Communications in Mathematics and Applications
Vol. 12, No. 3, pp. 417–443, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i3.1480

Research Article

Fuzzy Cone Normed Linear Space and Some Fixed Point Results for Weakly Compatible Mappings

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Received: November 3, 2020 **Accepted:** April 22, 2021

Abstract. In this paper, an idea of fuzzy cone normed linear space is introduced with underlying space is Felbin's type fuzzy Banach space. Some basic results as well as results in finite dimensional fuzzy cone normed linear space are studied. Lastly, some fixed point theorems for weakly compatible mappings are established in such spaces.

Keywords. Fuzzy real number; Fuzzy cone normed linear space

Mathematics Subject Classification (2020). 46S40; 03E72; 32C25

1. Introduction

After introducing the concept of fuzzy set by L.A. Zadeh [11] in 1965, many researchers extended its concept in the field of pure and applied mathematics. Especially in pure mathematics, to introduce notion of fuzzy set in functional analysis is one of the most important field of research in recent days. In this context, it was A.K. Katsaras [8] who first introduced the idea of fuzzy norm in 1984. On the other hand, Felbin [5] in 1992 introduced a different approach of fuzzy norm based on the concept of fuzzy metric space by Kaleva and Seikkela [7]. After that many authors developed these concepts in many ways (see [1], [4]).

In 2007, Huang and Zhang[6] introduced the notion of cone metric space by considering a real Banach space instead of the set of real numbers. Following this concept of cone, many

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authors have extended the results of fuzzy metric and fuzzy norm in cone setting (see [2], [3], [9], [10]). The aim of the present paper is to introduce an idea of fuzzy cone normed linear space by considering a fuzzy real Banach space and study some important finite dimensional results and fixed point results on it.

The organisation of the paper is as follows: In Section 2, some preliminary results are given which are used in this paper. In Section 3, an idea of fuzzy cone normed linear space is introduced and some basic results are proved. In Section 4, some results on finite dimensional fuzzy cone normed linear space are established. In Section 5, some fixed point theorems for weakly compatible mappings are established.

2. Preliminaries

In this section, some results are mentioned which are relevant to the main results of the paper.

Definition 2.1 ([5])**.** A fuzzy real number is a mapping $x : R \to [0,1]$ over the set R of all reals.

A fuzzy real number x is convex if $x(t) \geq \min(x(s), x(r))$ where $s \leq t \leq r$.
For $0 < \alpha \leq 1$, α-level set of a fuzzy real number x is defined by $\{t \in R \ : x(t) \geq \alpha\}$.
If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi-continuous convex normal fuzzy real number η (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
A fuzzy real number x is called non-negative if $x(t) = 0$, $\forall\ t < 0$.
Each real number r is considered as a fuzzy real number denoted by $\bar{r}$ and defined by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$.
Kaleva ([7], [5]) denoted the set of all convex, normal, upper semi-continuous fuzzy real numbers by $E(R(I))$ and the set of all non-negative, convex, normal, upper semi-continuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering "$\preceq$" in $E(R(I)$ is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0,1]$ where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$. The strict inequality in $E(R(I)$ is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0,1]$. Arithmetic operations $\oplus, \ominus, \odot$ on $E \times E$ are defined by

$$(\eta \oplus \delta)(t) = \sup_{s \in R} \min\{\eta(s), \delta(t-s)\}, \quad t \in R$$

$$(\eta \ominus \delta)(t) = \sup_{s \in R} \min\{\eta(s), \delta(s-t)\}, \quad t \in R$$

$$(\eta \odot \delta)(t) = \sup_{s \in R, s \neq \underline{0}} \min\left\{\eta(s), \delta\left(\frac{t}{s}\right)\right\}, \quad t \in R.$$

Proposition 2.2 ([5])**.** *Let* $\eta, \delta \in E(R(I))$ *and* $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$, $\alpha \in (0,1]$. *Then*

$$[\eta \oplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha, b^1_\alpha + b^2_\alpha]$$

$$[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha, b^1_\alpha - a^2_\alpha]$$
$$[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha, b^1_\alpha b^2_\alpha]$$
$$[\bar{1} \oslash \delta]_\alpha = \left[\frac{1}{b^2_\alpha}, \frac{1}{a^2_\alpha}\right], \quad a^2_\alpha > 0.$$

Definition 2.3 ([5])**.** A sequence $\{\eta_n\}$ in $E(R(I)$ is said to be convergent and converges to η denoted by $\lim_{n\to\infty} \eta_n = \eta$ if $\lim_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim_{n\to\infty} b^n_\alpha = b_\alpha$, where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall\ \alpha \in (0,1]$.

Note 2.4 ([5])**.** If $\eta, \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.

Note 2.5 ([5])**.** For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if $t = 0$ otherwise $t\eta(s) = \eta\left(\frac{s}{t}\right)$.

Definition of fuzzy norm on a linear space as introduced by Felbin [5] is given below:

Definition 2.6 ([5])**.** Let X be a vector space over R.

Let $\|\ \| : X \to R^*(I)$ and let the mappings $L, U : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $U(1,1) = 1$.
Write

$$[\|x\|]_\alpha = [\|x\|^1_\alpha, \|x\|^2_\alpha] \quad \text{for } x \in X, 0 < \alpha \le 1 \text{ and suppose for all } x \in X, x \ne \underline{0},$$

there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \le \alpha_0$,

(A) $\|x\|^2_\alpha < \infty$,

(B) $\inf \|x\|^1_\alpha > 0$.

The quadruple $(X, \|\ \|, L, U)$ is called a fuzzy normed linear space and $\|\ \|$ is a fuzzy norm if

(i) $\|x\| = \bar{0}$ if and only if $x = \underline{0}$;

(ii) $\|rx\| = |r|\|x\|$, $x \in X$, $r \in R$;

(iii) for all $x, y \in X$,

(a) whenever $s \le \|x\|^1_1$, $t \le \|y\|^1_1$ and $s + t \le \|x+y\|^1_1$,
$$\|x+y\|(s+t) \ge L(\|x\|(s), \|y\|(t)),$$

(b) whenever $s \ge \|x\|^1_1$, $t \ge \|y\|^1_1$ and $s + t \ge \|x+y\|^1_1$,
$$\|x+y\|(s+t) \le U(\|x\|(s), \|y\|(t)).$$

Remark 2.7 ([5])**.** Felbin proved that, if $L = \wedge$(Min) and $U = \vee$(Max) then the triangle inequality (iii) in the Definition 2.6 is equivalent to

$$\|x+y\| \preceq \|x\| \oplus \|y\|.$$

Further $\|\ \|^i_\alpha$, $i = 1,2$ are crisp norms on X for each $\alpha \in (0,1]$.

Definition 2.8 ([6])**.** Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{\theta\}$;

(ii) $a,b \in R$, $a,b \geq 0$, $x,y \in P \Rightarrow ax+by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y-x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ while $x << y$ will stand for $y-x \in \text{Int}P$, where $\text{Int}P$ denotes the interior of P.
The cone P is called normal if there is a number $K > 0$ such that for all $x,y \in E$ with $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.
The least positive number satisfying above is called the normal constant of P.
The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.
Equivalently, the cone P is regular if every decreasing sequence is bounded below is convergent.
It is clear that a regular cone is a normal cone.

3. Fuzzy Cone Normed Linear Space

In this section, an idea of fuzzy cone normed linear space is introduced and some basic results are proved.

Definition 3.1. Let $(E, \|\ \|)$ be a fuzzy real Banach space in the sense of $Felbin(L = \min, U = \max)$ where $\|\ \| : E \to R^*(I)$, $R^*(I)$ denotes the set of all non-negative fuzzy real numbers. For $c \in E$ and $r > 0$, define $B(c,r)$ by $B(c,r) = \{x \in E, \|c-x\| \prec \bar{r}\}$.

Definition 3.2. A point $x \in A \subset E$ is said to be an interior point of A if there exists $r > 0$ such that $B(c,r) \subset A$.

Definition 3.3. A subset P of fuzzy real Banach space E is called a fuzzy cone if

(i) P is closed (w.r.t Felbin fuzzy norm), non-empty and $P \neq \{\theta_E\}$; θ_E denotes the zero element in E.

(ii) $a,b \in R$, $a,b \geq 0$, $x,y \in P \Rightarrow ax+by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta_E$.

Given a fuzzy cone $P \subset E$, define a partial ordering $\leq$ with respect to P by $x \leq y$ iff $y-x \in P$ and $x < y$ indicates that $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y-x \in \text{Int}P$ where $\text{Int}P$ denotes the interior of P.

The fuzzy cone P is called fuzzy normal if there is a number $K > 0$ such that for all $x, y \in E$, with $\theta_E \le x \le y$ implies $\|x\| \preceq \bar{K} \odot \|y\|$, where $\|x\|, \|y\| \in R^*(I)$ and $\preceq$ is the ordering of fuzzy numbers. The least positive number satisfying above is called the normal constant of P.
The fuzzy cone P is called fuzzy regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots \le y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to \bar{0}$ as $n \to \infty$.
Equivalently, the fuzzy cone P is fuzzy regular if every decreasing sequence which is bounded below is convergent.
In the following, we always assume that P is a fuzzy cone in E with Int$P \neq \phi$ and $\le$ is a partial ordering with respect to P.

Definition 3.4. Let V be a linear space over the field R and $(E, \| \ \|)$ be a fuzzy real Banach space. The mapping $\| \ \|_p : V \to E$ is said to be a fuzzy cone norm if it satisfies the following conditions:
(CN1) $\|x\|_p \ge \theta_E$ and $\|x\|_p = \theta_E$ iff $x = \theta_V$; θ_V denotes the zero element in V;
(CN2) $\|\alpha x\|_p = |\alpha| \|x\|_p \ \forall \ x \in V, \ \alpha \in R$;
(CN3) $\|x + y\|_p \le \|x\|_p + \|y\|_p \ \forall \ x, y \in V$.
Then $\| \ \|_p$ is called a fuzzy cone norm on V and $(V, \| \ \|_p)$ is called a fuzzy cone normed linear space.

Example 3.5. Consider a fuzzy real Banach space $(E, \| \ \|)$ where $\| \ \| : E \to R^*(I)$ is defined by

$$\|x\|(t) = \begin{cases} 1 & \text{for } t = |x| \\ 0 & \text{otherwise} \end{cases}$$

where $E = R$, $P = [0, \infty)$ and $\le$ as the usual ordering.
Clearly $[\|x\|]_\alpha = [|x|, |x|]$, where $[\|x\|]_\alpha$ represents the $\alpha-$level set of fuzzy real number $\|x\|$.
Then P is closed w.r.t. fuzzy norm $\| \ \|$.
For, let $\{x_n\}$ be a sequence in P such that $\{x_n\}$ converges to x.

$$\Rightarrow \quad \|x_n - x\| \to \bar{0} \text{ as } n \to \infty$$

$$\Rightarrow \quad |x_n - x| \to 0 \text{ as } n \to \infty.$$

Since $x_n \ge 0 \ \forall \ n$. So $x \ge 0$ and $x \in P$.
It is obvious that $a, b \in R$, $a \ge 0$, $b \ge 0$ and $x, y \in P \Rightarrow ax + by \in P$.
So P is a fuzzy cone.
Now,

$$\theta_E \le x \le y \Rightarrow |x| \le |y| \Rightarrow \|x\| \preceq \bar{1} \odot \|y\|.$$

So P is a normal fuzzy cone with normal constant 1.
Now take $V = R^2$ and define $\| \ \|_p : V \to E$ as

$$\|x\|_p = \sqrt{x_1^2 + x_2^2} \quad \text{where } x = (x_1, x_2) \in V.$$

Then it is easy to verify that $\| \ \|_p$ satisfies the conditions (CN1) to (CN3). Hence $\| \ \|_p$ is a fuzzy cone norm on V.

Definition 3.6. Let $(V, \| \ \|_p)$ be a fuzzy cone normed linear space. Let $\{x_n\}$ be a sequence in V and $x \in V$. Then $\{x_n\}$ is said to be convergent and converges to x if for every $c \in E$ with $\theta_E \ll c$, there is a positive integer N such that for all $n \geq N$, $\|x_n - x\|_p \ll c$. We denote it by $\lim_{n\to\infty} x_n = x$.

Definition 3.7. Let $(V, \| \ \|_p)$ be a fuzzy cone normed linear space and $\{x_n\}$ be a sequence in V. Then $\{x_n\}$ is said to be Cauchy sequence if for every $c \in E$ with $\theta_E \ll c$ there is a positive integer N such that for all $m, n \geq N$, $\|x_n - x_m\|_p \ll c$.

Definition 3.8. Let $(V, \| \ \|_p)$ be a fuzzy cone normed linear space. Let $c \in E$ with $\theta_E \ll c$ and $b \in V$.
Define $B_p(b, c) = \{x \in V : \|x - b\|_p \ll c\}$.
Define $\beta = \{B_p(x, c); x \in V, \theta_E \ll c\}$.

Lemma 3.9. *For each $c_1, c_2 \in E$ with $\theta_E \ll c_1$ and $\theta_E \ll c_2$ there exists $\theta_E \ll c$ such that $c \ll c_1$ and $c \ll c_2$.*

Proof. Since $\theta_E \ll c_2$, for $r > 0$ such that $\|x\| \prec \bar{r}$

$$\Rightarrow \quad \|c_2 - (c_2 - x)\| \prec \bar{r}$$

$$\Rightarrow \quad c_2 - x \in B(c_2, r) \subset P.$$

Thus $c_2 - x \in \operatorname{int} P$.
i.e., $x \ll c_2$.
Choose n_0 such that $\frac{\|c_1\|_\alpha^2}{n_0} < r$, where $[\|c_1\|]_\alpha = [\|c_1\|_\alpha^1, \|c_1\|_\alpha^2]$, $[\bar{r}]_\alpha = [r, r]$, $\alpha \in (0, 1]$.
Let $c = \frac{c_1}{n_0}$. Then $c \in \operatorname{int} P$ and $c_1 - c = (1 - \frac{1}{n_0})c_1 \in \operatorname{int} P$. ($\lambda \operatorname{int} P \subset \operatorname{int} P$ for $\lambda > 0$)

$$\Rightarrow \quad c_1 - c \in \operatorname{int} P.$$

i.e., $c \ll c_1$.
Again $\|c\| = \frac{\|c_1\|}{n_0} \prec \bar{r}$ since $\frac{\|c_1\|_\alpha^2}{n_0} < r$ for $\alpha \in (0, 1]$.
So, $c \ll c_2$. □

Theorem 3.10. *Let $(V, \| \ \|_p)$ be a fuzzy cone normed linear space. Then $\tau_{f_c} = \{U \subset V; \forall x \in U, \exists B \in \beta$ such that $x \in B \subset U\}$ is a topology on $(V, \| \ \|_p)$.*

Proof. (i) $\phi, V \in \tau_{f_c}$.
(ii) Let $S, T \in \tau_{f_c}$ and $x \in S \cap T$.
$\Rightarrow$ there exists $c_1, c_2 \in E$ with $\theta_E \ll c_1$ and $\theta_E \ll c_2$ such that $B_p(x, c_1) \subset S$ and $B_p(x, c_2) \subset T$.
By Lemma 3.9, there exists $c \in E$ with $\theta_E \ll c$ such that $c \ll c_1$ and $c \ll c_2$.
Thus $B_p(x, c) \subset B_p(x, c_1)$ and $B_p(x, c) \subset B_p(x, c_2)$.

$$\Rightarrow \quad B_p(x, c) \subset B_p(x, c_1) \cap B_p(x, c_2) \subset S \cap T$$

$$\Rightarrow \quad S \cap T \in \tau_{f_c}.$$

(iii) Let $x \in \cup_{\alpha\in\triangle} U_\alpha$
$\Rightarrow x \in U_{\alpha_0}$ for some $\alpha_0 \in \triangle$.
Then there exists $c \in E$ with $\theta_E \ll c$ such that $B_p(x,c) \subset U_{\alpha_0} \subset \cup_{\alpha\in\triangle} U_\alpha$.
Hence $\cup_{\alpha\in\triangle} U_\alpha \in \tau_{f_c}$.
Thus τ_{f_c} is a topology on $(V, \|\ \|_p)$. □

Theorem 3.11. *Every fuzzy cone normed linear space* $(V, \|\ \|_p)$ *with normal constant* K *is Hausdroff topological space.*

Proof. Let $x, y \in V$ such that $x \neq y$.
We show that $B_p(x,c) \cap B_p(y,c) = \phi$ for some $c \in E$ with $\theta_E \ll c$.
If possible, let $z_c \in B_p(x,c) \cap B_p(y,c)$ for all $c \in E$ with $\theta_E \ll c$.

$$\Rightarrow \quad z_c \in B_p(x,c) \text{ and } z_c \in B_p(y,c)$$

$$\Rightarrow \quad \|z_c - x\|_p \ll c \text{ and } \|z_c - y\|_p \ll c$$

Now,

$$\|x-y\|_p \leq \|x - z_c\|_p + \|z_c - y\|_p \ll 2c \quad \text{for all } c \in E \text{ with } \theta_E \ll c.$$

Let $\|x-y\|_p = z$. Then $\theta_E \leq z \in E$.
Then

$$\theta_E \leq z \ll 2c \quad \forall\ c \gg \theta_E$$

$$\Rightarrow \quad \|z\| \preceq \bar{K} \odot \|2c\| \quad \forall\ c \gg \theta_E$$

$$\Rightarrow \quad \|z\|_\alpha^1 \leq K\|2c\|_\alpha^1 \text{ and } \|z\|_\alpha^2 \leq K\|2c\|_\alpha^2 \quad \forall\ c \gg \theta_E \ \ \forall\ \alpha \in (0,1].$$

Since $c \gg \theta_E$ is arbitrary, then $\|z\|_\alpha^1 = \|z\|_\alpha^2 = 0$ as $\forall \alpha \in (0,1]$.
$\Rightarrow \quad \|z\| = \bar{0}$
Thus $z = \theta_E$
i.e., $\|x-y\|_p = \theta_E$.
$\Rightarrow \quad x - y = \theta_V$.
So $x = y$ a contradiction.
Hence $(V, \|\ \|_p)$ is Hausdroff topological space. □

Definition 3.12. A subset B of V is said to be closed if any sequence $\{x_n\}$ in B converges to some point $x \in B$.

Definition 3.13. A subset F of V is said to be the closure of B if for any $x \in F$, there exists a sequence $\{x_n\}$ in B such that $\{x_n\}$ converges to x.

Definition 3.14. A subset C of V is said to be bounded if $C \subset B_p(b,c)$ for some $b \in V$ and $\theta_E \ll c$.

Definition 3.15. A subset F of V is said to be the compact if for any sequence $\{x_n\}$ in F, there exists a subsequence of $\{x_n\}$ which converges to some point in F.

Theorem 3.16. *Let $(V, \|\ \|_p)$ be a fuzzy cone normed linear space with normal constant K and $\{x_n\}$ be a sequence in V. Then $\{x_n\}$ converges to x iff $\|x_n - x\|_p \to \theta_E$ as $n \to \infty$.*

Proof. Suppose $\{x_n\}$ converges to x.
Then for every $c \in E$ with $c \gg \theta_E$, there exists a natural number N such that $\forall n \geq N$, $\|x_n - x\|_p \ll c$.
Let $\|x_n - x\|_p = y_n$. Then $\theta_E \leq y_n \in E$.
Then

$$
\begin{aligned}
& \theta_E \leq y_n \ll c \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N \\
\Rightarrow \quad & \|y_n\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N \\
\Rightarrow \quad & \|y_n\|_\alpha^1 \leq K\|c\|_\alpha^1 \text{ and } \|y_n\|_\alpha^2 \leq K\|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N \quad \forall\ \alpha \in (0,1].
\end{aligned}
$$

Since $c \gg \theta_E$ is arbitrary, then $\|y_n\|_\alpha^1, \|y_n\|_\alpha^2$ tends to 0 as $n \to \infty, \forall \alpha \in (0,1]$.

$$
\begin{aligned}
\Rightarrow \quad & \|y_n\| \to \bar{0} \text{ as } n \to \infty \\
\Rightarrow \quad & y_n \to \theta_E \text{ as } n \to \infty \\
\Rightarrow \quad & \|x_n - x\|_p \to \theta_E \text{ as } n \to \infty.
\end{aligned}
$$

Conversely, suppose that $\|x_n - x\|_p \to \theta_E$ as $n \to \infty$.
Let $\|x_n - x\|_p = c_n$. Then $c_n \in E \ \ \forall n$.
Since $c_n \to \theta_E$, for $r > 0$, there exists a natural number n_0 such that

$$
\begin{aligned}
& \|c_n\| \prec \bar{r} \quad \text{for all } n \geq n_0 \\
\Rightarrow \quad & \|c - (c - c_n)\| \prec \bar{r} \quad \text{for all } n \geq n_0 \\
\Rightarrow \quad & c - c_n \in B(c, r) \subset P \quad \text{for all } n \geq n_0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& c - c_n \in \operatorname{int} P \quad \text{for all } n \geq n_0 \\
\Rightarrow \quad & c_n \ll c \quad \text{for all } n \geq n_0 \\
\Rightarrow \quad & c_n \ll c \quad \text{for all } n \geq n_0 \\
\Rightarrow \quad & \|x_n - x\|_p \ll c \quad \text{for all } n \geq n_0 \\
\Rightarrow \quad & \{x_n\} \text{ converges to } x.
\end{aligned}
$$

□

Lemma 3.17. *Let $(V, \|\ \|_p)$ be a fuzzy cone normed linear space with normal constant K and $\{x_n\}$ be a convergent sequence in V, then its limit is unique.*

Proof. If possible, suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$.
Then for any $c \gg \theta_E$, there exists a natural number N such that for all $n \geq N$, $\|x_n - x\|_p \ll \frac{c}{2}$ and $\|x_n - y\|_p \ll \frac{c}{2}$.
Now,

$$\|x - y\|_p = \|x - x_n + x_n - y\|_p \leq \|x_n - x\|_p + \|x_n - y\|_p.$$

So,

$$\|x-y\|_p \ll \frac{c}{2}+\frac{c}{2}=c$$
$$\Rightarrow \quad \|x-y\|_p \ll c \quad \forall\ c \gg \theta_E$$

Let $\|x-y\|_p = z$. Then $\theta_E \le z \in E$.
Then

$$\theta_E \le z \ll c \quad \forall\ c \gg \theta_E \ \ \forall\ n \ge N$$
$$\Rightarrow \quad \|z\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \ \ \forall\ n \ge N$$
$$\Rightarrow \quad \|z\|_\alpha^1 \le K\|c\|_\alpha^1 \text{ and } \|z\|_\alpha^2 \le K\|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \ \ \forall\ n \ge N \ \ \forall\ \alpha \in (0,1].$$

Since $c \gg \theta_E$ is arbitrary, then

$$\|z\|_\alpha^1 = \|z\|_\alpha^2 = 0 \text{ as } n \to \infty, \quad \forall\ \alpha \in (0,1].$$

Thus

$$\|z\| = \bar{0}$$
$$\Rightarrow \quad z = \theta_E$$

i.e.,

$$\|x-y\|_p = \theta_E$$
$$\Rightarrow \quad x-y = \theta_V.$$

So $x = y$.
Hence the limit point is unique. □

Theorem 3.18. *Let $(V, \|\ \|_p)$ be a fuzzy cone normed linear space with normal constant K and $\{x_n\}$ be a sequence in V. Then $\{x_n\}$ is Cauchy iff $\|x_n - x_m\|_p \to \theta_E$ as $m,n \to \infty$.*

Proof. Suppose, $\{x_n\}$ is a Cauchy sequence.
Then for every $c \in E$ with $c \gg \theta_E$, there exists a natural number N such that $\forall m,n \ge N$, $\|x_n - x_m\|_p \ll c$.
Let $\|x_n - x_m\|_p = y_{n,m}$. Then $\theta_E \le y_{n,m} \in E$.
Then

$$\theta_E \le y_{n,m} \ll c \quad \forall\ c \gg \theta_E \ \ \forall\ m,n \ge N$$
$$\Rightarrow \quad \|y_{n,m}\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \ \ \forall\ m,n \ge N$$
$$\Rightarrow \quad \|y_{n,m}\|_\alpha^1 \le K\|c\|_\alpha^1 \text{ and } \|y_{n,m}\|_\alpha^2 \le K\|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \ \ \forall\ m,n \ge N \ \ \forall\ \alpha \in (0,1].$$

Since $c \gg \theta_E$ is arbitrary, then $\|y_{n,m}\|_\alpha^1, \|y_{n,m}\|_\alpha^2$ tends to 0 as $m,n \to \infty$, $\forall\ \alpha \in (0,1]$.

$$\Rightarrow \quad \|y_{n,m}\| \to \bar{0} \text{ as } m,n \to \infty$$
$$\Rightarrow \quad y_{n,m} \to \theta_E \text{ as } m,n \to \infty$$
$$\Rightarrow \quad \|x_n - x_m\|_p \to \theta_E \text{ as } m,n \to \infty.$$

Conversely, suppose that $\lim_{m,n\to\infty} \|x_n - x_m\|_p = \theta_E$.

Let $\|x_n - x_m\|_p = c_{n,m}$. Then $c_{n,m} \in E \;\; \forall n,m$.
Since $c_{n,m} \to \theta_E$, for $r > 0$, there exists a natural number n_0 such that

$$\begin{aligned} & \|c_n\| \prec \bar{r} \quad \text{for all } n,m \geq n_0 \\ \Rightarrow \quad & \|c - (c - c_{n,m})\| \prec \bar{r} \quad \text{for all } n,m \geq n_0 \\ \Rightarrow \quad & c - c_{n,m} \in B(c,r) \subset P \quad \text{for all } n,m \geq n_0. \end{aligned}$$

Thus

$$\begin{aligned} & c - c_{n,m} \in \operatorname{int} P \quad \text{for all } n,m \geq n_0 \\ \Rightarrow \quad & c_{n,m} \ll c \quad \text{for all } n,m \geq n_0 \\ \Rightarrow \quad & \|x_n - x_m\|_p \ll c \quad \text{for all } n,m \geq n_0 \\ \Rightarrow \quad & \{x_n\} \text{ is a Cauchy sequence.} \end{aligned}$$

□

Theorem 3.19. *Let $(V, \|\;\|_p)$ be a fuzzy cone normed linear space with normal constant K and $\{x_n\}$ be a sequence in V. If $\{x_n\}$ is convergent then it is Cauchy.*

Proof. Let $\{x_n\}$ converges to x.
Then for every $c \in E$ with $c \gg \theta_E$, there exists a natural number N such that $\forall\; m,n \geq N$, $\|x_n - x\|_p \ll \frac{c}{2}$ and $\|x_m - x\|_p \ll \frac{c}{2}$.
Now, $\|x_n - x_m\|_p \leq \|x_n - x\|_p + \|x_m - x\|_p$.
So, $\|x_n - x_m\|_p \ll \frac{c}{2} + \frac{c}{2} = c \;\; \forall\; m,n \geq N$.
Let $\|x_n - x_m\|_p = z_{n,m}$. Then $\theta_E \leq z_{n,m} \in E$.
Then

$$\begin{aligned} & \theta_E \leq z_{n,m} \ll c \quad \forall\; c \gg \theta_E \;\; \forall\; m,n \geq N \\ \Rightarrow \quad & \|z_{n,m}\| \preceq \bar{K} \odot \|c\| \quad \forall\; c \gg \theta_E \;\; \forall\; m,n \geq N \\ \Rightarrow \quad & \|z_{n,m}\|^1_\alpha \leq K\|c\|^1_\alpha \text{ and } \|z_{n,m}\|^2_\alpha \leq K\|c\|^2_\alpha \quad \forall\; c \gg \theta_E \;\; \forall\; m,n \geq N \;\; \forall\; \alpha \in (0,1]. \end{aligned}$$

Since $c \gg \theta_E$ is arbitrary, then $\|z_{n,m}\|^1_\alpha, \|z_{n,m}\|^2_\alpha$ tends to 0 as $m,n \to \infty$, $\forall\; \alpha \in (0,1]$.

$$\Rightarrow \quad \|z_{n,m}\| \to \bar{0} \text{ as } m,n \to \infty.$$

i.e.,

$$\begin{aligned} & z_{n,m} \to \theta_E \text{ as } m,n \to \infty \\ \Rightarrow \quad & \|x_n - x_m\|_p \to \theta_E \text{ as } m,n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.20. *Let $(V, \|\;\|_p)$ be a fuzzy cone normed linear space with normal constant K and $\{x_n\}$, $\{y_n\}$, are sequences in V. If $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$.*

Proof. Proof is straightforward. □

Theorem 3.21. *In a fuzzy cone normed linear space $(V, \|\;\|_p)$ with normal constant K, if $x_n \to x$ and $\lambda_n \to \lambda$ then $\lambda_n x_n \to \lambda x$.*

Proof.

$$\begin{aligned}\|\lambda_n x_n - \lambda x\|_p &= \|\lambda_n x_n - \lambda_n x + \lambda_n x - \lambda x\|_p \\ &\leq \|\lambda_n x_n - \lambda_n x\|_p + \|\lambda_n x - \lambda x\|_p \\ &= |\lambda_n| \|x_n - x\|_p + |\lambda_n - \lambda| \|x\|_p .\end{aligned}$$

Since $\|x_n - x\|_p \to \theta_E$, $|\lambda_n - \lambda| \to 0$ as $n \to \infty$.
For any $c \gg \theta_E$, there exists n_1 and n_2 such that $|\lambda_n| \|x_n - x\|_p \ll \frac{c}{2}$ and $|\lambda_n - \lambda| \|x\|_p \ll \frac{c}{2}$ for all $n \geq n_1$ and $n \geq n_2$, respectively.
So, $\|\lambda_n x_n - \lambda x\|_p \ll c$ for all $n \geq N = \max\{n_1, n_2\}$.
Let $\|\lambda_n x_n - \lambda x\|_p = y_n$. Then $\theta_E \leq y_n \in E$.
Then

$$\begin{aligned} & \theta_E \leq y_n \ll c \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N \\ \Rightarrow\quad & \|y_n\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N \\ \Rightarrow\quad & \|y_n\|_\alpha^1 \leq K \|c\|_\alpha^1 \text{ and } \|y_n\|_\alpha^2 \leq K \|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N \quad \forall\ \alpha \in (0,1].\end{aligned}$$

Since $c \gg \theta_E$ is arbitrary, then $\|y_n\|_\alpha^1, \|y_n\|_\alpha^2$ tends to 0 as $n \to \infty, \forall \alpha \in (0,1]$.

$$\begin{aligned} \Rightarrow\quad & \|y_n\| \to \bar{0} \text{ as } n \to \infty \\ \Rightarrow\quad & y_n \to \theta_E \text{ as } n \to \infty \\ \Rightarrow\quad & \|\lambda_n x_n - \lambda x\|_p \to \theta_E \text{ as } n \to \infty \\ \Rightarrow\quad & \lambda_n x_n \to \lambda x \text{ as } n \to \infty. \end{aligned}$$

□

Theorem 3.22. *Every compact fuzzy cone normed linear space $(V, \|\ \|_p)$ with normal constant K is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(V, \|\ \|_p)$. Since $(V, \|\ \|_p)$ is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$.
Let $\{x_{n_k}\}$ converges to $x \in V$. Then for every $c \in E$ with $c \gg \theta_E$, there exists a natural number n_1 such that $\forall n_k \geq n_1, \|x_{n_k} - x\|_p \ll \frac{c}{2}$. Since $\{x_n\}$ is Cauchy sequence, then for any $c \in E$ with $c \gg \theta_E$, there exists a natural number n_2 such that $\forall m, n \geq n_2 \ \ \|x_n - x_m\|_p \ll \frac{c}{2}$.
Let $N = \max\{n_1, n_2\}$.
Now,

$$\begin{aligned}\|x_n - x\|_p &= \|x_n - x_{n_k} + x_{n_k} - x\|_p \\ &\leq \|x_n - x_{n_k}\|_p + \|x_{n_k} - x\|_p \\ &\ll \frac{c}{2} + \frac{c}{2} \quad \forall\ n, n_k \geq N \\ &= c \quad \forall\ n, n_k \geq N.\end{aligned}$$

Let $\|x_n - x\|_p = y_n$. Then $\theta_E \preceq y_n \in E$.
Then

$$\theta_E \leq y_n \ll c \quad \forall\ c \gg \theta_E \quad \forall\ n \geq N$$

$\Rightarrow \quad \|y_n\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \ \ \forall\ n \geq N$

$\Rightarrow \quad \|y_n\|^1_\alpha \leq K\|c\|^1_\alpha$ and $\|y_n\|^2_\alpha \leq K\|c\|^2_\alpha \quad \forall\ c \gg \theta_E \ \ \forall\ n \geq N \ \ \forall\ \alpha \in (0,1]$.

Since $c \gg \theta_E$ is arbitrary, then $\|y_n\|^1_\alpha, \|y_n\|^2_\alpha$ tends to 0 as $n \to \infty, \forall \alpha \in (0,1]$.

$\Rightarrow \quad \|y_n\| \to \bar{0}$ as $n \to \infty$

$\Rightarrow \quad y_n \to \theta_E$ as $n \to \infty$

$\Rightarrow \quad \|x_n - x\|_p \to \theta_E$ as $n \to \infty$

$\Rightarrow \quad \{x_n\}$ converges to x.

Therefore, every Cauchy sequence in $(V, \|\ \|_p)$ is convergent.

Hence it is complete. □

Theorem 3.23. *Let $(V, \|\ \|_p)$ be a fuzzy cone normed linear space with normal constant K. Then every subsequence of a convergent sequence is convergent and it converges to the same limit.*

Proof. Let $\{x_n\}$ be a convergent sequence in V and converges to x.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Since $\{x_n\}$ converges to x, then for every $c \in E$ with $c \gg \theta_E$, there exists a natural number n_1 such that $\forall\ n \geq n_1$, $\|x_n - x\|_p \ll \frac{c}{2}$ i.e,

$$\|x_{n_k} - x\|_p \ll \frac{c}{2} \quad \forall\ n_k \geq n_1.$$

Hence $\{x_{n_k}\}$ converges to x.

If possible, suppose $\{x_{n_k}\}$ converges to y.

Then for every $c \in E$ with $c \gg \theta_E$, there exists a positive integer n_2 such that $\|x_{n_k} - y\|_p \ll \frac{c}{2}$ $\forall$ $n_k \geq n_2$.

Let $N = \max\{n_1, n_2\}$.

Now,

$$\begin{aligned}
\|x - y\|_p &= \|x - x_{n_k} + x_{n_k} - y\|_p \\
&\leq \|x_{n_k} - x\|_p + \|x_{n_k} - y\|_p \\
&\ll \frac{c}{2} + \frac{c}{2} \quad \forall\ n_k \geq N \\
&= c \quad \forall\ n_k \geq N.
\end{aligned}$$

Let $\|x - y\|_p = z$. Then $\theta_E \leq z \in E$.

Then

$$\theta_E \leq z \ll c \quad \forall\ c \gg \theta_E \ \ \forall\ n_k \geq N$$

$\Rightarrow \quad |z\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \ \ \forall\ n_k \geq N$

$\Rightarrow \quad \|z\|^1_\alpha \leq K\|c\|^1_\alpha$ and $\|z\|^2_\alpha \leq K\|c\|^2_\alpha \quad \forall\ c \gg \theta_E\ \forall\ n_k \geq N \ \ \forall\ \alpha \in (0,1]$.

Since $c \gg \theta_E$ is arbitrary, then $\|z\|^1_\alpha = \|z\|^2_\alpha = 0$ as $n_k \to \infty$, $\forall\ \alpha \in (0,1]$.

$\Rightarrow \quad \|z\| = \bar{0}$

Thus $z = \theta_E$.

i.e.,

$$\|x-y\|_p = \theta_E$$
$$\Rightarrow \quad x-y=\theta_V.$$

So $x=y$.
Hence proved. □

4. Finite Dimensional Fuzzy Cone Normed Linear Space

In this section, some finite dimensional results are studied on fuzzy cone normed linear space.

Lemma 4.1. *Let $\{x_1,x_2,\dots,x_n\}$ be a linearly independent set of vectors in a fuzzy cone normed linear space $(V,\|\ \|_p)$ where every elements of P are comparable and P is a normal fuzzy cone with normal constant K. Then there exists $c\in E$ with $\theta_E \ll c$, such that for every set of scalars $\alpha_1,\alpha_2,\dots,\alpha_n$,*

$$\sum_{i=1}^{n}|\alpha_i|c \le \|\alpha_1x_1+\alpha_2x_2+\ldots+\alpha_nx_n\|_p \tag{4.1}$$

Proof. Let $s=|\alpha_1|+|\alpha_2|+\ldots+|\alpha_n|$. If $s=0$ then $\alpha_i=0\ \forall\ i=1,2,\dots,n$ and the above relation (4.1) holds for any $c\in E$ with $\theta_E\ll c$.

Next, we suppose that $s>0$. Then (4.1) is equivalent to

$$\|\beta_1x_1+\beta_2x_2+\ldots+\beta_nx_n\|_p \ge c \tag{4.2}$$

for some $c\in E$ with $\theta_E\ll c$ and for all scalars β's with $\sum_{i=1}^{n}|\beta_i|=1$.
If possible suppose that (4.2) does not hold. Then there exists a sequence $\{y_m\}$ in V such that $\|y_m\|_p\to\theta_E$ as $m\to\infty$ where $y_m=\beta_1^m x_1+\beta_2^m x_2+\ldots+\beta_n^m x_n$ with $\sum_{i=1}^{n}|\beta_i^m|=1$.
Since $\sum_{i=1}^{n}|\beta_i^m|=1$ for $m=1,2,\dots$, we have $0\le|\beta_i^m|\le 1$ for $i=1,2,\dots,n$ and $m=1,2,\dots$. So for each fixed i, the sequence $\{\beta_i^m\}$ is bounded and hence $\{\beta_1^m\}$ has a convergent subsequence by Bolzano-Weierstrass theorem. Let β_1 denote the limit of that subsequence and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^m\}$ converges to β_2 (say).
Continuing in this way, after n steps we obtain a subsequence $\{y_{n,m}\}$ of $\{y_m\}$ of the form $y_{n,m}=\sum_{i=1}^{n}\delta_i^m x_i$ with $\sum_{i=1}^{n}|\delta_i^m|=1$ and $\delta_i^m\to\beta_i$ as $m\to\infty$ for each $i=1,2,\dots n$. So $\sum_{i=1}^{n}|\beta_i|=1$.
Let $y=\beta_1x_1+\beta_2x_2+\ldots+\beta_nx_n$. Since $\{x_1,x_2,\dots x_n\}$ is a linearly independent set and not all β_i's are zero. So $y\ne\theta_V$.
Now, $\|\delta_i^m x_i-\beta_i x_i\|_p=|\delta_i^m-\beta_i|\|x_i\|_p$.
Since $\delta_i^m\to\beta_i$ as $m\to\infty$ for each $i=1,2,\dots n$, $\|\delta_i^m x_i-\beta_i x_i\|_p\to\theta_E$.
Thus for any $c\in E$ with $\theta_E\ll c$, there exists a positive integer n_1 such that $\|\delta_i^m x_i-\beta_i x_i\|_p\ll\frac{c}{2n}$ for each $i=1,2,\dots n$ and $\forall\ m\ge n_1$.

Now,

$$\|y_{n,m} - y\|_p \le \|\delta_1^m x_1 - \beta_1 x_1\|_p + \|\delta_2^m x_2 - \beta_2 x_2\|_p + \ldots + \|\delta_n^m x_n - \beta_n x_n\|_p$$
$$\ll \frac{c}{2n} + \frac{c}{2n} + \ldots \frac{c}{2n} = \frac{c}{2} \quad \forall\ m \ge n_1. \tag{4.3}$$

Since $\{y_{n,m}\}$ is a subsequence of $\{y_m\}$ and $\|y_m\|_p \to \theta_E$ as $m \to \infty$, it follows that $\|y_{n,m}\|_p \to \theta_E$ as $m \to \infty$.

Thus for any $c \in E$ with $\theta_E \ll c$, there exists a positive integer n_2 such that

$$\|y_{n,m}\|_p \ll \frac{c}{2} \quad \forall\ m \ge n_2. \tag{4.4}$$

Let $N = \max\{n_1, n_2\}$. Then for all $m \ge N$, we have

$$\|y\|_p \le \|y - y_{n,m}\|_p + \|y_{n,m}\|_p$$
$$\ll \frac{c}{2} + \frac{c}{2} = c \quad \text{using (4.3) and (4.4).} \tag{4.5}$$

Let $\|y\|_p = z$. Then $\theta_E \le z \in E$.

Then

$$\theta_E \le z \ll c \quad \forall\ c \gg \theta_E$$
$$\Rightarrow \quad \|z\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E$$
$$\Rightarrow \quad \|z\|_\alpha^1 \le K\|c\|_\alpha^1 \text{ and } \|z\|_\alpha^2 \le K\|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \quad \forall\ \alpha \in (0,1].$$

Since $c \gg \theta_E$ is arbitrary, then

$$\|z\|_\alpha^1 = \|z\|_\alpha^2 = 0 \text{ as } \forall\ \alpha \in (0,1]$$
$$\Rightarrow \quad \|z\| = \bar{0}$$
$$\Rightarrow \quad z = \theta_E$$
$$\Rightarrow \quad \|y\|_p = \theta_E$$
$$\Rightarrow \quad y = \theta_V \text{ which is a contradiction.}$$

Hence the lemma is proved. □

Theorem 4.2. *Every finite dimensional fuzzy cone normed linear space $(V, \|\ \|_p)$ with normal fuzzy cone P with normal constant K is complete, provided every elements of P are comparable.*

Proof. Let $(V, \|\ \|_p)$ be a finite dimensional fuzzy cone normed linear space and P be a normal fuzzy cone with normal constant K.

Let $\{x_n\}$ be a Cauchy sequence in V.

Let $\dim V = m$ and $\{e_1, e_2, \ldots, e_m\}$ be a basis for V. Then each x_n has a unique representation as

$$x_n = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \ldots + \beta_m^{(n)} e_m$$

where $\beta_1^{(n)}$, $\beta_2^{(n)}, \ldots, \beta_m^{(n)}$ are scalars for each $n = 1, 2, \ldots$.

Since $\{x_n\}$ is Cauchy sequence, for any $e \in E$ with $\theta_E \ll e$, there exists a positive integer N such that

$$\|x_n - x_k\|_p \ll e \quad \forall\ n, k \ge N \tag{4.6}$$

Then by Lemma 4.1, there exists $c \in E$ with $\theta_E \ll c$, such that

$$c\sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| \leq \| \sum_{i=1}^{m} (\beta_i^{(n)} - \beta_i^{(k)}) e_i \|_p = \|x_n - x_k\|_p \tag{4.7}$$

Let $\|x_n - x_k\|_p = u_{n,k}$. Then $\theta_E \leq u_{n,k} \in E$.

Then from (4.6) and (4.7), we have

$$c\sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| \leq \left\| \sum_{i=1}^{m} (\beta_i^{(n)} - \beta_i^{(k)}) e_i \right\|_p = u_{n,k} \ll e \quad \forall\, e \gg \theta_E \;\; \forall\, n,k \geq N$$

$$\Rightarrow \quad \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| c \ll e \quad \forall\, \theta_E \ll e \;\; \forall\, n,k \geq N$$

$$\Rightarrow \quad \| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| c \| \leq \bar{K} \odot \|e\| \quad \forall\, e \gg \theta_E$$

$$\Rightarrow \quad \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| \|c\|_\alpha^1 \leq K \|e\|_\alpha^1 \text{ and}$$

$$\sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| \|c\|_\alpha^2 \leq K \|e\|_\alpha^2 \quad \forall\, e \gg \theta_E, \;\; \forall\, \alpha \in (0,1].$$

Since $e \gg \theta_E$ is arbitrary, then

$$\left\| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| c \right\|_\alpha^1 = \left\| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| c \right\|_\alpha^2 = 0 \text{ as } \forall\, \alpha \in (0,1] \;\; \forall\, n,k \geq N$$

$$\Rightarrow \quad \| \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| c \| = \bar{0} \quad \forall\, n,k \geq N$$

$$\Rightarrow \quad \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| c = \theta_E \quad \forall\, n,k \geq N$$

$$\Rightarrow \quad \sum_{i=1}^{m} |\beta_i^{(n)} - \beta_i^{(k)}| = 0 \quad \forall\, n,k > N \;\; (c \in \operatorname{Int} P)$$

$$\Rightarrow \quad |\beta_i^{(n)} - \beta_i^{(k)}| = 0 \quad \forall\, n,k > N \text{ and for each } i = 1,2,\dots m.$$

It follows that each of the sequences $\{\beta_i^{(n)}\}$ is Cauchy sequence of scalars. Thus each $\{\beta_i^{(n)}\}$ converges and denote their limits by β_i for each $i = 1,2,\dots m$.

Let $x = \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_m e_m$. Clearly, $x \in V$.

Now,

$$\|x_n - x\|_p = \left\| \sum_{i=1}^{m} (\beta_i^{(n)} - \beta_i) e_i \right\|_p$$

$$\leq |\beta_1^{(n)} - \beta_1| \|e_1\|_p + |\beta_2^{(n)} - \beta_2| \|e_2\|_p + \dots + |\beta_m^{(n)} - \beta_m| \|e_m\|_p$$

Since $|\beta_i^{(n)} - \beta_i| \to 0$ as $n \to \infty$, for each $i = 1,2,\dots m$.

For any $c \gg \theta_E$, there exists n_i such that

$$|\beta_i^{(n)} - \beta_i| \|e_i\|_p \ll \frac{c}{m} \quad \text{for all } n \geq n_i \text{ and } i = 1,2,\dots m.$$

So,

$$\|x_n - x\|_p \ll \frac{c}{m} + \frac{c}{m} + \ldots + \frac{c}{m} \quad \text{for all } n \geq N = \max\{n_1, n_2, \ldots n_m\}$$

$$\Rightarrow \quad \|x_n - x\|_p \ll c \quad \text{for all } n \geq N.$$

Let $\|x_n - x\|_p = y_n$. Then $\theta_E \leq y_n \in E$.

Then

$$\theta_E \leq y_n \ll c \quad \forall\ c \gg \theta_E \ \ n \geq N$$

$$\Rightarrow \quad \|y_n\| \preceq \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \ \ n \geq N$$

$$\Rightarrow \quad \|y_n\|_\alpha^1 \leq K\|c\|_\alpha^1 \text{ and } \|y_n\|_\alpha^2 \leq K\|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \ \ \forall\ \alpha \in (0,1] \ \ n \geq N.$$

Since $c \gg \theta_E$ is arbitrary, then

$$\|y_n\|_\alpha^1 = \|y_n\|_\alpha^2 = 0 \text{ as } \forall\ \alpha \in (0,1] \ \ n \geq N$$

$$\Rightarrow \quad \|y_n\| = \bar{0} \quad \forall\ n \geq N$$

$$\Rightarrow \quad y_n = \theta_E \quad \forall\ n \geq N$$

$$\Rightarrow \quad \|x_n - x\|_p \to \theta_E \text{ as } n \to \infty.$$

Thus every Cauchy sequence $\{x_n\}$ in V is convergent.

Hence $(V, \|\ \|_p)$ is complete. □

Theorem 4.3. *Let $(V, \|\ \|_p)$ be a finite dimensional fuzzy cone normed linear space where every elements of P are comparable and P be a normal fuzzy cone with normal constant K. A subset M of V is compact iff M is closed and bounded, provided every elements of P are comparable.*

Proof. Let M be a compact subset of V.

We will now show that M is closed and bounded.

Let $x \in \bar{M}$, then $\exists$ a sequence $\{x_n\}$ in M such that $\lim_{n\to\infty} x_n = x$.

Since M is compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a point in M. Again $\{x_n\} \to x$, so $\{x_{n_k}\} \to x$ and hence $x \in M$. So M is closed.

If possible, suppose that M is not bounded, then for each $c_n \in E$ with $\theta_E \ll c_n$ and

$$\|c_n\|_\alpha^i < \|c_{n+1}\|_\alpha^i \quad \text{for } i = 1,2$$

where

$$[\|c_n\|]_\alpha = [\|c_n\|_\alpha^1, \|c_n\|_\alpha^2], \quad \alpha \in (0,1],\ \exists\ y_n \in M \text{ such that } c_n < \|y_n - b\|_p \tag{4.8}$$

for some fixed $b \in M$.

Clearly $\{y_n\}$ is a sequence in M. Since M is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n_j}\}$ such that $\{y_{n_j}\}$ converges to $y \in M$.

i.e.

$$\|y_{n_j} - y\|_p \to \theta_E \text{ as } j \to \infty \tag{4.9}$$

Now,

$$c_{n_j} < \|y_{n_j} - b\|_p \leq \|y_{n_j} - y\|_p + \|y - b\|_p.$$

Let $\|y-b\|_p = e$ and $\|y_{n_j} - y\|_p = z_{n_j}$.
Then $\theta_E \le e, z_{n_j} \in E$ and $\|z_{n_j}\| \to \bar{0}$ as $j \to \infty$ (from (4.9)).
Then

$$
\begin{aligned}
& c_{n_j} \le z_{n_j} + e \\
\Rightarrow \quad & \|c_{n_j}\| \preceq \bar{K} \odot \|z_{n_j} + e\| \qquad \text{(since P is a normal fuzzy cone with normal constant K)} \\
\Rightarrow \quad & \|c_{n_j}\| \preceq \bar{K}[\|z_{n_j}\| \oplus \|e\|] \\
\Rightarrow \quad & \|c_{n_j}\|_\alpha^1 \le K\|z_{n_j}\|_\alpha^1 + K\|e\|_\alpha^1 \ \text{ and } \ \|c_{n_j}\|_\alpha^2 \le K\|z_{n_j}\|_\alpha^2 + K\|e\|_\alpha^2
\end{aligned}
$$

where $[\|c_{n_j}\|]_\alpha = [\|c_{n_j}\|_\alpha^1, \|c_{n_j}\|_\alpha^2]$, $[\|z_{n_j}\|]_\alpha = [\|z_{n_j}\|_\alpha^1, \|z_{n_j}\|_\alpha^2]$, $[\|e\|]_\alpha = [\|e\|_\alpha^1, \|e\|_\alpha^2]$, $\alpha \in (0,1]$.
Taking limit as $j \to \infty$, $\|c_{n_j}\|_\alpha^i \to \infty$, $\|z_{n_j}\|_\alpha^i \to 0$, $i = 1,2$.
Thus we get $\|e\|_\alpha^1 = \|e\|_\alpha^2 = \infty$, which is a contradiction as $\|e\|_\alpha^2$ must be less than infinity i.e, $\|e\|_\alpha^2 < \infty$.
Hence M is bounded.
Conversely, suppose that M is closed and bounded.
Let $\dim V = n$ and $\{e_1, e_2, \dots, e_n\}$ be a basis for V.
Choose $\{x_m\}$ in M. Since M is bounded, there exists $e \in E$ with $\theta_E \ll e$ such that

$$\|x_m - b\|_p \ll e \quad \text{for some } b \in V \text{ and } \forall\ m. \tag{4.10}$$

Now,

$$
\begin{aligned}
\|x_m\|_p = \|x_m - b + b\|_p &\le \|x_m - b\|_p + \|b\|_p \\
&\ll\ e + \|b\|_p \qquad \text{(using (4.10))}
\end{aligned} \tag{4.11}
$$

Let $x_m = \beta_1^{(m)} e_1 + \beta_2^{(m)} e_2 + \dots + \beta_n^{(m)} e_n$ where $\beta_1^{(m)}, \beta_2^{(m)}, \dots, \beta_n^{(m)}$ are scalars for each m.
Then, by Lemma 4.1, there exists $c \in E$ with $\theta_E \ll c$, such that

$$
\begin{aligned}
& c \sum_{j=1}^{n} |\beta_j^{(m)}| \le \| \sum_{j=1}^{n} \beta_j^{(m)} e_j \|_p = \|x_m\|_p \\
\Rightarrow \quad & c \sum_{j=1}^{n} |\beta_j^{(m)}| \le \|x_m\|_p \ll e + \|b\|_p
\end{aligned} \tag{4.12}
$$

Let $\|b\|_p = z$. Then $\theta_E \le z \in E$.
So,

$$
\begin{aligned}
& c \sum_{j=1}^{n} |\beta_j^{(m)}| \ll e + z \\
\Rightarrow \quad & \|c\| \sum_{j=1}^{n} |\beta_j^{(m)}| \preceq \bar{K} \odot \|e + z\| \\
\Rightarrow \quad & \|c\| \sum_{j=1}^{n} |\beta_j^{(m)}| \preceq K[\|e\| \oplus \|z\|]
\end{aligned}
$$

So,

$$\|c\|_\alpha^1 \sum_{j=1}^{n} |\beta_j^{(m)}| \le \eta_\alpha^1$$

and

$$\|c\|_\alpha^2 \sum_{j=1}^{n} |\beta_j^{(m)}| \le \eta_\alpha^2,$$

where $[\|c\|]_\alpha = [\|c\|_\alpha^1, \|c\|_\alpha^2]$, $[\|e\|]_\alpha = [\|e\|_\alpha^1, \|e\|_\alpha^2]$, $[\|z\|]_\alpha = [\|z\|_\alpha^1, \|z\|_\alpha^2]$, $[\|\eta\|]_\alpha = [\|\eta\|_\alpha^1, \|\eta\|_\alpha^2]$, $\eta_\alpha^i = K\|e\|_\alpha^i + K\|z\|_\alpha^i$, $i = 1,2$.

So,

$$\sum_{j=1}^{n} |\beta_j^{(m)}| \le \frac{\eta_\alpha^1}{\|c\|_\alpha^1} \quad \text{and} \quad \sum_{j=1}^{n} |\beta_j^{(m)}| \le \frac{\eta_\alpha^2}{\|c\|_\alpha^2}$$

It follows that each sequences $\{\beta_j^{(m)}\}$ $(j = 1,2,\dots,n)$ are bounded.

By Bolzano-Weiestrass theorem, each of sequences $\{\beta_j^{(m)}\}$ $(j = 1,2,\dots,n)$ has a convergent subsequence say $\{\beta_j^{m_k}\}$ $\forall\ j = 1,2,\dots,n$.

Let $x_{m_k} = \beta_1^{m_k} e_1 + \beta_2^{m_k} e_2 + \ldots + \beta_n^{m_k} e_n$, where $\{\beta_1^{m_k}\}, \{\beta_2^{m_k}\}, \dots, \{\beta_n^{m_k}\}$ are all convergent sequence of scalars and $\beta_j = \lim_{k\to\infty} \beta_j^{m_k}$, $j = 1,2,\dots,n$.

Let $x = \beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_n e_n$.

Now,

$$\|x_{m_k} - x\|_p = \left\| \sum_{j=1}^{n} (\beta_j^{m_k} - \beta_j) e_j \right\|_p \le \sum_{j=1}^{n} |\beta_j^{m_k} - \beta_j| \|e_j\|_p$$

Since $|\beta_j^{m_k} - \beta_j| \to 0$ as $k \to \infty$, for each $j = 1,2,\dots n$.

For any $c \gg \theta_E$, there exists n_j such that

$$|\beta_j^{m_k} - \beta_j| \|e_j\|_p \ll \frac{c}{n} \quad \text{for all } k \ge n_j \text{ and } j = 1,2,\dots n$$

$$\Rightarrow \quad \|x_{m_k} - x\|_p \ll c \quad \forall\ k \ge N = \max\{n_1, n_2, \dots, n_n\}.$$

Let $\|x_{m_k} - x\|_p = y_{m_k}$. Then $\theta_E \le y_{m_k} \in E$.

So,

$$y_{m_k} \ll c \quad \forall\ k \ge N$$

$$\Rightarrow \quad \|y_{m_k}\| \le \bar{K} \odot \|c\| \quad \forall\ c \gg \theta_E \ \ \forall\ k \ge N$$

$$\Rightarrow \quad \|y_{m_k}\|_\alpha^1 \le K\|c\|_\alpha^1 \text{ and } \|y_{m_k}\|_\alpha^2 \le K\|c\|_\alpha^2 \quad \forall\ c \gg \theta_E \ \ \forall\ \alpha \in (0,1] \ \ \forall\ k \ge N.$$

Since $c \gg \theta_E$ is arbitrary,

$$\|y_{m_k}\|_\alpha^1 \to 0 \text{ and } \|y_{m_k}\|_\alpha^2 \to 0 \text{ as } k \to \infty$$

$$\Rightarrow \quad \|y_{m_k}\| \to \bar{0} \text{ as } k \to \infty$$

$$\Rightarrow \quad y_{m_k} \to \theta_E \text{ as } k \to \infty$$

$$\Rightarrow \quad \|x_{m_k} - x\|_p \to \theta_E \text{ as } k \to \infty$$

i.e., $\{x_{m_k}\}$ is a convergent subsequence of $\{x_m\}$ and converges to x. Since M is closed, $x \in M$. Thus every sequence in M has a convergent subsequence and converges to an element of M. Hence M is compact. □

5. Fixed Point Theorems

In this section, we study some fixed point theorems for weakly compatible mappings in fuzzy cone normed linear space.

Definition 5.1. Let A and S be mappings from a fuzzy cone normed linear space $(V,\|\ \|_p)$ into itself. Then A and S are said to be weakly compatible if they commute at their coincident point, that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

Theorem 5.2. *Let $(V,\|\ \|_p)$ be a complete fuzzy cone normed linear space with normal constant K and $A,B,S,T : V \to V$ are mappings satisfying:*

(i) *$A(V) \subset B(V)$ and $A(V)$ is a complete subspace of V.*

(ii) *(A,S) and (B,T) are weakly compatible*

$$a\|Sx - Ty\|_p + b\|Ax - Sx\|_p + c\|By - Ty\|_p \leq \|Ax - By\|_p,$$

where $a > 1$, $0 \leq b < 1$, $c \geq -1, a + b + c > 1$.

Then A,S,B,T have a unique common fixed point in V.

Proof. Let $x_0 \in V$. Define a sequence $\{x_n\}$ in X by

$$Ax_{2n+1} = Tx_{2n} = y_{2n},\ Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}.$$

Then,

$$\begin{aligned}
&a\|Sx_{2n+1} - T_{2n+2}\|_p + b\|Ax_{2n+1} - Sx_{2n+1}\|_p + c\|B_{2n+2} - T_{2n+2}\|_p \leq \|Ax_{2n+1} - B_{2n+2}\|_p \\
\Rightarrow\ & a\|y_{2n+1} - y_{2n+2}\|_p + b\|y_{2n} - y_{2n+1}\|_p + c\|y_{2n+1} - y_{2n+2}\|_p \leq \|y_{2n} - y_{2n+1}\|_p \\
\Rightarrow\ & (a+c)\|y_{2n+1} - y_{2n+2}\|_p \leq (1-b)\|y_{2n} - y_{2n+1}\|_p \\
\Rightarrow\ & \|y_{2n+1} - y_{2n+2}\|_p \leq \frac{1-b}{a+c}\|y_{2n} - y_{2n+1}\|_p .
\end{aligned}$$

Let $\lambda = \frac{1-b}{a+c}$. Then $0 < \lambda < 1$.

Thus for any natural number n,

$$\begin{aligned}
\|y_{n+1} - y_{n+2}\|_p &\leq \lambda\|y_n - y_{n+1}\|_p \\
&\leq \lambda^2\|y_{n-1} - y_n\|_p \\
&\ \vdots \\
&\leq \lambda^{n+1}\|y_0 - y_1\|_p .
\end{aligned}$$

For $m > n$,

$$\begin{aligned}
\|y_m - y_n\|_p &\leq \|y_m - y_{m-1}\|_p + \|y_{m-1} - y_{m-2}\|_p + \cdots + \|y_{n+1} - y_n\|_p \\
&\leq (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n)\|y_0 - y_1\|_p \\
&\leq \frac{\lambda^n}{1-\lambda}\|y_0 - y_1\|_p .
\end{aligned}$$

Let $\|y_m - y_n\|_p = z_{m,n}$, $\|y_0 - y_1\|_p = e$. Then $z_{m,n}$, $e \in E$.

So,

$$z_{m,n} \leq \frac{\lambda^n}{1-\lambda}e$$

$$\Rightarrow \quad \|z_{m,n}\| \preceq \bar{K} \odot \|\frac{\lambda^n}{1-\lambda} e\| \quad (K \text{ is the normal constant})$$

$$\Rightarrow \quad \|z_{m,n}\|_\alpha^1 \leq K \frac{\lambda^n}{1-\lambda} \|e\|_\alpha^1$$

and

$$\|z_{m,n}\|_\alpha^2 \leq K \frac{\lambda^n}{1-\lambda} \|e\|_\alpha^2$$

where

$$[\|e\|]_\alpha = [\|e\|_\alpha^1, \|e\|_\alpha^2], [\|z_{m,n}\|]_\alpha = [\|z_{m,n}\|_\alpha^1, \|z_{m,n}\|_\alpha^2]$$

$$\Rightarrow \quad \|z_{m,n}\|_\alpha^i \to 0 \text{ as } m,n \to \infty \text{ for } i = 1,2$$

$$\Rightarrow \quad \|z_{m,n}\| \to \bar{0} \text{ as } m,n \to \infty$$

$$\Rightarrow \quad \|y_m - y_n\|_p \to \theta_E \text{ as } m,n \to \infty$$

So $\{y_n\}$ is a Cauchy sequence in $A(V)$. Since $A(V)$ is complete, there exists $z \in A(V)$ such that $\{y_n\}$ converges to z. Since $z \in A(V)$, there exists $u \in V$ such that $Au = z$. Since $\{y_n\}$ converges to z, all its subsequences $\{y_{2n}\}, \{y_{2n-1}\}$ converges to z.

Now,

$$a\|Su - Tx_{2n}\|_p + b\|Su - Au\|_p + c\|Bx_{2n} - Tx_{2n}\|_p \leq \|Au - Bx_{2n}\|_p$$

$$\Rightarrow \quad a\|Su - y_{2n}\|_p + b\|Su - z\|_p + c\|y_{2n-1} - y_{2n}\|_p \leq \|Au - y_{2n-1}\|_p = \|z - y_{2n-1}\|_p$$

Taking limit as $n \to \infty$, we get

$$a\|Su - z\|_p + b\|Su - z\|_p + \theta_E \leq \theta_E$$

$$\Rightarrow \quad (a+b)\|Su - z\|_p \leq \theta_E.$$

Since $(a+b) > 0$,

$$\|Su - z\|_p = \theta_E$$

$$\Rightarrow \quad Su = z.$$

So, $Au = Su = z$.

Since (A,S) are weakly compatible, $SAu = ASu \Rightarrow Sz = Az$.

Since $Au \in A(V)$ and $A(V) \subset B(V)$, there exists $v \in V$ such that $Bv = Au$.

So, $Bv = Au = Su = z$.

Now,

$$a\|Su - Tv\|_p + b\|Au - Su\|_p + c\|Bv - Tv\|_p \leq \|Au - Bv\|_p$$

$$\Rightarrow \quad a\|z - Tv\|_p + b\|z - z\|_p + c\|z - Tv\|_p \leq \|z - z\|_p$$

$$\Rightarrow \quad (a+c)\|z - Tv\|_p \leq \theta_E$$

Since $(a+c) > 0$,

$$\|z - Tv\|_p = \theta_E$$

$$\Rightarrow \quad Tv = z.$$

So, $Bv = Tv = z$.

Since (B,T) are weakly compatible, $BTv = TBv \Rightarrow Bz = Tz$.
Now, we will show that z is a fixed point of A.
For,

$$\begin{aligned} & a\|Sz - Tv\|_p + b\|Az - Sz\|_p + c\|Bv - Tv\|_p \le \|Az - Bv\|_p \\ \Rightarrow\ & a\|Az - z\|_p + \theta_E + \theta_E \le \|Az - z\|_p \\ \Rightarrow\ & (a-1)\|Az - z\|_p \le \theta_E \end{aligned}$$

Since $a > 1$,

$$\begin{aligned} & \|Az - z\|_p = \theta_E \\ \Rightarrow\ & Az = z. \end{aligned}$$

So, $Az = Sz = z$.
Now,

$$\begin{aligned} & a\|Sz - Tz\|_p + b\|Az - Sz\|_p + c\|Bz - Tz\|_p \le \|Az - Bz\|_p \\ \Rightarrow\ & a\|z - Bz\|_p + \theta_E + \theta_E \le \|z - Bz\|_p \\ \Rightarrow\ & (a-1)\|z - Bz\|_p \le \theta_E \end{aligned}$$

Since $a > 1$,

$$\begin{aligned} & \|z - Bz\|_p = \theta_E \\ \Rightarrow\ & Bz = z. \end{aligned}$$

So, $Bz = Tz = z$.
Thus $Az = Sz = Bz = Tz = z$.

Uniqueness: If there exists $w \in V$ such that $Aw = Sw = Bw = Tw = w$.
Now,

$$\begin{aligned} & a\|Sz - Tw\|_p + b\|Az - Sz\|_p + c\|Bw - Tw\|_p \le \|Az - Bw\|_p \\ \Rightarrow\ & a\|z - w\|_p + \theta_E + \theta_E \le \|z - w\|_p \\ \Rightarrow\ & (a-1)\|z - w\|_p \le \theta_E \end{aligned}$$

Since $a > 1$,

$$\begin{aligned} & \|z - w\|_p = \theta_E \\ \Rightarrow\ & z = w. \end{aligned}$$

Thus z is the unique common fixed point of A,S,B,T. □

Theorem 5.3. *Let $(V, \|\ \|_p)$ be a complete fuzzy cone normed linear space with normal constant K and $A,B,S,T : V \to V$ are mappings satisfying:*

(i) *$A(V) \subset B(V)$ and $A(V)$ is a complete subspace of V.*

(ii) *(A,S) and (B,T) are weakly compatible.*

$a[\|Sx - Ty\|_p + \|Ax - Sx\|_p] + b[\|By - Ty\|_p + \|Sx - By\|_p] \le \|Ax - By\|_p$, *where* $\frac{1}{2} < a < 1, b > \frac{1}{2}$

Then A,S,B,T have a unique common fixed point in V.

Proof. Let $x_0 \in V$. Define a sequence $\{x_n\}$ in X by $Ax_{2n+1} = Tx_{2n} = y_{2n}$, $Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}$. Then,

$$\begin{aligned}
& a[\|Sx_{2n+1} - T_{2n+2}\|_p + \|Ax_{2n+1} - Sx_{2n+1}\|_p] + b[\|B_{2n+2} - T_{2n+2}\|_p + \|S_{2n+1} - B_{2n+2}\|_p] \\
& \quad \le \|Ax_{2n+1} - B_{2n+2}\|_p \\
\Rightarrow \quad & a[\|y_{2n+1} - y_{2n+2}\|_p + \|y_{2n} - y_{2n+1}\|_p] + b[\|y_{2n+1} - y_{2n+2}\|_p + \|y_{2n+1} - y_{2n+1}\|_p] \\
& \quad \le \|y_{2n} - y_{2n+1}\|_p \\
\Rightarrow \quad & (a+b)\|y_{2n+1} - y_{2n+2}\|_p \le (1-a)\|y_{2n} - y_{2n+1}\|_p \\
\Rightarrow \quad & \|y_{2n+1} - y_{2n+2}\|_p \le \frac{1-a}{a+b}\|y_{2n} - y_{2n+1}\|_p
\end{aligned}$$

Let $\lambda = \frac{1-a}{a+b}$. Since $\frac{1}{2} < a < 1$, $2a > 1 \Rightarrow 2a + b > 1 \Rightarrow a + b > 1 - a \Rightarrow \frac{1-a}{a+b} < 1$. Then $0 < \lambda < 1$. Thus for any natural number n,

$$\begin{aligned}
\|y_{n+1} - y_{n+2}\|_p &\le \lambda\|y_n - y_{n+1}\|_p \\
&\le \lambda^2\|y_{n-1} - y_n\|_p \\
&\vdots \\
&\le \lambda^{n+1}\|y_0 - y_1\|_p.
\end{aligned}$$

For $m > n$,

$$\begin{aligned}
\|y_m - y_n\|_p &\le \|y_m - y_{m-1}\|_p + \|y_{m-1} - y_{m-2}\|_p + \dots + \|y_{n+1} - y_n\|_p \\
&\le (\lambda^{m-1} + \lambda^{m-2} + \dots + \lambda^n)\|y_0 - y_1\|_p \\
&\le \frac{\lambda^n}{1-\lambda}\|y_0 - y_1\|_p.
\end{aligned}$$

Let $\|y_m - y_n\|_p = z_{m,n}$, $\|y_0 - y_1\|_p = e$. Then $z_{m,n}$, $e \in E$.

So,

$$\begin{aligned}
& z_{m,n} \le \frac{\lambda^n}{1-\lambda} e \\
\Rightarrow \quad & \|z_{m,n}\| \preceq \bar{K} \odot \left\|\frac{\lambda^n}{1-\lambda} e\right\| \quad (K \text{ is the normal constant}) \\
\Rightarrow \quad & \|z_{m,n}\|_\alpha^1 \le K \frac{\lambda^n}{1-\lambda}\|e\|_\alpha^1
\end{aligned}$$

and

$$\|z_{m,n}\|_\alpha^2 \le K \frac{\lambda^n}{1-\lambda}\|e\|_\alpha^2$$

where

$$\begin{aligned}
& [\|e\|]_\alpha = [\|e\|_\alpha^1, \|e\|_\alpha^2], \\
& [\|z_{m,n}\|]_\alpha = [\|z_{m,n}\|_\alpha^1, \|z_{m,n}\|_\alpha^2] \\
\Rightarrow \quad & \|z_{m,n}\|_\alpha^i \to 0 \text{ as } m, n \to \infty \text{ for } i = 1, 2 \\
\Rightarrow \quad & \|z_{m,n}\| \to \bar{0} \text{ as } m, n \to \infty \\
\Rightarrow \quad & \|y_m - y_n\|_p \to \theta_E \text{ as } m, n \to \infty.
\end{aligned}$$

So $\{y_n\}$ is a Cauchy sequence in $A(V)$. Since $A(V)$ is complete, there exists $z \in A(V)$ such that $\{y_n\}$ converges to z. Since $z \in A(V)$, there exists $u \in V$ such that $Au = z$. Since $\{y_n\}$ converges to z, all its subsequences $\{y_{2n}\}$, $\{y_{2n-1}\}$ converges to z.
Now,

$$\begin{aligned} & a[\|Su - Tx_{2n}\|_p + \|Au - Su\|_p] + b[\|Bx_{2n} - Tx_{2n}\|_p + \|Su - Bx_{2n}\|_p] \le \|Au - Bx_{2n}\|_p \\ \Rightarrow \quad & a[\|Su - y_{2n}\|_p + \|z - Su\|_p] + b[\|y_{2n-1} - y_{2n}\|_p + \|Su - x_{2n-1}\|_p] \\ & \quad \le \|Au - y_{2n-1}\|_p = \|z - y_{2n-1}\|_p \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} & a[\|Su - z\|_p + \|Su - z\|_p] + b[\theta_E + \|Su - z\|_p] \le \theta_E \\ \Rightarrow \quad & (2a + b)\|Su - z\|_p \le \theta_E \end{aligned}$$

Since $(2a + b) > 1 > 0$,

$$\begin{aligned} & \|Su - z\|_p = \theta_E \\ \Rightarrow \quad & Su = z. \end{aligned}$$

So, $Au = Su = z$.
Since (A,S) are weakly compatible, $SAu = ASu \Rightarrow Sz = Az$.
Since $Au \in A(V)$ and $A(V) \subset B(V)$, there exists $v \in V$ such that $Bv = Au$.
So, $Bv = Au = Su = z$.
Now,

$$\begin{aligned} & a[\|Su - Tv\|_p + \|Au - Su\|_p] + b[\|Bv - Tv\|_p + \|Su - Bv\|_p] \le \|Au - Bv\|_p \\ \Rightarrow \quad & a[\|z - Tv\|_p + \theta_E] + b[\|z - Tv\|_p + \theta_E] \le \|z - z\|_p \\ \Rightarrow \quad & (a + b)\|z - Tv\|_p \le \theta_E \end{aligned}$$

Since $(a + b) > 0$,

$$\begin{aligned} & \|z - Tv\|_p = \theta_E \\ \Rightarrow \quad & Tv = z. \end{aligned}$$

So, $Bv = Tv = z$.
Since (B,T) are weakly compatible, $BTv = TBv \Rightarrow Bz = Tz$.
Now we will show that z is a fixed point of A.
For,

$$\begin{aligned} & a[\|Sz - Tv\|_p + \|Az - Sz\|_p] + b[\|Bv - Tv\|_p + \|Sz - Bv\|_p] \le \|Az - Bv\|_p \\ \Rightarrow \quad & a[\|Az - z\|_p + \theta_E] + b[\theta_E + \|Az - z\|_p] \le \|Az - z\|_p \\ \Rightarrow \quad & (a + b)\|Az - z\|_p \le \|Az - z\|_p \end{aligned}$$

Since $(a + b) > 1$,

$$\begin{aligned} & \|Az - z\|_p = \theta_E \\ \Rightarrow \quad & Az = z. \end{aligned}$$

So, $Az = Sz = z$.

Now,

$$a[\|Sz - Tz\|_p + \|Az - Sz\|_p] + b[\|Bz - Tz\|_p + \|Sz - Bz\|_p] \leq \|Az - Bz\|_p$$

$$\Rightarrow \quad a[\|z - Bz\|_p + \theta_E] + b[\theta_E + \|z - Bz\|_p] \leq \|z - Bz\|_p$$

$$\Rightarrow \quad (a+b)\|z - Bz\|_p \leq \|z - Bz\|_p$$

Since $(a+b) > 1$,

$$\|z - Bz\|_p = \theta_E$$

$$\Rightarrow \quad Bz = z.$$

So, $Bz = Tz = z$.

Thus $Az = Sz = Bz = Tz = z$.

Uniqueness: If there exists $w \in V$ such that $Aw = Sw = Bw = Tw = w$.

Now,

$$a[\|Sz - Tw\|_p + \|Az - Sz\|_p] + b[\|Bw - Tw\|_p + \|Sz - Bw\|_p] \leq \|Az - Bw\|_p$$

$$\Rightarrow \quad a[\|z - w\|_p + \theta_E] + b[\theta_E + \|z - w\|_p \leq \|z - w\|_p$$

$$\Rightarrow \quad (a+b)\|z - w\|_p \leq \|z - w\|_p$$

Since $(a+b) > 1$,

$$\|z - w\|_p = \theta_E$$

$$\Rightarrow \quad z = w.$$

Thus z is the unique common fixed point of A, S, B, T. □

Example 5.4. Consider a fuzzy real Banach space $(E, \| \ \|)$ where $\| \ \| : E \to R^*(I)$ is defined by

$$\|x\|(t) = \begin{cases} 1 & \text{for } t = \sqrt{x_1^2 + x_2^2} \\ 0 & \text{otherwise} \end{cases}$$

where $E = R^2$, $P = \{(x_1, x_2), x_1, x_2 \geq 0\}$ and $\leq$ as the usual ordering.

Then $(V, \| \ \|_p)$ is a fuzzy cone normed linear space with normal constant 1 where $V = R$ and $\| \ \|_p : V \to E$ as $(|x|, |x|)$ where $x \in V$.

Consider the mapping A, S, B, T from V to V defined by

$$Ax = \begin{cases} x+1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \qquad Bx = \begin{cases} x+2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$$Sx = \begin{cases} \frac{x+1}{3} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \qquad Tx = \begin{cases} \frac{x+2}{3} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Here $0, -1$ and $0, -2$ are the points of coincidence of A, S and B, T, respectively. Note that $AS(0) = SA(0)$, $AS(-1) = SA(-1)$ and $BT(0) = TB(0)$, $BT(-2) = TB(-2)$.

Thus, the mappings (A, S) and (B, T) are weakly compatible. Now, we show that the above mappings satisfies the condition of Theorem 5.2 for $a = \frac{3}{2}$, $b = \frac{1}{2}$, $c = -\frac{1}{2}$.

Case 1. $x = 0,\ y = 0$.
Then the condition $a\|Sx - Ty\|_p + b\|Ax - Sx\|_p + c\|By - Ty\|_p \le \|Ax - By\|_p$ trivially holds.

Case 2. $x \neq 0, y = 0$.
Then

$$\begin{aligned}
&a\|Sx - Ty\|_p + b\|Ax - Sx\|_p + c\|By - Ty\|_p \\
&\quad = \frac{3}{2}\left\|\frac{x+1}{3} - 0\right\|_p + \frac{1}{2}\left\|x + 1 - \frac{x+1}{3}\right\|_p - \frac{1}{2}\|0 - 0\|_p \\
&\quad = \frac{1}{2}\|x+1\|_p + \frac{1}{3}\|x+1\|_p \\
&\quad = \frac{5}{6}\|x+1\|_p \\
&\quad = \frac{5}{6}(|x+1|, |x+1|) \\
&\quad \le (|x+1|, |x+1|) \\
&\quad = \|Ax - By\|_p
\end{aligned}$$

Case 3. $x = 0,\ y \neq 0$.
Then

$$\begin{aligned}
&a\|Sx - Ty\|_p + b\|Ax - Sx\|_p + c\|By - Ty\|_p \\
&\quad = \frac{3}{2}\left\|0 - \frac{y+2}{3}\right\|_p + \frac{1}{2}\|0 - 0\|_p - \frac{1}{2}\left\|y + 2 - \frac{y+2}{3}\right\|_p \\
&\quad = \frac{1}{2}\|y+2\|_p - \frac{1}{3}\|y+2\|_p \\
&\quad = \frac{1}{6}\|y+2\|_p \\
&\quad = \frac{1}{6}(|y+2|, |y+2|) \\
&\quad \le (|y+2|, |y+2|) \\
&\quad = \|Ax - By\|_p
\end{aligned}$$

Case 4. $x \neq 0, y \neq 0$.
Then

$$\begin{aligned}
&a\|Sx - Ty\|_p + b\|Ax - Sx\|_p + c\|By - Ty\|_p \\
&\quad = \frac{3}{2}\left\|\frac{x+1}{3} - \frac{y+2}{3}\right\|_p + \frac{1}{2}\left\|x + 1 - \frac{x+1}{3}\right\|_p - \frac{1}{2}\left\|y + 2 - \frac{y+2}{3}\right\|_p \\
&\quad = \frac{1}{2}\|x - y - 1\|_p + \frac{1}{3}\|x+1\|_p - \frac{1}{3}\|y+2\|_p \\
&\quad = \frac{1}{2}\|x - y - 1\|_p + \frac{1}{3}[\|x+1\|_p - \|y+2\|_p]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\|x-y-1\|_p + \frac{1}{3}\|x-y-1\|_p \qquad (|x|-|y| \le |x-y|) \\
&= \frac{5}{6}\|x-y-1\|_p \\
&= \frac{5}{6}(|x-y-1|, |x-y-1|) \\
&\le (|x-y-1|, |x-y-1|) \\
&= \|Ax-By\|_p
\end{aligned}
$$

Thus, the mappings A,S,B,T satisfies the conditions of Theorem 5.2.
Consequently 0 is the unique common fixed point of A,S,B,T.

6. Conclusion

In this paper, idea of fuzzy cone normed linear space is introduced in a different approach considering a fuzzy real Banach space in the sense of *Felbin*($L = \min, U = \min$). Some fundamental results and fixed point results are established in such spaces. There is huge scope of further study such as operator norm, fixed point theory, approximation theory etc. in such spaces. Since fuzzy mathematics along with the classical ones are constantly developing, so this concept of fuzzy cone normed liear space can also play an important part inthe new fuzzy era.

Acknowledgement

The authors are grateful to reviewers for their valuable suggestions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] T. Bag and S.K. Samanta, Finite dimensional fuzzy normed linear spaces, *Annals of Fuzzy Mathematics and Informatics* **6**(2) (2013), 271 – 283, `http://www.afmi.or.kr/papers/2013/Vol-06_No-02/AFMI-6-2(227--453)/AFMI-6-2(271--283)-H-120903.pdf`.

[2] T. Bag, Finite dimensional fuzzy cone normed linear spaces, *International Journal of Mathematics and Scientific Computing* **3**(1) (2013), 9 – 14, `https://www.veltech.edu.in/wp-content/uploads/2016/04/Paper-04-13.pdf`.

[3] T. Bag, Fuzzy cone metric spaces and fixed point theorems of contractive mappings, *Annals of Fuzzy Mathematics and Informatics* **6**(3) (2013), 657 – 668, `http://www.afmi.or.kr/papers/2013/Vol-06_No-03/AFMI-6-3(455--798)/AFMI-6-3(657--668)-H-121022R1.pdf`.

[4] S.-C. Cheng and J.N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, in: *First International Conference on Fuzzy Theory and Technology Proceedings, Abstracts and Summaries*, P.P. Wang, J. Dai and J.C.-Y. Tyan (eds.), pp. 193 – 197 (1992), https://creighton.pure.elsevier.com/en/publications/fuzzy-linear-operators-and-fuzzy-normed-linear-spaces.

[5] C. Felbin, Finite dimensional fuzzy normed linear spaces, *Fuzzy Sets and Systems* **48** (1992), 239 – 248, DOI: 10.1016/0165-0114(92)90338-5.

[6] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications* **332** (2007), 1468 – 1476, DOI: 10.1016/j.jmaa.2005.03.087.

[7] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets and Systems* **12** (1984), 215 – 229, DOI: 10.1016/0165-0114(84)90069-1.

[8] A.K. Katsaras, Fuzzy topological vector spaces, *Fuzzy Sets and Systems* **12** (1984), 143 – 154, DOI: 10.1016/0165-0114(84)90034-4.

[9] T. Öner, M. Kandemir and B. Tanay, Fuzzy cone metric spaces, *Journal of Nonlinear Sciences and Applications* **8** (2015), 610 – 616, DOI: 10.22436/jnsa.008.05.13.

[10] P. Tamang and T. Bag, Some results on finite dimensional fuzzy cone normed linear space, *Annals of Fuzzy Mathematics and Informatics* **13**(1) (2017), 123 – 134, http://www.afmi.or.kr/papers/2017/Vol-13_No-01/PDF/AFMI-13-1(123-134)-H-160422-1R2.pdf.

[11] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.