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Research Article

On Soft ω-Connectedness in Soft Topological Spaces

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Abstract. This article aims to define soft ω-connected space and soft ω-disconnected space in soft topological spaces. We study the characteristics of these spaces with appropriate examples and discuss their relation with soft connected and soft disconnected spaces.

Keywords. Soft ω-open set; Soft ω-closed set; Soft ω-separated sets; Soft ω-connected space; Soft ω-disconnected space

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1. Introduction

To solve the complicated real life problems in engineering, social science and other fields, several theories like fuzzy set theory [13], theory of interval mathematics [12] etc. have been introduced. But all these theories had some drawbacks. To avoid these drawbacks or inadequacy of the parametrization tool, Molodtsov [5] used an adequate parametrization. He initiated the basic notion of *soft set theory* in 1999 and presented the first result of the theory. He has attracted many researchers to work on this theory. Maji *et al*. [4] applied this theory in 2003, to solve problems in decision making.

As topology is prominent in various branches of mathematics. So, Shabir and Naz [9] formulated the idea of *soft topological spaces*. Later, Hussain and Ahmad [3] studied

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the *properties of soft topologial spaces*. Chen [1] investigated the *properties of soft semi-open sets and soft semi-closed sets*. Connectedness [7] is a powerful aid in topology. Hussain and Ahmad [2] defined and explored the properties of soft connected space in soft topological spaces. Sundaram and John [10] has introduced ω-closed set in topology. Motivating from his idea, Paul [6] proposed soft ω closed sets in soft topological spaces. Inspiring from the idea of Paul [6], and Hussain and Ahmad [3], we define soft ω-connectedness in soft topological spaces with the help of some illustrations.

2. Preliminaries

In this section, we recall some basic definitions which are helpful while proving our main results. Throughout this paper, we shall denote I_U by universal set, Δ by parameter set and $(I_U,\widetilde{\tau},\Delta)$ by soft topological space.

Definition 2.1 ([9])**.** Suppose I_U be an initial universal set, Δ be a parameter set and $\mathcal{P}(I_U)$ denote the power set of I_U. A pair (M_Δ,Δ) is called a soft set over I_U, if M_Δ is a mapping given by $M_\Delta:\Delta\to\mathcal{P}(I_U)$.

Definition 2.2 ([9])**.** For two soft sets (M_{Δ_a},Δ_a) and (N_{Δ_b},Δ_b) over a common universe I_U, where Δ_a ad Δ_b are subsets of Δ, then (M_{Δ_a},Δ_a) is a soft subset of (N_{Δ_b},Δ_b) if

(i) $\Delta_a\subseteq\Delta_b$, and

(ii) for all $\delta_{a_1}\in\Delta_a$, $M_{\Delta_a}(\delta_{a_1})$ and $N_{\Delta_b}(\delta_{a_1})$ are identical approximations.

We write $(M_{\Delta_a},\Delta_a)\widetilde{\subseteq}(N_{\Delta_b},\Delta_b)$.

Remark 2.3 ([9])**.** (M_{Δ_a},Δ_a) is soft superset of (N_{Δ_b},Δ_b), if (N_{Δ_b},Δ_b) is a soft subset of (M_{Δ_a},Δ_a). We denote it by $(M_{\Delta_a},\Delta_a)\widetilde{\supseteq}(N_{\Delta_b},\Delta_b)$.

Definition 2.4 ([9])**.** Two soft sets (M_{Δ_a},Δ_a) and (N_{Δ_b},Δ_b) over a common universe I_U are equal if (M_{Δ_a},Δ_a) is a soft subset of (N_{Δ_b},Δ_b) and (N_{Δ_b},Δ_b) is soft subset of (M_{Δ_a},Δ_a).

Definition 2.5 ([9])**.** Let (M_Δ,Δ) be a soft set over I_U, then (M_Δ,Δ) is

(i) null soft set denoted by $\widetilde{\phi}$ if for all $\delta_1\in\Delta$, $M_\Delta(\delta_1)=\phi$.

(ii) absolute soft set denoted by $\widetilde{I}_U$ if for all $\delta_1\in\Delta, M_\Delta(\delta_1)=I_U$.

Definition 2.6 ([11])**.** A soft set (M_Δ,Δ) over I_U is said to be a soft point if there is exactly one $\delta_1\in\Delta$ such that $M_\Delta(\delta_1)=\{\eta\}$, for some $\eta\in I_U$ and $M_\Delta(\delta)=\phi$ for all $\delta\in\Delta\backslash\{\delta_1\}$. Such a soft point is denoted by $M^{\eta}_{\Delta_{\delta_1}}$. The collection of all soft points of a soft set (M_Δ,Δ) is denoted by $SP(M_\Delta,\Delta)$.

Definition 2.7 ([11])**.** A soft point $M^{\eta}_{\Delta_\delta}$ is said to belong to a soft set (N_Δ,Δ) if $\delta\in\Delta$, $\eta\in I_U$ and $M_\Delta(\delta)=\{\eta\}\subseteq N_\Delta(\delta)$ and we write $M^{\eta}_{\Delta_\delta}\widetilde{\in}(N_\Delta,\Delta)$. Thus, any soft point belongs to absolute soft set.

Definition 2.8 ([9]). The union of two soft sets (M_{Δ_a},Δ_a) and (N_{Δ_b},Δ_b) over the common universe I_U is the soft set (H_{Δ_c},Δ_c), where $\Delta_c=\Delta_a\cup\Delta_b$ and for all $\delta_c\in\Delta_c$,

$$H_{\Delta_c}(\delta)=\begin{cases} M_{\Delta_a}(\delta) & \text{if } \delta\in\Delta_a-\Delta_b\\ N_{\Delta_b}(\delta) & \text{if } \delta\in\Delta_b-\Delta_a\\ M_{\Delta_a}(\delta)\cup N_{\Delta_b} & \text{if } \delta\in\Delta_a\cap\Delta_b\,. \end{cases}$$

We write $(M_{\Delta_a},\Delta_a)\tilde{\cup}(N_{\Delta_b},\Delta_b)=(H_{\Delta_c},\Delta_c)$.

Definition 2.9 ([9]). The intersection (H_{Δ_c},Δ_c) of two soft sets (M_{Δ_a},Δ_a) and (N_{Δ_b},Δ_b) over a common universe I_U, denoted by $(M_{\Delta_a},\Delta_a)\tilde{\cap}(N_{\Delta_b},\Delta_b)$, is defined as $\Delta_c=\Delta_a\cap\Delta_b$, and $H_{\Delta_c}(\delta)=M_{\Delta_a}(\delta)\cap N_{\Delta_b}(\delta)$ for all $\delta\in\Delta_c$.

Definition 2.10 ([9]). The relative complement of a soft set (M_Δ,Δ) denoted by $(M_\Delta,\Delta)^c$ and is defined by $(M_\Delta,\Delta)^c=(M_\Delta^c,\Delta)$ where $M_\Delta^c:\Delta\to\mathcal{P}(I_U)$ is a mapping defined by

$M_\Delta^c(\delta)=I_U-M_\Delta(\delta)$, for all $\delta\in\Delta$.

Definition 2.11 ([9]). Let (M_Δ,Δ) be a soft set over I_U and $I_V(\neq\phi)\subseteq I_U$. Then, the sub soft set of (M_Δ,Δ) over I_V, denoted by $(M_\Delta^{I_V},\Delta)$, is defined as:

$M_\Delta^{I_V}(\delta)=I_V\cap M_\Delta(\delta)$, for all $\delta\in\Delta$.

In other words, $(M_\Delta^{I_V},\Delta)=\tilde{I}_V\tilde{\cap}(M_\Delta,\Delta)$.

Definition 2.12 ([9]). Let I_U be an initial universal set, Δ be the non-empty set of parameters and $\tilde{\tau}$ be the collection of soft sets over I_U, then $\tilde{\tau}$ is a soft topology on I_U, if

(i) $\tilde{\phi},\tilde{I}_U\tilde{\in}\tilde{\tau}$,

(ii) union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$,

(iii) intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

Then, the triplet $(I_U,\tilde{\tau},\Delta)$ is called a soft topological space over I_U. The members of $\tilde{\tau}$ are called soft open sets and complements of them are called soft closed sets in I_U.

Definition 2.13. [9]] Let $(I_U,\tilde{\tau},\Delta)$ be a soft topological space over I_U and I_V be a non-empty subset of I_U. Then, $\tilde{\tau}^{I_V}=\{(M_\Delta^{I_V},\Delta):(M_\Delta,\Delta)\tilde{\in}\tilde{\tau}\}$ is the soft relative topology on I_V and $(I_V,\tilde{\tau}^{I_V},\Delta)$ is called a soft subspace of $(I_U,\tilde{\tau},\Delta)$.

Definition 2.14 ([9]). Let $(I_U,\tilde{\tau},\Delta)$ be a soft topological space and (M_Δ,Δ) be a soft set over I_U, then the soft closure of (M_Δ,Δ), denoted by $\overline{(M_\Delta,\Delta)}$ is defined as the intersection of all soft closed supersets of (M_Δ,Δ).

Remark 2.15 ([9]). If (M_Δ,Δ) is soft closed set, then $\overline{(M_\Delta,\Delta)}=(M_\Delta,\Delta)$.

Definition 2.16 ([2]). Two soft sets (M_Δ,Δ) and (N_Δ,Δ) are said to be *soft disjoint* if $(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}$.

Definition 2.17 ([7]). Let $(I_U,\widetilde{\tau},\Delta)$ be a soft topological space. Two non-null soft sets (M_Δ,Δ) and (N_Δ,Δ) over I_U are *soft separated sets* if $\overline{(M_\Delta,\Delta)}\widetilde{\cap}(N_\Delta,\Delta)=\widetilde{\phi}$ and $(M_\Delta,\Delta)\widetilde{\cap}\overline{(N_\Delta,\Delta)}=\widetilde{\phi}$.

Definition 2.18 ([7]). A soft topological space $(I_U,\widetilde{\tau},\Delta)$ is called *soft disconnected* if we can write $\widetilde{I}_U=(M_\Delta,\Delta)\widetilde{\cup}(N_\Delta,\Delta)$ such that $\overline{(M_\Delta,\Delta)}\widetilde{\cap}(N_\Delta,\Delta)=\widetilde{\phi}$ and $(M_\Delta,\Delta)\widetilde{\cap}\overline{(N_\Delta,\Delta)}=\widetilde{\phi}$, otherwise the space is soft connected space.

Remark 2.19.
(i) Soft discrete topological space with a non-singleton set in $SP(\widetilde{I}_U)$ is always soft disconnected space.
(ii) Soft indiscrete topological space is soft connected space.

3. Properties of Soft ω-Open Set and Soft ω-Closed Set

This section contains the characteristics of soft ω-open sets and soft ω-closed sets with some suitable examples.

Definition 3.1 ([8]). Let (W^0_Δ,Δ) be a soft set over I_U. Then, (W^0_Δ,Δ) is *soft ω-open set* if for any soft semi-closed set (C_Δ,Δ) contained in (W^0_Δ,Δ), we have $(C_\Delta,\Delta)\widetilde{\subseteq}(W^0_\Delta,\Delta)^\circ$. The set of all soft ω-open sets is denoted by $G_{s\omega}(\widetilde{I}_U)$.

Example 3.1. Let $I_U=\{\eta_1,\eta_2\}$, $\Delta=\{\delta_1,\delta_2\}$ and $\widetilde{\tau}=\{\widetilde{\phi},\widetilde{I}_U,\{(\delta_1,\{\eta_1\}),(\delta_2,\{\eta_2\})\}\}$, then $(W^0_\Delta,\Delta)=\{(\delta_1,\phi),(\delta_2,\{\eta_2\})\}\widetilde{\in}G_{s\omega}(\widetilde{I}_U)$.

Proposition 3.2 ([8]). *If* $(W^0_\Delta,\Delta)\in\widetilde{\tau}$, *then* $(W^0_\Delta,\Delta)\in G_{s\omega}(\widetilde{I}_U)$. *But if* $(W^0_\Delta,\Delta)\in G_{s\omega}(\widetilde{I}_U)$, *then it is not necessary that* $(W^0_\Delta,\Delta)\in\widetilde{\tau}$ *which can be seen by Example* 3.1.

Proposition 3.3 ([8]).
(i) *If* $(W^\lambda_\Delta,\Delta)\widetilde{\in}G_{s\omega}(\widetilde{I}_U)$, *then* $\widetilde{\cup}_\lambda(W^\lambda_\Delta,\Delta)\widetilde{\in}G_{s\omega}(\widetilde{I}_U)$.
(ii) *If* (W^1_Δ,Δ) *and* $(W^2_\Delta,\Delta)\widetilde{\in}G_{s\omega}(\widetilde{I}_U)$, *then* $(W^1_\Delta,\Delta)\widetilde{\cap}(W^2_\Delta,\Delta)\widetilde{\in}G_{s\omega}(\widetilde{I}_U)$.

Definition 3.4 ([8]). Consider a soft set (W_Δ,Δ) over I_U. Then, (W_Δ,Δ) is soft ω-closed set if for any soft semi-open set (O_Δ,Δ) containing (W_Δ,Δ), we have $\overline{(W_\Delta,\Delta)}\widetilde{\subseteq}(O_\Delta,\Delta)$. The collection of all soft ω-closed sets is denoted by $F_{s\omega}(\widetilde{I}_U)$.

Example 3.2. Let $I_U=\{\eta_1,\eta_2\}$, $\Delta=\{\delta_1,\delta_2\}$ and $\widetilde{\tau}=\{\widetilde{\phi},\widetilde{I}_U,\{(\delta_1,\{\eta_1\}),(\delta_2,I_U)\}\}$. Then, $(W_\Delta,\Delta)=\{(\delta_1,I_U),(\delta_2,\{\eta_1\})\}\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$.

Proposition 3.5 ([8]). *If* $(W_\Delta,\Delta)\widetilde{\in}\widetilde{\tau}^c$, *then* $(W_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$. *But the converse is not true by Example* 3.2.

Proposition 3.6 ([8]).
(i) *If* $(M_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$ *and* $(N_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$, *then* $(M_\Delta,\Delta)\widetilde{\cup}(N_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$.
(ii) *If* $(W^\lambda_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$, *then* $\widetilde{\cap}_\lambda(W^\lambda_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)$.

Definition 3.7 ([8]). Consider a soft set (W_Δ,Δ) over I_U. Then, the soft ω-closure of (W_Δ,Δ), denoted by $\{\overline{(W_\Delta,\Delta)}\}_\omega$, is defined as the intersection of all soft ω-closed supersets of (W_Δ,Δ) i.e., $\{\overline{(W_\Delta,\Delta)}\}_\omega=\widetilde{\cap}\{(F_\Delta,\Delta):(W_\Delta,\Delta)\widetilde{\subseteq}(F_\Delta,\Delta),(W_\Delta,\Delta)\widetilde{\in}F_{s\omega}(\widetilde{I}_U)\}$.

Example 3.3. Let $I_U = \{\eta_1, \eta_2\}$, $\Delta = \{\delta_1, \delta_2\}$, $\tilde{\tau} = \{\tilde{\phi}, \tilde{I}_U, \{(\delta_1, \{\eta_2\}), (\delta_2, \phi)\}\}$ be a soft topology on I_U and $(W_\Delta, \Delta) = \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_2\})\}$ be a soft set over I_U, then

$$\{\overline{(W_\Delta, \Delta)}\}_\omega = \{(\delta_1, \{\eta_1\}), (\delta_2, I_U)\}.$$

Remark 3.8 ([8]). (i) $(W_\Delta, \Delta) \tilde{\subseteq} \{\overline{(W_\Delta, \Delta)}\}_\omega$.

(ii) $\{\overline{(W_\Delta, \Delta)}\}_\omega \tilde{\in} F_{s\omega}(\tilde{I}_U)$.

Lemma 3.9 ([8]). *$\{\overline{(W_\Delta, \Delta)}\}_\omega$ is the smallest soft ω-closed set containing (W_Δ, Δ).*

Lemma 3.10 ([8]). *A soft set $(W_\Delta, \Delta) \tilde{\in} F_{s\omega}(\tilde{I}_U)$ if and only if $\{\overline{(W_\Delta, \Delta)}\}_\omega = (W_\Delta, \Delta)$.*

Lemma 3.11 ([8]). *Consider a soft topological space $(I_U, \tilde{\tau}, \Delta)$, (W_Δ, Δ) and (W'_Δ, Δ) are soft sets over I_U, then*

(i) $\{\overline{\tilde{\phi}}\}_\omega = \tilde{\phi}$ *and* $\{\overline{\tilde{I}_U}\}_\omega = \tilde{I}_U$.

(ii) $[\overline{\{\overline{(W_\Delta, \Delta)}\}_\omega}]_\omega = \{\overline{(W_\Delta, \Delta)}\}_\omega$.

(iii) *If* $(W_\Delta, \Delta) \tilde{\subseteq} (W'_\Delta, \Delta)$, *then* $\{\overline{(W_\Delta, \Delta)}\}_\omega \tilde{\subseteq} \{\overline{(W'_\Delta, \Delta)}\}_\omega$.

(iv) $\{\overline{(W_\Delta \cup W'_\Delta, \Delta)}\}_\omega = \{\overline{(W_\Delta, \Delta)}\}_\omega \tilde{\cup} \{\overline{(W'_\Delta, \Delta)}\}_\omega$.

(v) $\{\overline{(W_\Delta \cap W'_\Delta, \Delta)}\}_\omega \tilde{\subseteq} \{\overline{(W_\Delta, \Delta)}\}_\omega \tilde{\cap} \{\overline{(W'_\Delta, \Delta)}\}_\omega$.

(vi) $\{\overline{(W_\Delta, \Delta)}\}_\omega \tilde{\subseteq} \overline{(W_\Delta, \Delta)}$.

4. Soft ω-Separated Sets

In this section, we introduce the concept of soft ω-separated sets and study their main properties. We consider the various examples to show the relation of soft ω-separated sets with soft separated sets and soft ω-closed sets.

Definition 4.1. Consider two non null soft sets (M_Δ, Δ) and (N_Δ, Δ) in soft topological space $(I_U, \tilde{\tau}, \Delta)$, then (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets if and only if

$$\{\overline{(M_\Delta, \Delta)}\}_\omega \tilde{\cap} (N_\Delta, \Delta) = \tilde{\phi} \text{ and } \{\overline{(N_\Delta, \Delta)}\}_\omega \tilde{\cap} (M_\Delta, \Delta) = \tilde{\phi}.$$

Example 4.1. Let $I_U = \{\eta_1, \eta_2\}$, $\Delta = \{\delta_1, \delta_2\}$ and $\tilde{\tau} = \{\tilde{\phi}, \tilde{I}_U, \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_2\})\}\}$. Consider two soft sets $(M_\Delta, \Delta) = \{(\delta_1, \{\eta_1\}), (\delta_2, \phi)\}$ and $(N_\Delta, \Delta) = \{(\delta_1, \phi), (\delta_2, \{\eta_2\})\}$ over I_U, then $\{\overline{(M_\Delta, \Delta)}\}_\omega = \{(\delta_1, I_U), (\delta_2, \{\eta_1\})\}$ and $\{\overline{(N_\Delta, \Delta)}\}_\omega = \{(\delta_1, \{\eta_2\}), (\delta_2, I_U)\}$ such that

$$\{\overline{(M_\Delta, \Delta)}\}_\omega \tilde{\cap} (N_\Delta, \Delta) = \tilde{\phi} \text{ and } \{\overline{(N_\Delta, \Delta)}\}_\omega \tilde{\cap} (M_\Delta, \Delta) = \tilde{\phi}.$$

Thus, (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets.

Remark 4.2. (i) If (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets, then they are disjoint.

Proof. As (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets, thus

$$\{\overline{(M_\Delta, \Delta)}\}_\omega \tilde{\cap} (N_\Delta, \Delta) = \tilde{\phi} \text{ and } \{\overline{(N_\Delta, \Delta)}\}_\omega \tilde{\cap} (M_\Delta, \Delta) = \tilde{\phi}.$$

Now,

$$(M_\Delta, \Delta) \tilde{\subseteq} \{\overline{(M_\Delta, \Delta)}\}_\omega$$

$$\Rightarrow \quad (M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)\tilde{\subseteq}\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}$$

$$\Rightarrow \quad (M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}.$$

Thus,(M_Δ,Δ) and (N_Δ,Δ) are disjoint soft sets.

(ii) If two soft sets are disjoint, then they need not be soft ω-separated sets.

Example. Consider a *soft topological space* as in Example 4.1. Let $(M_\Delta,\Delta)=\{(\delta_1,\{\eta_1\}),(\delta_2,\{\eta_2\})\}$ and $(N_\Delta,\Delta)=\{(\delta_1,\{\eta_2\}),(\delta_2,\{\eta_1\})\}$ be two soft sets over I_U, then (M_Δ,Δ) and (N_Δ,Δ) are disjoint soft sets but they are not soft ω-separated sets as $\{\overline{(M_\Delta,\Delta)}\}_\omega=\tilde{I}_U$ and $\{\overline{(N_\Delta,\Delta)}\}_\omega=\{(\delta_1,\{\eta_2\}),(\delta_2,\{\eta_1\})\}$ and $\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)\neq\tilde{\phi}$.

Theorem 4.3. *If* $(M_\Delta,\Delta)\tilde{\in}F_{s\omega}(\tilde{I}_U)$ *and* $(N_\Delta,\Delta)\tilde{\in}F_{s\omega}(\tilde{I}_U)$*, then* (M_Δ,Δ) *and* (N_Δ,Δ) *are soft* ω*-separated if and only if they are disjoint.*

Proof. Let (M_Δ,Δ) and (N_Δ,Δ) be two soft ω-closed disjoint sets over I_U, then by lemma 3.10, we have

$$\{\overline{(M_\Delta,\Delta)}\}_\omega=(M_\Delta,\Delta) \text{ and } \{\overline{(N_\Delta,\Delta)}\}_\omega=(N_\Delta,\Delta).$$

As (M_Δ,Δ) and (N_Δ,Δ) are disjoint soft sets, thus

$$\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)=(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}$$

and

$$\{\overline{(N_\Delta,\Delta)}\}_\omega\tilde{\cap}(M_\Delta,\Delta)=(N_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)=\tilde{\phi}.$$

Thus, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets.

Conversely, if two soft sets are ω-separated, then we already have proved that they are disjoint from Remark 4.2(i). □

Remark 4.4. If two soft sets are soft ω-separated, then they need not be soft ω-closed set which can be seen from Example 4.1.

Theorem 4.5. *Two soft separated sets are soft* ω*-separated sets.*

Proof. Let (M_Δ,Δ) and (N_Δ,Δ) be two soft separated sets over I_U, then $\overline{(M_\Delta,\Delta)}\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}$ and $\overline{(N_\Delta,\Delta)}\tilde{\cap}(M_\Delta,\Delta)=\tilde{\phi}$.

As

$$\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\subseteq}\overline{(M_\Delta,\Delta)}$$

$$\Rightarrow \quad \{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)\tilde{\subseteq}\overline{(M_\Delta,\Delta)}\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}$$

and similarly,

$$\{\overline{(N_\Delta,\Delta)}\}_\omega\tilde{\cap}(M_\Delta,\Delta)\tilde{\subseteq}\overline{(N_\Delta,\Delta)}\tilde{\cap}(M_\Delta,\Delta)=\tilde{\phi}.$$

Thus, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets. □

Remark 4.6. The converse of above theorem is not true i.e., two soft ω-separated sets are not necessarily soft separated sets by the following example:

Let $I_U = \{\eta_1, \eta_2\}$, $\Delta = \{\delta_1, \delta_2\}$ and $\widetilde{\tau} = \{\widetilde{\phi}, \widetilde{I}_U\}$, then $(I_U, \widetilde{\tau}, \Delta)$ is a soft topological space. Consider two soft sets $(M_\Delta, \Delta) = \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_2\})\}$ and $(N_\Delta, \Delta) = \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_1\})\}$ over I_U. Then, $\{\overline{(M_\Delta, \Delta)}\}_\omega = \{(\delta_1, \{\eta_1\}), (\delta_2, \{\eta_2\})\}$ and $\{\overline{(N_\Delta, \Delta)}\}_\omega = \{(\delta_1, \{\eta_2\}), (\delta_2, \{\eta_1\})\}$. Thus, (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets. But they are not soft separated sets as $\overline{(M_\Delta, \Delta)} = \widetilde{I}_U$ and $\overline{(N_\Delta, \Delta)} = \widetilde{I}_U$.

Theorem 4.7. *If (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets, then*

(i) *if $(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$ is soft ω-closed, then (M_Δ, Δ) and (N_Δ, Δ) are soft ω-closed sets.*

(ii) *if $(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$ is soft ω-open, then (M_Δ, Δ) and (N_Δ, Δ) are soft ω-open sets.*

Proof. Since (M_Δ, Δ) and (N_Δ, Δ) are soft ω-separated sets, therefore $\{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\cap} (N_\Delta, \Delta) = \widetilde{\phi}$ and $(M_\Delta, \Delta) \widetilde{\cap} \{\overline{(N_\Delta, \Delta)}\}_\omega = \widetilde{\phi}$

(i) If $(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$ is soft ω-closed set, then

$$\{\overline{(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)}\}_\omega = (M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$$
$$\Rightarrow \quad \{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\cup} \{\overline{(N_\Delta, \Delta)}\}_\omega = (M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$$

Now,

$$\{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\subseteq} \{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\cup} \{\overline{(N_\Delta, \Delta)}\}_\omega = (M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$$
$$\Rightarrow \quad \{\{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\cap} \{(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)\} = \{\overline{(M_\Delta, \Delta)}\}_\omega$$
$$\Rightarrow \quad \{\{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\cap} (M_\Delta, \Delta)\} \widetilde{\cup} \{\{\overline{(M_\Delta, \Delta)}\}_\omega \widetilde{\cap} (N_\Delta, \Delta)\} = \{\overline{(M_\Delta, \Delta)}\}_\omega$$
$$\Rightarrow \quad (M_\Delta, \Delta)\widetilde{\cup}\widetilde{\phi} = \{\overline{(M_\Delta, \Delta)}\}_\omega \qquad (\because (M_\Delta, \Delta) \widetilde{\subseteq} \{\overline{(M_\Delta, \Delta)}\}_\omega)$$
$$\Rightarrow \quad (M_\Delta, \Delta) = \{\overline{(M_\Delta, \Delta)}\}_\omega.$$

Therefore (M_Δ, Δ) is soft ω-closed.

Similarly, we can show that (N_Δ, Δ) is also soft ω-closed.

(ii) Let $(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)$ be soft ω-open set. Since $\{\overline{(N_\Delta, \Delta)}\}_\omega$ is soft ω-closed set, therefore $\{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c$ is soft ω-open set. Thus,

$$\{(M_\Delta, \Delta)\widetilde{\cup}(N_\Delta, \Delta)\} \widetilde{\cap} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c \text{ is soft } \omega\text{-open set} \quad \text{(by using Proposition 3.3(ii))}$$
$$\Rightarrow \quad \{(M_\Delta, \Delta) \widetilde{\cap} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c\} \widetilde{\cup} \{(N_\Delta, \Delta) \widetilde{\cap} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c\} \text{ is soft } \omega\text{-open set} \tag{4.1}$$

As

$$(M_\Delta, \Delta) \widetilde{\cap} \{\overline{(N_\Delta, \Delta)}\}_\omega = \widetilde{\phi}$$
$$\Rightarrow \quad (M_\Delta, \Delta) \widetilde{\subseteq} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c$$
$$\Rightarrow \quad (M_\Delta, \Delta) \widetilde{\cap} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c = (M_\Delta, \Delta). \tag{4.2}$$

Also,

$$(N_\Delta, \Delta) \widetilde{\subseteq} \{\overline{(N_\Delta, \Delta)}\}_\omega$$
$$\Rightarrow \quad \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c \widetilde{\subseteq} (N_\Delta, \Delta)^c$$
$$\Rightarrow \quad (N_\Delta, \Delta) \widetilde{\cap} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c \widetilde{\subseteq} \widetilde{\phi}$$
$$\Rightarrow \quad (N_\Delta, \Delta) \widetilde{\cap} \{\{\overline{(N_\Delta, \Delta)}\}_\omega\}^c = \widetilde{\phi} \tag{4.3}$$

Thus, using (4.5) and (4.6) in (4.4), we have $(M_\Delta,\Delta)\widetilde{\cup}\widetilde{\phi}=(M_\Delta,\Delta)$ is soft ω-open. Similarly, it can be shown that (N_Δ,Δ) is also soft ω-open set. □

Theorem 4.8. *Two soft disjoint sets (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated if and only if they are soft ω-open and soft ω-closed sets in $(M_\Delta\cup N_\Delta,\Delta)$ with soft relative topology.*

Proof. Let (M_Δ,Δ) and (N_Δ,Δ) be two disjoint soft ω-separated sets. Thus,

$$\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(N_\Delta,\Delta)=\widetilde{\phi}\ \text{ and }\ (M_\Delta,\Delta)\,\widetilde{\cap}\,\{\overline{(N_\Delta,\Delta)}\}_\omega=\widetilde{\phi}$$

Let $(M_\Delta\cup N_\Delta,\Delta)=(V_\Delta,\Delta)$, then

$$\begin{aligned}\{\overline{(M_\Delta,\Delta)}\}_\omega^{V_\Delta}&=\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(V_\Delta,\Delta)\\&=\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,\{(M_\Delta,\Delta)\widetilde{\cup}(N_\Delta,\Delta)\}\\&=\{\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(M_\Delta,\Delta)\}\,\widetilde{\cup}\,\{\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(N_\Delta,\Delta)\}\\&=(M_\Delta,\Delta)\widetilde{\cup}\widetilde{\phi}=(M_\Delta,\Delta).\end{aligned}$$

Thus, (M_Δ,Δ) is soft ω-closed in (V_Δ,Δ). Similarly, it can be shown that (N_Δ,Δ) is also soft ω-closed in (V_Δ,Δ).

Since $(M_\Delta,\Delta)\widetilde{\cap}(N_\Delta,\Delta)=\widetilde{\phi}$ and $(M_\Delta,\Delta)\widetilde{\cup}(N_\Delta,\Delta)=(V_\Delta,\Delta)$, therefore $(M_\Delta,\Delta)=(V_\Delta,\Delta)-(N_\Delta,\Delta)$ and $(N_\Delta,\Delta)=(V_\Delta,\Delta)-(M_\Delta,\Delta)$.

Now, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-closed set in (V_Δ,Δ), thus $(M_\Delta,\Delta)=(V_\Delta,\Delta)-(N_\Delta,\Delta)$ and $(N_\Delta,\Delta)=(V_\Delta,\Delta)-(M_\Delta,\Delta)$ are soft ω-open set in (V_Δ,Δ). Thus, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-open and soft ω-closed in (V_Δ,Δ).

Conversely, let (M_Δ,Δ) and (N_Δ,Δ) be two disjoint soft sets which are both soft ω-open as well as soft ω-closed in $(V_\Delta,\Delta)=(M_\Delta\cup N_\Delta,\Delta)$.

Now,

$$\begin{aligned}&(M_\Delta,\Delta)\widetilde{\subseteq}(V_\Delta,\Delta)\ \text{ and }\ (N_\Delta,\Delta)\widetilde{\subseteq}(V_\Delta,\Delta)\\\Rightarrow\ &\{\overline{(M_\Delta,\Delta)}\}_\omega^{V_\Delta}=(M_\Delta,\Delta)\ \text{ and }\ \{\overline{(N_\Delta,\Delta)}\}_\omega^{V_\Delta}=(N_\Delta,\Delta)\\\Rightarrow\ &\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(V_\Delta,\Delta)=(M_\Delta,\Delta)\ \text{ and }\ \{\overline{(N_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(V_\Delta,\Delta)=(N_\Delta,\Delta)\end{aligned}$$

Thus,

$$\begin{aligned}(M_\Delta,\Delta)\widetilde{\cap}(N_\Delta,\Delta)&=\{\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(V_\Delta,\Delta)\}\,\widetilde{\cap}\,(N_\Delta,\Delta)\\&=\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,\{(V_\Delta,\Delta)\widetilde{\cap}(N_\Delta,\Delta)\}\\&=\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(N_\Delta,\Delta)\end{aligned}$$

$$\Rightarrow\quad\{\overline{(M_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(N_\Delta,\Delta)=\widetilde{\phi}$$

Similarly, $\{\overline{(N_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(M_\Delta,\Delta)=\widetilde{\phi}$. Thus, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets. □

Theorem 4.9. *If (F_Δ,Δ) and (G_Δ,Δ) are soft ω-separated sets and $(M_\Delta,\Delta)\widetilde{\subseteq}(F_\Delta,\Delta)$ and $(N_\Delta,\Delta)\widetilde{\subseteq}(G_\Delta,\Delta)$, then (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets.*

Proof. Since (F_Δ,Δ) and (G_Δ,Δ) are soft ω-separated sets, therefore $\{\overline{(F_\Delta,\Delta)}\}_\omega\,\widetilde{\cap}\,(G_\Delta,\Delta)=\widetilde{\phi}$ and $(F_\Delta,\Delta)\,\widetilde{\cap}\,\{\overline{(G_\Delta,\Delta)}\}_\omega=\widetilde{\phi}$.

As

$$(M_\Delta,\Delta)\widetilde{\subseteq}(F_\Delta,\Delta) \text{ and } (N_\Delta,\Delta)\widetilde{\subseteq}(G_\Delta,\Delta)$$

$$\Rightarrow \quad \{\overline{(M_\Delta,\Delta)}\}_\omega \widetilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega \text{ and } \{\overline{(N_\Delta,\Delta)}\}_\omega \widetilde{\subseteq} \{\overline{(G_\Delta,\Delta)}\}_\omega$$

$$\Rightarrow \quad \{\overline{(M_\Delta,\Delta)}\}_\omega \widetilde{\cap} (N_\Delta,\Delta) \widetilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega \widetilde{\cap} (N_\Delta,\Delta) \widetilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega \widetilde{\cap} (G_\Delta,\Delta) = \widetilde{\phi}$$

$$\Rightarrow \quad \{\overline{(M_\Delta,\Delta)}\}_\omega \widetilde{\cap} (N_\Delta,\Delta) = \widetilde{\phi}$$

Similarly, $\{\overline{(N_\Delta,\Delta)}\}_\omega \widetilde{\cap} (M_\Delta,\Delta) = \widetilde{\phi}$. Thus, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets. □

Theorem 4.10. *If (M_Δ,Δ) and (N_Δ,Δ) are soft ω-closed sets, then $(M_\Delta,\Delta)\widetilde{\cap}(N^c_\Delta,\Delta)$ and $(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)$ are soft ω-separated sets.*

Proof. Since (M_Δ,Δ) and (N_Δ,Δ) are soft ω-closed sets, therefore

$$\overline{(M_\Delta,\Delta)_\omega} = (M_\Delta,\Delta) \text{ and } \overline{(N_\Delta,\Delta)_\omega} = (N_\Delta,\Delta).$$

Now,

$$\begin{aligned}\{\overline{(M_\Delta,\Delta)\widetilde{\cap}(N^c_\Delta,\Delta)}\}_\omega \widetilde{\cap} \{(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)\} &\widetilde{\subseteq} \{\overline{(M_\Delta,\Delta)_\omega} \widetilde{\cap} \overline{(N^c_\Delta,\Delta)_\omega}\} \widetilde{\cap} \{(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)\} \\ &= \{(M_\Delta,\Delta)\widetilde{\cap}\overline{(N^c_\Delta,\Delta)_\omega}\} \widetilde{\cap} \{(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)\} \\ &= \{(M_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)\} \widetilde{\cap} \{\overline{(N^c_\Delta,\Delta)_\omega} \widetilde{\cap} (N_\Delta,\Delta)\} \\ &= \widetilde{\phi}.\end{aligned}$$

Similarly, $\{(M_\Delta,\Delta)\widetilde{\cap}(N^c_\Delta,\Delta)\} \widetilde{\cap} \{\overline{(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)}\}_\omega = \widetilde{\phi}$.
Thus, $(M_\Delta,\Delta)\widetilde{\cap}(N^c_\Delta,\Delta)$ and $(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)$ are soft ω-separated sets. □

Remark 4.11. If (M_Δ,Δ) and (N_Δ,Δ) are soft ω-open sets, then $(M_\Delta,\Delta)\widetilde{\cap}(N^c_\Delta,\Delta)$ and $(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)$ are soft ω-separated sets.

Proof. As (M_Δ,Δ) and (N_Δ,Δ) are soft ω-open sets, thus $(M_\Delta,\Delta)^c$ and $(N_\Delta,\Delta)^c$ are soft ω-closed sets. Using above theorem, we have $(M_\Delta,\Delta)\widetilde{\cap}(N^c_\Delta,\Delta)$ and $(N_\Delta,\Delta)\widetilde{\cap}(M^c_\Delta,\Delta)$ are soft ω-separated sets. □

Theorem 4.12. *If (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets of soft topological space $(I_U,\widetilde{\tau},\Delta)$ such that $\widetilde{I}_U = (M_\Delta,\Delta)\widetilde{\cup}(N_\Delta,\Delta)$, then $(M_\Delta,\Delta)^c$ and $(N_\Delta,\Delta)^c$ are also soft ω-separated sets.*

Proof. Since (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets, therefore

$$\overline{(M_\Delta,\Delta)_\omega}\widetilde{\cap}(N_\Delta,\Delta) = \widetilde{\phi} \text{ and } \overline{(N_\Delta,\Delta)_\omega}\widetilde{\cap}(M_\Delta,\Delta) = \widetilde{\phi}.$$

As

$$\widetilde{I}_U = (M_\Delta,\Delta)\widetilde{\cup}(N_\Delta,\Delta) \text{ and } (M_\Delta,\Delta)\widetilde{\cap}(N_\Delta,\Delta) = \widetilde{\phi},$$

thus we have

$$(M_\Delta,\Delta) = (N_\Delta,\Delta)^c \text{ and } (N_\Delta,\Delta) = (M_\Delta,\Delta)^c.$$

Now,

$$\{\overline{(M_\Delta,\Delta)^c}\}_\omega \widetilde{\cap} (N_\Delta,\Delta)^c = \{\overline{(N_\Delta,\Delta)}\}_\omega \widetilde{\cap} (M_\Delta,\Delta) = \widetilde{\phi}$$

and

$$\{\overline{(N_\Delta,\Delta)^c}\}_\omega \tilde{\cap} (M_\Delta,\Delta)^c = \{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\cap} (N_\Delta,\Delta) = \tilde{\phi}.$$

Therefore, $(M_\Delta,\Delta)^c$ and $(N_\Delta,\Delta)^c$ are soft ω-separated sets. □

Theorem 4.13. *If (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets of soft topological space $(I_U,\tilde{\tau},\Delta)$ such that $\tilde{I}_U = (M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$, then for every soft set $(F_\Delta,\Delta)\tilde{\subseteq}\tilde{I}_U$, we have*

$$\{\overline{(F_\Delta,\Delta)}\}_\omega = [\{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)}\}_\omega \tilde{\cap} (M_\Delta,\Delta)] \tilde{\cup} [\{\overline{(F_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)}\}_\omega \tilde{\cap} (N_\Delta,\Delta)]$$

Proof. We know that

$$(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$$

$$\Rightarrow \{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)}\}_\omega \tilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega$$

$$\Rightarrow \{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)}\}_\omega \tilde{\cap}(M_\Delta,\Delta) \tilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega \tilde{\cap}(M_\Delta,\Delta) \tilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega . \tag{4.4}$$

Similarly, we have

$$\{\overline{(F_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)}\}_\omega \tilde{\cap}(N_\Delta,\Delta) \tilde{\subseteq} \overline{(F_\Delta,\Delta)}_\omega \tilde{\cap}(N_\Delta,\Delta) \tilde{\subseteq} \{\overline{(F_\Delta,\Delta)}\}_\omega . \tag{4.5}$$

From (4.4) and (4.5), we have

$$[\{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)}\}_\omega \tilde{\cap}(M_\Delta,\Delta)] \tilde{\cup} [\{\overline{(F_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)}\}_\omega \tilde{\cap}(N_\Delta,\Delta)] \tilde{\subseteq} \overline{(F_\Delta,\Delta)}_\omega . \tag{4.6}$$

Now,

$$\begin{aligned}(F_\Delta,\Delta) &= (F_\Delta,\Delta)\tilde{\cap}\tilde{I}_U \\ &= (F_\Delta,\Delta)\tilde{\cap}\{(M_\Delta,\Delta)\tilde{\cup}(M_\Delta,\Delta)^c\} \\ &= \{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)\}\tilde{\cup}\{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c\}\end{aligned}$$

therefore

$$\{\overline{(F_\Delta,\Delta)}\}_\omega = \{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)}\}_\omega \tilde{\cup} \{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c}\}_\omega$$

$$\Rightarrow \{\overline{(F_\Delta,\Delta)}\}_\omega \tilde{\cap}(M_\Delta,\Delta) \tilde{\subseteq} [\{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)}\}_\omega \tilde{\cap}(M_\Delta,\Delta)] \tilde{\cup} [\{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c}\}_\omega]. \tag{4.7}$$

As (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets, thus by theorem 4.11, $(M_\Delta,\Delta)^c$ and $(N_\Delta,\Delta)^c$ are also soft ω-separated sets, therefore

$$\{\overline{(M_\Delta,\Delta)^c}\}_\omega \tilde{\cap}(N_\Delta,\Delta)^c = \tilde{\phi} \text{ and } \{\overline{(N_\Delta,\Delta)^c}\}_\omega \tilde{\cap}(M_\Delta,\Delta)^c = \tilde{\phi}$$

$$\Rightarrow \{\overline{(M_\Delta,\Delta)^c}\}_\omega \tilde{\subseteq}(N_\Delta,\Delta)$$

$$\Rightarrow (M_\Delta,\Delta)^c \tilde{\subseteq}(N_\Delta,\Delta) \quad [\text{As } (M_\Delta,\Delta)^c \tilde{\subseteq} \{\overline{(M_\Delta,\Delta)^c}\}_\omega].$$

Now,

$$(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c \tilde{\subseteq}(F_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)$$

$$\Rightarrow \{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c}\}_\omega \tilde{\subseteq} \{\overline{(F_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)}_\omega . \tag{4.8}$$

Also,

$$(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c \tilde{\subseteq}(M_\Delta,\Delta)^c$$

$$\Rightarrow \{\overline{(F_\Delta,\Delta)\tilde{\cap}(M_\Delta,\Delta)^c}\}_\omega \tilde{\subseteq} \{\overline{(M_\Delta,\Delta)^c}\}_\omega \tilde{\subseteq}(N_\Delta,\Delta). \tag{4.9}$$

From (4.8) and (4.9), we have

$$\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)^c}\}_\omega \tilde{\sqsubseteq} \{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta). \tag{4.10}$$

From (4.7) and (4.10), we have

$$\begin{aligned}\{\overline{(F_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta) \tilde{\sqsubseteq} & [\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \\ & \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)].\end{aligned} \tag{4.11}$$

Similarly,

$$\{\overline{(F_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta) \tilde{\sqsubseteq} [\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)]. \tag{4.12}$$

Taking union of (4.11) and (4.12), we have

$$\begin{aligned}& \overline{(F_\Delta,\Delta)}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \tilde{\sqcup}[\overline{(F_\Delta,\Delta)}_\omega \tilde{\sqcap}(N_\Delta,\Delta)] \\ & \quad \tilde{\sqsubseteq}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)] \\ \Rightarrow \quad & \{\overline{(F_\Delta,\Delta)}\}_\omega \tilde{\sqcap}\{(M_\Delta,\Delta)\tilde{\sqcup}(N_\Delta,\Delta)\} \tilde{\sqsubseteq}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \\ & \quad \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)] \\ \Rightarrow \quad & \{\overline{(F,P')}\}_\omega \tilde{\sqcap} \ \tilde{I}_U \tilde{\sqsubseteq} \ [\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)] \\ \Rightarrow \quad & \{\overline{(F_\Delta,\Delta)}\}_\omega \tilde{\sqsubseteq}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)].\end{aligned} \tag{4.13}$$

From (4.6) and (4.13), we get

$$\{\overline{(F_\Delta,\Delta)}\}_\omega = [\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(M_\Delta,\Delta)] \tilde{\sqcup}[\{\overline{(F_\Delta,\Delta)\tilde{\sqcap}(N_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta). \qquad \square$$

Theorem 4.14. *Two soft sets (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets if and only if there exist two soft ω-open sets (F_Δ,Δ) and (G_Δ,Δ) such that $(M_\Delta,\Delta)\tilde{\sqsubseteq}(F_\Delta,\Delta)$, $(N_\Delta,\Delta)\tilde{\sqsubseteq}(G_\Delta,\Delta)$ and $(M_\Delta,\Delta)\tilde{\sqcap}(G_\Delta,\Delta)=\tilde{\phi}$ and $(N_\Delta,\Delta)\tilde{\sqcap}(F_\Delta,\Delta)=\tilde{\phi}$.*

Proof. Let (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets. Then,

$$(M_\Delta,\Delta)\tilde{\sqcap}\{\overline{(N_\Delta,\Delta)}\}_\omega = \tilde{\phi} \ \text{ and } \ \{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\sqcap}(N_\Delta,\Delta)=\tilde{\phi}.$$

Let $(G_\Delta,\Delta)=\{\overline{(M_\Delta,\Delta)}_\omega\}^c$ and $(F_\Delta,\Delta)=\{\overline{(N_\Delta,\Delta)}_\omega\}^c$. Then, (F_Δ,Δ) and (G_Δ,Δ) are soft ω-open sets such that

$$(M_\Delta,\Delta)\tilde{\sqsubseteq}(F_\Delta,\Delta),(N_\Delta,\Delta)\tilde{\sqsubseteq}(G_\Delta,\Delta)$$

and

$$\begin{aligned}& (M_\Delta,\Delta)\tilde{\sqcap}(G_\Delta,\Delta)=(M_\Delta,\Delta)\tilde{\sqcap}\{\overline{(M_\Delta,\Delta)}_\omega\}^c \tilde{\sqsubseteq}(M_\Delta,\Delta)\tilde{\sqcap}(M_\Delta,\Delta)^c=\tilde{\phi} \\ \Rightarrow \quad & (M_\Delta,\Delta)\tilde{\sqcap}(G_\Delta,\Delta)=\tilde{\phi}.\end{aligned}$$

Similarly, we have $(N_\Delta,\Delta)\tilde{\sqcap}(F_\Delta,\Delta)=\tilde{\phi}$.

Conversely, let (F_Δ,Δ) and (G_Δ,Δ) be two soft ω-open sets such that $(M_\Delta,\Delta)\tilde{\sqsubseteq}(F_\Delta,\Delta)$, $(N_\Delta,\Delta)\tilde{\sqsubseteq}(G_\Delta,\Delta)$, $(M_\Delta,\Delta)\tilde{\sqcap}(G_\Delta,\Delta)=\tilde{\phi}$ and $(N_\Delta,\Delta)\tilde{\sqcap}(F_\Delta,\Delta)=\tilde{\phi}$.

Since $(F_\Delta,\Delta)^c$ and $(G_\Delta,\Delta)^c$ are soft ω-closed sets, therefore $\{\overline{(F_\Delta,\Delta)^c}\}_\omega=(F_\Delta,\Delta)$ and $\{\overline{(G_\Delta,\Delta)^c}\}_\omega=(G_\Delta,\Delta)$. Now,

$$\begin{aligned}& (M_\Delta,\Delta)\tilde{\sqsubseteq}(G_\Delta,\Delta)^c \\ \Rightarrow \quad & \{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\sqsubseteq}\{\overline{(G_\Delta,\Delta)^c}\}_\omega=(G_\Delta,\Delta)^c \tilde{\sqsubseteq}(N_\Delta,\Delta)^c\end{aligned}$$

$$\Rightarrow \quad \{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\cap} (N_\Delta,\Delta) = \tilde{\phi}$$

and

$$(N_\Delta,\Delta) \tilde{\subseteq} (F_\Delta,\Delta)^c$$
$$\Rightarrow \quad \{\overline{(N_\Delta,\Delta)}\}_\omega \tilde{\subseteq} \{\overline{(F_\Delta,\Delta)^c}\}_\omega = (F_\Delta,\Delta)^c \tilde{\subseteq} (M_\Delta,\Delta)^c$$
$$\Rightarrow \quad (M_\Delta,\Delta) \tilde{\cap} \{\overline{(N_\Delta,\Delta)}\}_\omega = \tilde{\phi}.$$

Thus, (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets. □

5. Soft ω-Disconnected and Soft ω-Connected Space

In this section, we establish a new concept of soft ω-connected and soft ω-disconnected spaces. We further discuss some properties of these spaces with some suitable examples.

Definition 5.1. A soft topological space $(I_U,\tilde{\tau},\Delta)$ is soft ω-disconnected space if we can write $\tilde{I}_U$ as union of two non-null soft ω-separated sets, i.e.,

$$\tilde{I}_U = (M_\Delta,\Delta) \tilde{\cup} (N_\Delta,\Delta),$$

where $(M_\Delta,\Delta) \neq \tilde{\phi}, (N_\Delta,\Delta) \neq \tilde{\phi}$ and $\{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\cap} (N_\Delta,\Delta) = \tilde{\phi}$ and $(M_\Delta,\Delta) \tilde{\cap} \{\overline{(N_\Delta,\Delta)}\}_\omega = \tilde{\phi}$, otherwise the soft topological space $(I_U,\tilde{\tau},\Delta)$ is called soft ω-connected space.

Remark 5.2. (i) An indiscrete soft topological space with non-singleton set in $SP(\tilde{I}_U)$ is always soft ω-disconnected.

(ii) A discrete soft topological space with non-singleton set in $SP(\tilde{I}_U)$ is always soft ω-disconnected.

Example 5.1. Consider $I_U = \{\eta_1,\eta_2\}, \Delta = \{\delta_1,\delta_2\}$ and $\tilde{\tau} = \{\tilde{\phi}, \tilde{I}_U, \{(\delta_1,\{\eta_1\}),(\delta_2,\phi)\}, \{(\delta_1,\{\eta_2\}),(\delta_2,I_U)\}\}$, then $(I_U,\tilde{\tau},\Delta)$ is soft ω-disconnected space as we can write

$$\tilde{I}_U = (M_\Delta,\Delta) \tilde{\cup} (N_\Delta,\Delta),$$

where $(M_\Delta,\Delta) = \{(\delta_1,\{\eta_2\}),(\delta_2,I_U)\}$ and $(N_\Delta,\Delta) = \{(\delta_1,\{\eta_1\}),(\delta_2,\phi)\}$ such that $\{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\cap} (N_\Delta,\Delta) = \tilde{\phi}$ and $(M_\Delta,\Delta) \tilde{\cap} \{\overline{(N_\Delta,\Delta)}\}_\omega = \tilde{\phi}$.

Example 5.2. Consider $I_U = \{\eta_1,\eta_2\}, \Delta = \{\delta_1,\delta_2\}$ and $\tilde{\tau} = \{\tilde{\phi}, \tilde{I}_U, \{(\delta_1,\{\eta_1\}),(\delta_2,\phi)\}\}$.
Then, all the possible soft subsets of $\tilde{I}_U$ are:

$(A_{1_\Delta},\Delta) = \{(\delta_1,\{\eta_1\}),(\delta_2,\phi)\}$; $(A_{2_\Delta},\Delta) = \{(\delta_1,\{\eta_2\}),(\delta_2,\phi)\}$;
$(A_{3_\Delta},\Delta) = \{(\delta_1,\{\eta_1\}),(\delta_2,\{\eta_1\})\}$; $(A_{4_\Delta},\Delta) = \{(\delta_1,\{\eta_1\}),(\delta_2,\{\eta_2\})\}$;
$(A_{5_\Delta},\Delta) = \{(\delta_1,\{\eta_1\}),(\delta_2,I_U)\}$; $(A_{6_\Delta},\Delta) = \{(\delta_1,\{\eta_2\}),(\delta_2,\{\eta_1\})\}$;
$(A_{7_\Delta},\Delta) = \{(\delta_1,\{\eta_2\}),(\delta_2,\{\eta_2\})\}$; $(A_{8_\Delta},\Delta) = \{(\delta_1,\{\eta_2\}),(\delta_2,I_U)\}$;
$(A_{9_\Delta},\Delta) = \{(\delta_1,\phi),(\delta_2,\{\eta_1\})\}$; $(A_{10_\Delta},\Delta) = \{(\delta_1,\phi),(\delta_2,\{\eta_2\})\}$;
$(A_{11_\Delta},\Delta) = \{(\delta_1,I_U),(\delta_2,\{\eta_1\})\}$; $(A_{12_\Delta},\Delta) = \{(\delta_1,I_U),(\delta_2,\{\eta_2\})\}$;
$(A_{13_\Delta},\Delta) = \tilde{I}_U$; $(A_{14_\Delta},\Delta) = \phi$;
$(A_{15_\Delta},\Delta) = \{(\delta_1,\phi),(\delta_2,I_U)\}$; $(A_{16_\Delta},\Delta) = \{(\delta_1,I_U),(\delta_2,\phi)\}$.

Now,

$$\{\overline{(A1_\Delta,\Delta)}\}_\omega = \{\overline{(A3_\Delta,\Delta)}\}_\omega = \{\overline{(A4_\Delta,\Delta)}\}_\omega = \{\overline{(A5_\Delta,\Delta)}\}_\omega = \{\overline{(A9_\Delta,\Delta)}\}_\omega = \{\overline{(A10_\Delta,\Delta)}\}_\omega$$

$$= \{\overline{(A11_\Delta,\Delta)}\}_\omega = \{\overline{(A12_\Delta,\Delta)}\}_\omega = \{\overline{(A13_\Delta,\Delta)}\}_\omega = \{\overline{(A16_\Delta,\Delta)}\}_\omega = \widetilde{I}_U$$

and

$$\{\overline{(A2_\Delta,\Delta)}\}_\omega = \{\overline{(A6_\Delta,\Delta)}\}_\omega = \{\overline{(A7_\Delta,\Delta)}\}_\omega = \{\overline{(A8_\Delta,\Delta)}\}_\omega = \{\overline{(A15_\Delta,\Delta)}\}_\omega = \{(\delta_1,\{\eta_2\}),(\delta_2,I_U)\}.$$

Now, non-null disjoint pair of soft sets whose union is $\widetilde{I}_U$ are:

$(A1_\Delta,\Delta)$ and $(A8_\Delta,\Delta)$; $(A2_\Delta,\Delta)$ and $(A5_\Delta,\Delta)$; $(A3_\Delta,\Delta)$ and $(A7_\Delta,\Delta)$; $(A4_\Delta,\Delta)$ and $(A6_\Delta,\Delta)$; $(A9_\Delta,\Delta)$ and $(A12_\Delta,\Delta)$; $(A10_\Delta,\Delta)$ and $(A11_\Delta,\Delta)$; $(A15_\Delta,\Delta)$ and $(A16_\Delta,\Delta)$.

As for each pair, we have $\{\overline{(Ai_\Delta,\Delta)}\}_\omega \widetilde{\cap} (Aj_\Delta,\Delta) \neq \phi$. Thus, we cannot able to write $\widetilde{I}_U$ as union of two non null soft ω-separated sets. Hence, $(I_U,\widetilde{\tau},\Delta)$ is soft ω-connected space.

Theorem 5.3. *Every soft disconnected space is a soft ω-disconnected space.*

Proof. Consider a soft disconnected topological space $(I_U,\widetilde{\tau},\Delta)$, then by definition,

$$\widetilde{I}_U = (M_\Delta,\Delta) \widetilde{\cup} (N_\Delta,\Delta),$$

where $(M_\Delta,\Delta) \neq \widetilde{\phi}$ and $(N_\Delta,\Delta) \neq \widetilde{\phi}$ such that (M_Δ,Δ) and (N_Δ,Δ) are soft separable sets. Using Theorem 4.5, we have (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separable sets such that $\widetilde{I}_U = (M_\Delta,\Delta) \widetilde{\cup} (N_\Delta,\Delta)$. Thus, $(I_U,\widetilde{\tau},\Delta)$ is soft ω-disconnected space. □

Remark 5.4. The converse of above theorem is not true in general i.e., every soft ω-disconnected space is not necessarily soft disconnected, which can be clearly seen from example in Remark 4.6.

Theorem 5.5. *A soft topological space $(I_U,\widetilde{\tau},\Delta)$ is soft ω-disconnected if and only if there exists a non-null proper soft subset of I_U which is both soft ω-open and soft ω-closed.*

Proof. Consider a soft ω-disconnected space $(I_U,\widetilde{\tau},\Delta)$, then by definition, there exists non-null soft sets (M_Δ,Δ) and (N_Δ,Δ), where $\widetilde{I}_U = (M_\Delta,\Delta) \widetilde{\cup} (N_\Delta,\Delta)$ such that $\{\overline{(M_\Delta,\Delta)}\}_\omega \widetilde{\cap} (N_\Delta,\Delta) = \widetilde{\phi}$ and $(N_\Delta,\Delta) \widetilde{\cap} \{\overline{(N_\Delta,\Delta)}\}_\omega = \widetilde{\phi}$. This implies that $(M_\Delta,\Delta) \widetilde{\cap} (N_\Delta,\Delta) = \widetilde{\phi}$ and $\widetilde{I}_U = (M_\Delta,\Delta) \widetilde{\cup} (N_\Delta,\Delta)$. Thus, $(M_\Delta,\Delta) = (N^c_\Delta,\Delta)$ and $(N_\Delta,\Delta) = (M^c_\Delta,\Delta)$.
Now,

$$\{\overline{(M_\Delta,\Delta)}\}_\omega \widetilde{\cap} (N_\Delta,\Delta) = \widetilde{\phi} \text{ and } (M_\Delta,\Delta) \widetilde{\cap} \{\overline{(N_\Delta,\Delta)}\}_\omega = \widetilde{\phi}$$

$$\Rightarrow \quad \{\overline{(M_\Delta,\Delta)}\}_\omega \widetilde{\subseteq} (N^c_\Delta,\Delta) = (M_\Delta,\Delta) \text{ and } \{\overline{(N_\Delta,\Delta)}\}_\omega \widetilde{\subseteq} (M^c_\Delta,\Delta) = (N_\Delta,\Delta).$$

But we have $(M_\Delta,\Delta) \widetilde{\subseteq} \{\overline{(M_\Delta,\Delta)}\}_\omega$ and $(N_\Delta,\Delta) \widetilde{\subseteq} \{\overline{(N_\Delta,\Delta)}\}_\omega$.
Thus, $(M_\Delta,\Delta) = \{\overline{(M_\Delta,\Delta)}\}_\omega$ and $(N_\Delta,\Delta) = \{\overline{(N_\Delta,\Delta)}\}_\omega$.
This implies that (M_Δ,Δ) and (N_Δ,Δ) are soft ω-closed sets and hence $(M^c_\Delta,\Delta) = (N_\Delta,\Delta)$ and $(N^c_\Delta,\Delta) = (M_\Delta,\Delta)$ are soft ω-open sets. Thus, we have non-null proper soft subset of I_U which is soft ω-open as well as soft ω-closed sets.
Conversely, let (M_Δ,Δ) be the non-null proper soft subset of I_U which is both soft ω-open as well as soft ω-closed. Then, $(M^c_\Delta,\Delta) = (N_\Delta,\Delta)$ is non-null proper subset of I_U which is also soft ω-open as well as soft ω-closed. Also, $(M_\Delta,\Delta) \widetilde{\cap} (N_\Delta,\Delta) = \widetilde{\phi}$. By using Theorem 4.3, we have (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets such that $\widetilde{I}_U = (M_\Delta,\Delta) \widetilde{\cup} (N_\Delta,\Delta)$. Thus, $(I_U,\widetilde{\tau},\Delta)$ is soft ω-disconnected space. □

Theorem 5.6. *Soft ω-disconnectedness is not a Hereditary Property i.e., subspace of soft ω-disconnected space need not be soft ω-disconnected.*

Example. Suppose $I_U = \{\eta_1,\eta_2,\eta_3,\eta_4,\eta_5\}$, $\Delta = \{\delta_1,\delta_2\}$ and $\tilde{\tau} = \{\tilde{\phi},\tilde{I}_U,\{(\delta_1,\{\eta_1\}),(\delta_2,\{\eta_1\})\}$, $\{(\delta_1,\{\eta_3,\eta_4\}),(\delta_2,\{\eta_3,\eta_4\})\},\{(\delta_1,\{\eta_1,\eta_3,\eta_4\}),(\delta_2,\{\eta_1,\eta_3,\eta_4\})\}$, $\{(\delta_1,\{\eta_2,\eta_3,\eta_4,\eta_5\}),(\delta_2,\{\eta_2,\eta_3,\eta_4,\eta_5\})\}\}$, then $(I_U,\tilde{\tau},\Delta)$ is soft ω-disconnected space.
But if we take $I_V = \{\eta_2,\eta_4,\eta_5\} \subseteq I_U$, then $\tilde{\tau}_{I_V} = \{\tilde{\phi},\tilde{I}_V,\{(\delta_1,\{\eta_4\}),(\delta_2,\{\eta_4\})\}\}$, which is not a soft ω-disconnected space. Thus, subspace of soft ω-disconnected need not be soft ω-disconnected.

Theorem 5.7. *A soft topological space $(I_U,\tilde{\tau},\Delta)$ is soft ω-connected if and only if $\tilde{\phi}$ and $\tilde{I}_U$ are the only soft ω-open and soft ω-closed sets.*

Proof. Consider a soft ω-connected space $(I_U,\tilde{\tau},\Delta)$. If possible, let $(M_\Delta,\Delta) \neq \tilde{I}_U$ be a non-null soft set which is both soft ω-open and soft ω-closed.

If $(N_\Delta,\Delta) = (M^c_\Delta,\Delta)$, then (N_Δ,Δ) is also both soft ω-open and soft ω-closed.

Since $(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta) = \tilde{\phi}$, $\{\overline{(M_\Delta,\Delta)}\}_\omega = (M_\Delta,\Delta)$ and $\{\overline{(N_\Delta,\Delta)}\}_\omega = (N_\Delta,\Delta)$, thus $\{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\cap}(N_\Delta,\Delta)$ $= \tilde{\phi}$ and $(M_\Delta,\Delta)\tilde{\cap}\{\overline{(N_\Delta,\Delta)}\}_\omega = \tilde{\phi}$. This implies that $(I_U,\tilde{\tau},\Delta)$ is soft ω-disconnected space, which is a contradiction. Thus, $\tilde{\phi}$ and $\tilde{I}_U$ are the only soft ω-open and soft ω-closed sets.

Conversely, suppose $\tilde{\phi}$ and $\tilde{I}_U$ are the only soft sets which is both ω-open and soft ω-closed. We claim that $(I_U,\tilde{\tau},\Delta)$ is soft ω-connected space. Let, if possible, it is not soft ω-connected, then $\tilde{I}_U = (M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$, where $(M_\Delta,\Delta) \neq \tilde{\phi},(N_\Delta,\Delta) \neq \tilde{\phi}$ such that

$$\begin{aligned}
&\{\overline{(M_\Delta,\Delta)}\}_\omega \tilde{\cap}(N_\Delta,\Delta) = \tilde{\phi} \text{ and } (M_\Delta,\Delta)\tilde{\cap}\{\overline{(N_\Delta,\Delta)}\}_\omega = \tilde{\phi}\\
\Rightarrow\ & (M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta) = \tilde{\phi}\\
\Rightarrow\ & (M_\Delta,\Delta) = (N^c_\Delta,\Delta) \text{ and } (N_\Delta,\Delta) = (M^c_\Delta,\Delta)
\end{aligned}$$

As

$$\begin{aligned}
&(M_\Delta,\Delta)\tilde{\subseteq}\{\overline{(M_\Delta,\Delta)}\}_\omega\\
\Rightarrow\ & (M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)\tilde{\subseteq}\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cup}(N_\Delta,\Delta)\\
\Rightarrow\ & \{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cup}(N_\Delta,\Delta) = \tilde{I}_U
\end{aligned}$$

But

$$\begin{aligned}
&\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta) = \tilde{\phi}\\
\Rightarrow\ & \{\overline{(M_\Delta,\Delta)}\}_\omega = (N^c_\Delta,\Delta) = (M_\Delta,\Delta)
\end{aligned}$$

Thus, (M_Δ,Δ) is soft ω-closed set. Similarly, (N_Δ,Δ) is also soft ω-closed set. This implies that $(M^c_\Delta,\Delta) = (N_\Delta,\Delta)$ and $(N^c_\Delta,\Delta) = (M_\Delta,\Delta)$ are soft ω-open sets. Thus, we have a non-null proper soft set which is both soft ω-open and soft ω-closed, which is a contradiction to the given fact. Thus, $(I_U,\tilde{\tau},\Delta)$ is soft ω-connected space. □

Definition 5.8. Let $(I_U,\tilde{\tau},\Delta)$ be a soft topological space and (M_Δ,Δ) be a soft set over I_U. Then, (M_Δ,Δ) is *soft ω-connected set* if it is soft ω-connected as a soft subspace.

Theorem 5.9. *If (F_Δ,Δ) is soft ω-connected set in soft topological space $(I_U,\tilde{\tau},\Delta)$ and $(F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$ such that (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets, then either $(F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)$ or $(F_\Delta,\Delta)\tilde{\subseteq}(N_\Delta,\Delta)$.*

Proof. As

$$\begin{aligned} & (F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta) \\ \Rightarrow\quad & (F_\Delta,\Delta)=(F_\Delta,\Delta)\tilde{\cap}\{(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)\}=\{(M_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)\}\tilde{\cup}\{(F_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)\} \end{aligned}$$

i.e.,

$$(F_\Delta,\Delta)=(A_{1_\Delta},\Delta)\tilde{\cup}(A2_\Delta,\Delta),$$

where $(A_{1_\Delta},\Delta)=(M_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)$ and $(A_{2_\Delta},\Delta)=(N_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)$. Now,

$$\begin{aligned} \{\overline{(A_{1_\Delta},\Delta)}\}_\omega\tilde{\cap}(A_{2_\Delta},\Delta) &= \{\overline{(M_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)}\}_\omega\tilde{\cap}\{(N_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)\} \\ &\tilde{\subseteq}\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}\{\overline{(F_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)=\tilde{\phi}. \end{aligned}$$

Similarly, $(A_{1_\Delta},\Delta)\tilde{\cap}\{\overline{(A_{2_\Delta},\Delta)}\}_\omega=\tilde{\phi}$. But (F_Δ,Δ) is soft ω-connected set, thus we have either $(A_{1_\Delta},\Delta)=\tilde{\phi}$ or $(A_{2_\Delta},\Delta)=\tilde{\phi}$.
If $(A_{1_\Delta},\Delta)=\tilde{\phi}$, then $(M_\Delta,\Delta)\tilde{\cap}(F_\Delta,\Delta)=\tilde{\phi}$ and as $(F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$, thus we have $(F_\Delta,\Delta)\tilde{\subseteq}(N_\Delta,\Delta)$. Similarly, if $(A_{2_\Delta},\Delta)=\phi$, then we have $(F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)$. □

Theorem 5.10. *If (F_Δ,Δ) is soft ω-connected set and $(F_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)\tilde{\subseteq}\{\overline{(F_\Delta,\Delta)}\}_\omega$, then (G_Δ,Δ) is soft ω-connected and hence $\{\overline{(F_\Delta,\Delta)}\}_\omega$ is also soft ω-connected set.*

Proof. Let, if possible, (G_Δ,Δ) is soft ω-disconnected set, then

$$(G_\Delta,\Delta)=(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta),\ \text{ where } (M_\Delta,\Delta)\neq\tilde{\phi},(N_\Delta,\Delta)\neq\tilde{\phi}$$

such that

$$\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}\ \text{ and }\ (M_\Delta,\Delta)\tilde{\cap}\{\overline{(N_\Delta,\Delta)}\}_\omega=\tilde{\phi}.$$

Now,

$$\begin{aligned} & (F_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta) \\ \Rightarrow\quad & (F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta), \end{aligned}$$

where (M_Δ,Δ) and (N_Δ,Δ) are soft ω-separated sets. Therefore, by above theorem, we have either $(F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)$ or $(F_\Delta,\Delta)\tilde{\subseteq}(N_\Delta,\Delta)$. If $(F_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)$,then

$$\begin{aligned} & \{\overline{(F_\Delta,\Delta)}\}_\omega\tilde{\subseteq}\{\overline{(M_\Delta,\Delta)}\}_\omega \\ \Rightarrow\quad & \{\overline{(F_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)\tilde{\subseteq}\{\overline{(M_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi} \\ \Rightarrow\quad & \{\overline{(F_\Delta,\Delta)}\}_\omega\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi} \\ \Rightarrow\quad & (N_\Delta,\Delta)\tilde{\subseteq}\{\overline{(F_\Delta,\Delta)}_\omega\}^c \end{aligned}$$

but $(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)\tilde{\subseteq}\{\overline{(F_\Delta,\Delta)}\}_\omega$. This implies that $(N_\Delta,\Delta)=\tilde{\phi}$, which is a contradiction. Similarly, if $(F_\Delta,\Delta)\tilde{\subseteq}(N_\Delta,\Delta)$, then we get $(M_\Delta,\Delta)=\tilde{\phi}$, which is again a contradiction. Thus, (G_Δ,Δ) is soft ω-connected set.
Now, if we take $(G_\Delta,\Delta)=\{\overline{(F_\Delta,\Delta)}\}_\omega$, then $\{\overline{(F_\Delta,\Delta)}\}_\omega$ is soft ω-connected set. □

Theorem 5.11. *If (M_Δ,Δ) and (N_Δ,Δ) are two soft ω-connected sets such that*

$$(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)\neq\tilde{\phi},\ \textit{then}\ (M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)\ \textit{is also soft}\ \omega\textit{-connected set.}$$

Proof. Let, if possible, $(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$ be soft ω-disconnected set, therefore

$$(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)=(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta),$$

where $(F_\Delta,\Delta)\neq\tilde{\phi}$, $(G_\Delta,\Delta)\neq\tilde{\phi}$ such that (F_Δ,Δ) and (G_Δ,Δ) are soft ω-separated sets.
Since

$$(M_\Delta,\Delta)\tilde{\subseteq}(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)=(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta).$$

Therefore

$$(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta).$$

Thus, by Theorem 5.9, we have either $(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ or $(M_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$.
Similarly, either $(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ or $(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$.
Thus, we have four choices

either $(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ or $(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$
or $(M_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ or $(M_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$.

If $(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ or $(M_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$, then

$$\begin{aligned}
&(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)\text{ or }(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)\\
\Rightarrow\ &(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)\text{ or }(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)\\
\Rightarrow\ &(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta)=(F_\Delta,\Delta)\text{ or }(F_\Delta,\Delta)\tilde{\cup}(G_\Delta,\Delta)=(G_\Delta,\Delta)\\
\Rightarrow\ &(G_\Delta,\Delta)=\tilde{\phi}\text{ or }(F_\Delta,\Delta)=\tilde{\phi},\ \text{ which is a contradiction.}
\end{aligned}$$

If $(M_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$ or $(M_\Delta,\Delta)\tilde{\subseteq}(G_\Delta,\Delta)$ and $(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)$, then in both the cases,

$$\begin{aligned}
&(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)\tilde{\subseteq}(F_\Delta,\Delta)\tilde{\cap}(G_\Delta,\Delta)=\tilde{\phi}\\
\Rightarrow\ &(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi},
\end{aligned}$$

which is again a contradiction to the given hypothesis that $(M_\Delta,\Delta)\tilde{\cap}(N_\Delta,\Delta)\neq\tilde{\phi}$. Thus, we have $(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$ is *soft ω-connected set*. □

Theorem 5.12. *If $(I_U,\tilde{\tau},\Delta)$ is soft ω-connected space, then it is soft connected also.*

Proof. Let, if possible, $(I_U,\tilde{\tau},\Delta)$ is soft disconnected space, then by definition, there exists non-null soft sets (M_Δ,Δ) and (N_Δ,Δ), where $\tilde{I}_U=(M_\Delta,\Delta)\tilde{\cup}(N_\Delta,\Delta)$ such that

$$\overline{(M_\Delta,\Delta)}\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}\text{ and }(M_\Delta,\Delta)\tilde{\cap}\overline{(N_\Delta,\Delta)}=\tilde{\phi}.$$

Since

$$\overline{(M_\Delta,\Delta)}_\omega\tilde{\subseteq}\overline{(M_\Delta,\Delta)}$$

therefore

$$\overline{(M_\Delta,\Delta)}_\omega\tilde{\cap}(N_\Delta,\Delta)\tilde{\subseteq}\overline{(M_\Delta,\Delta)}\tilde{\cap}(N_\Delta,\Delta)=\tilde{\phi}$$

$\Rightarrow \quad \overline{(M_\Delta,\Delta)_\omega}\,\widetilde{\cap}\,(N_\Delta,\Delta)=\widetilde{\phi}.$

Similarly, $(M_\Delta,\Delta)\widetilde{\cap}\overline{(N_\Delta,\Delta)_\omega}=\widetilde{\phi}$. This implies that the space $(I_U,\widetilde{\tau},\Delta)$ is soft ω-disconnected, which is a contradiction. Thus, the space $(I_U,\widetilde{\tau},\Delta)$ is soft connected. □

Remark 5.13. If $(I_U,\widetilde{\tau},\Delta)$ is *soft ω-connected space* and $I_V\subseteq I_U$, then $(I_V,\widetilde{\tau}_{I_V},\Delta)$ need not be soft ω-connected space which can be seen from the following example. In Example 5.2, $(I_U,\widetilde{\tau},\Delta)$ is *soft ω-connected space*. Let $I_V=\{\eta_2\}\subseteq I_U$, then $\widetilde{\tau}_{I_V}=\{\widetilde{\phi},\widetilde{I}_V\}$. Clearly, $(I_V,\widetilde{\tau}_{I_V},\Delta)$ is *soft ω-disconnected space*.

6. Concluding Remarks

Soft set theory is a wide mathematical aid for handling vagueness and uncertainty. In this paper, some basic concepts of soft set and soft topological spaces are considered. We define soft ω-connectedness and soft ω-disconnectedness in soft topological spaces and define its relation with soft connectedness and soft disconnectedness. We further discuss some properties of soft ω-connected and soft ω-disconnected sets with suitable examples.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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