematical Analysis and Application* **368** (2010), 551 – 558, DOI: 10.1016/j.jmaa.2010.03.026.

[14] H. Minkowski, Allgemeine Lehrsätze über die convexen Polyeder, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1897** (1897), 198 – 220.

[15] H. Minkowski, Volumen und Oberfläche, *Mathematische Annalen* **57** (1903), 447 – 495, DOI: 10.1007/BF01445180.

[16] G. Monge, Mémoire sur le calcul intégral des équations aux différences partielles, *Mémoires de l'Académie des Sciences Paris, France: Imprimerie Royale* (1784), 118 – 192.

[17] S. T. Niel and W. X.-Jia, *Handbook of Geometric Analysis*, **1**, Higher Education Press, Somerville, USA (2008).

[18] B. F. Nteumagne and R. J. Moitsheki, Optimal systems and group invariant solution for a model arising in financial mathematics, *Mathematical Modelling and Analysis* **14**(4) (2009), 495 – 502, DOI: 10.3846/1392-6292.2009.14.495-502.

[19] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Text in Mathematics, Springer-Verlag, New York (1993).

[20] L. V. Ovsyannikov, *Group Analysis of Differential Equations*, Nauka, Moscow (1978), [in Russian, English translation New York (1982)].

[21] M. Ruggieri and M. P. Speciale, Optimal system and new approximate solutions of a generalized Ames's equation, *Symmetry* **11** (2019), 1230, DOI: 10.3390/sym11101230.

[22] M. Umair and T. Feroze, Exact solutions of Zabolotskaya-Khokhlov equation using optimal system, *International Journal of Geometric Methods in Modern Physics* **15** (2018), 1850111, DOI: 10.1142/S0219887818501116.

[23] M. Yourdkhany, M. Nadjafikhah and M. Toomanian, Preliminary group classification and some exact solutions of 2-Hessian equation, *Bulletin of the Iranian Mathematical Society* **47** (2021), 977 – 994, DOI: 10.1007/s41980-020-00424-3.