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Order Bounded Elements of Topological *-algebras

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Abstract. Several different notions of *bounded* element of a topological *-algebra $\mathfrak{A}$ are considered. The case where boundedness is defined via the natural order of $\mathfrak{A}$ is examined in more details and it is proved that under certain circumstances (in particular, when $\mathfrak{A}$ possesses sufficiently many *-representations) *order boundedness* is equivalent to *spectral boundedness*.

1. **Introduction and preliminaries**

Let $\mathfrak{A}$ be a *topological *-algebra* (i.e., a *-algebra equipped with a locally convex topology τ such that for each $a \in \mathfrak{A}$ the mappings $x \mapsto ax$, $a \mapsto xa$ and the involution * are continuous in $\mathfrak{A}[\tau]$). The notion of *bounded element* of $\mathfrak{A}$ was first introduced by Allan [1] with the aim of developing a spectral theory for topological *-algebras. This definition was suggested by the successful spectral analysis for closed operators in Hilbert space $\mathscr{H}$: a complex number λ is in the spectrum $\sigma(T)$ of a closed operator T if $T - \lambda I$ has no inverse in the *-algebra $\mathscr{B}(\mathscr{H})$ of bounded operators. What makes this definition particularly significant is the fact that $\sigma(T)$ is compact if, and only if, T is a bounded operator. Allan's definition sounds as follows: an element x of the topological *-algebra $\mathfrak{A}[\tau]$ is *Allan bounded* if there exists $\lambda \neq 0$ such that the set $\{(\lambda^{-1}x)^n;\ n = 1, 2, \dots\}$ is a bounded subset of $\mathfrak{A}[\tau]$.

There are, however, several other possibilities for defining bounded elements. For instance, one may say that x is *left τ-bounded*, if there exists $\gamma_x > 0$ such that

$$p_\alpha(xy) \le \gamma_x p_\alpha(y), \quad \forall\, \alpha \in \Delta;\ \forall\, y \in \mathfrak{A},$$

where $\{p_\alpha; \alpha \in \Delta\}$ is a directed family of seminorms defining the topology τ of $\mathfrak{A}$ [5]; or *spectrally bounded* if its spectrum is a bounded subset of the complex plane.

Moreover some attempts to extend this notion to the larger set-up of locally convex quasi *-algebras [9, 11, 12] or locally convex partial *-algebras [3, 4] has been done. But in these cases, Allan's notion cannot be adopted, since powers of a given element x need not be defined.

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In all cases, what one expects when dealing with bounded elements is that they are realized by bounded operators by *any* (continuous, in a certain sense) *-representation of $\mathfrak{A}$ in Hilbert space. This could be a reasonable definition in itself, if we were sure that $\mathfrak{A}$ possesses sufficiently many *-representations in Hilbert space.

Bounded elements in purely algebraic terms have been considered by Vidav [15] and Schmüdgen [8] with respect to some (positive) wedge. We extend this purely algebraic definition by considering as strongly positive elements those belonging to the τ-closure in $\mathfrak{A}$ of the, say, algebraic cone of positive elements of a *-algebra. The main result is that *order bounded* elements, as we will call them, allow equivalent characterizations in terms of continuous positive linear functionals and also in terms of *-representations, that, in the case the positive wedge is a cone, are sufficiently many to separate points of $\mathfrak{A}$.

The following preliminary definitions will be needed in the sequel. For more details we refer to [7, 2].

Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{D}$ a dense subspace of $\mathscr{H}$. We denote by $\mathscr{L}^\dagger(\mathscr{D})$ the set of all (closable) linear operators X such that $D(X)=\mathscr{D}$, $D(X^*)\supseteq\mathscr{D}$ and $X\mathscr{D}\subset\mathscr{D}$, $X^*\mathscr{D}\subset\mathscr{D}$. The set $\mathscr{L}^\dagger(\mathscr{D})$ is a *-algebra with respect to the ordinary operations of addition, multiplication by scalars and multiplication and the involution $X\mapsto X^\dagger := X^*\restriction_{\mathscr{D}}$. We put $I_{\mathscr{D}}=I\restriction_{\mathscr{D}}$. Then $I_{\mathscr{D}}$ is the unit of $\mathscr{L}^\dagger(\mathscr{D})$. A *-subalgebra of $\mathscr{L}^\dagger(\mathscr{D})$ is called an *O*-algebra* [7].

Let $\mathfrak{A}$ be a *-algebra and $\mathscr{D}_\pi$ a dense domain in a certain Hilbert space $\mathscr{H}_\pi$. A linear map π from $\mathfrak{A}$ into $\mathscr{L}^\dagger(\mathscr{D}_\pi)$ such that:

(i) $\pi(a^*)=\pi(a)^\dagger$, $\forall\, a\in\mathfrak{A}$,

(ii) if $a,b\in\mathfrak{A}$, then $\pi(ab)=\pi(a)\pi(b)$, is called a **-representation* of $\mathfrak{A}$. Moreover, if $\mathfrak{A}$ has a unit $e\in\mathfrak{A}$, we assume $\pi(e)=I_{\mathscr{D}_\pi}$, the identity of $\mathscr{D}_\pi$.

A *-representation π of a topological *-algebra $\mathfrak{A}[\tau]$ is said to be a (τ,τ_{w})-continuous if, for every $\xi,\eta\in\mathscr{D}_\pi$, there exists a τ-continuous seminorm p on $\mathfrak{A}$ such that

$$|\langle\pi(a)\xi|\eta\rangle|\le p(a),\quad \forall\, a\in\mathfrak{A}.$$

A linear functional ω on $\mathfrak{A}$ is called positive if $\omega(a^*a)\ge 0$, for every $a\in\mathfrak{A}$. To every positive linear functional ω on $\mathfrak{A}$ there corresponds a Hilbert space $\mathscr{H}_\omega$ and a linear map λ_ω from $\mathfrak{A}$ into a dense subspace $\lambda_\omega(\mathfrak{A})\subset\mathscr{H}_\omega$ and a *-representation π_ω acting on a dense domain $\mathscr{D}_{\pi_\omega}$ such that $\lambda_\omega(\mathfrak{A})\subset\mathscr{D}_{\pi_\omega}\subset\mathscr{H}_\pi$ and

$$\omega(b^*xa)=\langle\pi_\omega(x)\lambda_\omega(a)|\lambda_\omega(b)\rangle,\quad \forall\, a,b,x\in\mathfrak{A}.$$

The representation π_ω can be taken to be closed [7]. If $\mathfrak{A}$ has a unit e, then there exists a vector ξ_ω such that $\lambda_\omega(\mathfrak{A})=\{\pi_\omega(a)\xi_\omega,\, a\in\mathfrak{A}\}$ and

$$\omega(x)=\langle\pi_\omega(x)\xi_\omega|\xi_\omega\rangle,\quad \forall\, x\in\mathfrak{A}.$$

We will refer to π_ω as to the GNS *-representation of $\mathfrak{A}$ defined by ω.

2. Topological algebras with sufficiently many *-representations

Throughout this paper we will consider only topological *-algebras possessing sufficiently many continuous *-representations. More precisely

Definition 2.1. A topological *-algebra $\mathfrak{A}[\tau]$ is called *faithfully representable*, shortly an FR*-algebra*, if for every $x \in \mathfrak{A} \setminus \{0\}$ there exists a $(\tau, \tau_{\mathrm{w}})$-continuous *-representation π of $\mathfrak{A}$ such that $\pi(x) \neq 0$.

We denote by $\mathrm{Rep}_c(\mathfrak{A})$ the family of all $(\tau, \tau_{\mathrm{w}})$-continuous *-representation of $\mathfrak{A}$.

The next result is easily proved.

Lemma 2.2. *Let $\mathfrak{A}[\tau]$ be a topological *-algebra. The following statements are equivalent.*

(i) $\mathfrak{A}$ *is an* FR*-*algebra.*

(ii) *For every $x \in \mathfrak{A} \setminus \{0\}$, there exists a τ-continuous positive linear functional ω such that $\omega(x^*x) > 0$*

3. Order bounded elements

3.1. *Order structure*

Let $\mathfrak{A}$ be a *-algebra. We denote by

$$\mathfrak{A}^+_{\mathrm{alg}} = \left\{ \sum_{k=1}^{n} x_k^* x_k,\ x_k \in \mathfrak{A},\ n \in \mathbb{N} \right\}$$

the set (wedge) of positive elements of $\mathfrak{A}$.

If $\mathfrak{A}[\tau]$ is a topological *-algebra, *strongly positive* elements of $\mathfrak{A}$ are then defined as members of $\overline{\mathfrak{A}^+_{\mathrm{alg}}}^{\tau}$. We put $\mathfrak{A}^+ := \overline{\mathfrak{A}^+_{\mathrm{alg}}}^{\tau}$.

The set $\mathfrak{A}^+$ is an *m-admissible wedge* in the sense of Schmüdgen [7, Sect. 1.4]; i.e.,

(1) $e \in \mathfrak{A}^+$;

(2) $x + y \in \mathfrak{A}^+$, $\forall\ x, y \in \mathfrak{A}^+$;

(3) $\lambda x \in \mathfrak{A}^+$, $\forall\ x \in \mathfrak{A}^+$, $\lambda \geq 0$;

(4) $a^* x a \in \mathfrak{A}^+$, $\forall\ x \in \mathfrak{A}^+$, $a \in \mathfrak{A}$.

$\mathfrak{A}^+$ defines an order on the real vector space $\mathfrak{A}_h = \{x \in \mathfrak{A} : x = x^*\}$ by $x \leq y \Leftrightarrow y - x \in \mathfrak{A}^+$.

The following statement is easily proved.

Proposition 3.1. *If $x \geq 0$, then $\pi(x) \geq 0$, for every $\pi \in \mathrm{Rep}_c(\mathfrak{A})$.*

Theorem 3.2. *Assume that $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$. For every $a \in \mathfrak{A}^+$, $a \neq 0$, there exists a τ-continuous linear functional ω on $\mathfrak{A}$ with the properties*

(a) *$\omega(x) \geq 0$, $\forall\ x \in \mathfrak{A}^+$;*

(b) $\omega(a) > 0$.

Proof. Consider the real vector space $\mathfrak{A}_h$. The set $\{a\}$ is obviously convex and compact and does not intersect $(-\mathfrak{A}^+)$. Hence by [6, Chapter 2, §5, Proposition 4], there exists a closed hyperplane separating these two sets. Let $g(x) = 0$ be the equation of this hyperplane. Then, either $g(a) > 0$ and $g(-\mathfrak{A}^+) < 0$ (in which case we take $\omega = g$) or the contrary (in this case we take $\omega = -g$). □

Definition 3.3. A linear functional ω on $\mathfrak{A}$ is called *strongly positive* if $\omega(x) \geq 0$, $\forall\ x \in \mathfrak{A}^+$.

Clearly, if ω is positive and τ-continuous, then it is strongly positive.

The set of strongly positive linear functionals on $\mathfrak{A}$ will be denoted by $\mathscr{P}(\mathfrak{A})$, while $\mathscr{P}_c(\mathfrak{A})$ will denote the subset of $\mathscr{P}(\mathfrak{A})$ consisting of its τ-continuous elements.

Definition 3.4. A family of strongly positive linear functionals $\mathscr{F}$ on $\mathfrak{A}[\tau]$ is called *sufficient* if for every $x \in \mathfrak{A}^+$, $x \neq 0$ there exists $\omega \in \mathscr{F}$ such that $\omega(x) > 0$.

Corollary 3.5. *Let $\mathfrak{A}[\tau]$ be a topological *-algebra. The following statements are equivalent.*

(i) $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$, *i.e.* $\mathfrak{A}^+$ *is a cone.*
(ii) $\mathscr{P}_c(\mathfrak{A})$ *is sufficient.*
(iii) $\mathfrak{A}[\tau]$ *is an* FR**-algebra.*

Proof. (i)⇒(ii) is Theorem 3.2. As for (ii)⇒(i), if $x \in \mathfrak{A}^+ \cap (-\mathfrak{A}^+)$ and $\omega \in \mathscr{P}_c(\mathfrak{A})$, then $\omega(-x) = -\omega(x) \geq 0$. Hence $\omega(x) = 0$. Since ω is arbitrary, it follows that $x = 0$. (ii) ⟺ (iii) follows from Lemma 2.2. Finally we prove that (iii)⟺(i). Let $x \in \mathfrak{A}^+ \cap (-\mathfrak{A}^+)$, $x \neq 0$. Then there exist $\pi \in \mathrm{Rep}_c(\mathfrak{A})$ and $\xi \in \mathscr{D}_\pi$ such that $\langle \pi(x)\xi|\xi\rangle \neq 0$. Since x is the limit of a net of elements of $\mathfrak{A}^+_{\mathrm{alg}}$, we get $\langle \pi(x)\xi|\xi\rangle > 0$. Similarly, $\langle \pi(-x)\xi|\xi\rangle > 0$. This is a contradiction. □

Proposition 3.6. *Let $\mathfrak{A}[\tau]$ be an* FR**-algebra with $\mathscr{P}_c(\mathfrak{A})$ sufficient. Assume that the following condition* (P) *holds*

If $y \in \mathfrak{A}$ and $\omega(a^* y a) \geq 0$, for every $\omega \in \mathscr{P}_c(\mathfrak{A})$ and $a \in \mathfrak{A}$, then $y \in \mathfrak{A}^+$.

Then, for an element $x \in \mathfrak{A}$, the following statements are equivalent.

(i) $x \in \mathfrak{A}^+$;
(ii) $\omega(x) \geq 0$, *for every* $\omega \in \mathscr{P}_c(\mathfrak{A})$
(iii) $\pi(x) \geq 0$, *for every* $\pi \in \mathrm{Rep}_c(\mathfrak{A})$.

Proof. (i)⇒(ii) is a trivial consequence of the definition of strongly positive element and of the continuity of every $\omega \in \mathscr{P}_c(\mathfrak{A})$ w. r. to τ.
(ii)⇒(iii): Let π be (τ, τ_w)-continuous *-representation π of $\mathfrak{A}$. Define $\omega_\xi(x) := \langle \pi(x)\xi|\xi\rangle$ with $\xi \in \mathscr{D}_\pi$, $\|\xi\| = 1$. Then $\omega_\xi \in \mathscr{P}_c(\mathfrak{A})$, since

$$|\omega_\xi(x)| = |\langle \pi(x)\xi|\xi\rangle| \leq p(x)$$

for some τ-continuous seminorm p on $\mathfrak{A}$. Then, if a satisfies (ii), $\langle\pi(a)\xi|\xi\rangle \geq 0$, for every $\xi \in \mathcal{D}_\pi$.
(iii)⇒(i): Let $\omega \in \mathscr{P}_c(\mathfrak{A})$ and let π_ω be the corresponding GNS representation. Then, π_ω is (τ, τ_w)-continuous. Indeed,

$$|\langle\pi_\omega(x)\lambda_\omega(a)|\lambda_\omega(b)\rangle| = |\omega(b^*xa)| \leq p(x), \quad \forall\, x \in \mathfrak{A};\, a, b \in \mathfrak{A},$$

for some τ-continuous seminorm p on $\mathfrak{A}$ (due to the continuity of ω and of the multiplications). If (iii) holds, then $\pi_\omega(x) \geq 0$. This implies that $\omega(a^*xa) \geq 0$, for every $a \in \mathfrak{A}$. The statement then follows from the assumption (P). □

Remark 3.7. If $\mathfrak{A}$ has a unit, (P) is equivalent to the following

(P′) If $y \in \mathfrak{A}$ and $\omega(y) \geq 0$, for every $\omega \in \mathscr{P}_c(\mathfrak{A})$, then $y \in \mathfrak{A}^+$.

Remark 3.8. The condition (P) together with $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$ implies that, for every nonzero $x \in \mathfrak{A}$, there exists $\omega \in \mathscr{P}_c(\mathfrak{A})$ such that $\omega(x) \neq 0$. Indeed, if $\omega(x) = 0$ for every $\omega \in \mathscr{P}_c(\mathfrak{A})$, then $x \in \mathfrak{A}^+$ and also $-x \in \mathfrak{A}^+$; hence $x = 0$.

3.2. *Order bounded elements*

Let $\mathfrak{A}[\tau]$ be a topological *-algebra with unit e. As we have seen in Section 3.1, $\mathfrak{A}[\tau]$ has a natural order related to the topology τ. This order can be used to define *bounded* elements. In what follows, we will assume that $\mathfrak{A}$ has a unit e.

Let $x \in \mathfrak{A}$; put $\Re(x) = \frac{1}{2}(x + x^*)$, $\Im(x) = \frac{1}{2i}(x - x^*)$. Then $\Re x, \Im(x) \in \mathfrak{A}_h$ (the set of selfadjoint elements of $\mathfrak{A}$) and $x = \Re(x) + i\Im(x)$.

Definition 3.9. An element $x \in \mathfrak{A}$ is called *order bounded* if there exists $\gamma \geq 0$ such that

$$\pm\Re(x) \leq \gamma e; \qquad \pm\Im(x) \leq \gamma e.$$

We denote by $\mathfrak{A}_b$ the family of order bounded elements.

Proposition 3.10. *The following statements hold:*

(1) $\alpha x + \beta y \in \mathfrak{A}_b$, $\forall\, x, y \in \mathfrak{A}_b$, $\alpha, \beta \in \mathbb{C}$.
(2) $x \in \mathfrak{A}_b \Leftrightarrow x^* \in \mathfrak{A}_b$.
(3) $x, y \in \mathfrak{A}_b \Rightarrow xy \in \mathfrak{A}_b$.
(4) $a \in \mathfrak{A}_b \Leftrightarrow aa^* \in \mathfrak{A}_b$.

*Hence, $\mathfrak{A}_b$ is a *-algebra.*

Proof. See [8, Lemma 2.1]. □

For $x \in \mathfrak{A}_h$, put

$$\|x\|_b := \inf\{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}.$$

$\|\cdot\|_b$ is a seminorm on the real vector space $(\mathfrak{A}_b)_h$.

Lemma 3.11. *If $\mathfrak{A} \cap (-\mathfrak{A}^+) = \{0\}$, $\|\cdot\|_b$ is a norm on $(\mathfrak{A}_b)_h$.*

Proof. Put $E = \{\gamma > 0 : -\gamma e \leq x \leq \gamma e\}$. If $\inf E = 0$, then for every $\epsilon > 0$ there exists $\gamma_\epsilon \in E$ such that $\gamma_\epsilon < \epsilon$. This implies that $-\epsilon e \leq x \leq \epsilon e$. If $\omega \in \mathscr{P}_c(\mathfrak{A})$, we get $-\epsilon\, \omega(e) \leq \omega(x) \leq \epsilon\, \omega(e)$ (we may suppose $\omega(e) > 0$ for every $\omega \in \mathscr{P}_c(\mathfrak{A})$, since the Cauchy-Schwarz inequality implies that, if $\omega(e) = 0$, $\omega \equiv 0$). Hence, $\omega(x) = 0$. By the sufficiency of $\mathscr{P}_c(\mathfrak{A})$, it follows that $x = 0$. □

Proposition 3.12. *If $x \in \mathfrak{A}_b$, then $\pi(x)$ is a bounded operator, for every (τ, τ_w)-continuous *-representation of $\mathfrak{A}$. Moreover, if $x = x^*$, $\|\pi(x)\| \leq \|x\|_b$.*

Proof. It follows easily from Proposition 3.1 and from the definitions. □

Theorem 3.13. *Let $\mathfrak{A}[\tau]$ be a topological *-algebra with unit e and assume that condition* (P) *holds. For $x \in \mathfrak{A}$, the following statements are equivalent.*

(i) *x is order bounded.*

(ii) *There exists $\gamma_x > 0$ such that*

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \quad \forall\, \omega \in \mathscr{P}_c(\mathfrak{A}),\, a \in \mathfrak{A}.$$

(iii) *There exists $\gamma_x > 0$ such that*

$$|\omega(b^*xa)| \leq \gamma_x \omega(a^*a)^{1/2} \omega(b^*b)^{1/2}, \quad \forall\, \omega \in \mathscr{P}_c(\mathfrak{A}),\, a, b \in \mathfrak{A}.$$

(iv) *$\pi(x)$ is bounded, for every $\pi \in \mathrm{Rep}_c(\mathfrak{A})$, and*

$$\sup\{\|\overline{\pi(x)}\|,\, \pi \in \mathrm{Rep}_c(\mathfrak{A})\} < \infty\}.$$

Proof. It is sufficient to consider the case $x = x^*$ and again we suppose $\omega(e) > 0$, for every $\omega \in \mathscr{P}_c(\mathfrak{A})$.

(i)⇒ (ii): If $x = x^*$ is order bounded, then also x^2 is order bounded. Thus, for some $\mu > 0$, $a^*x^2a \leq \mu^2 a^*a$, for every $a \in \mathfrak{A}$. Hence,

$$|\omega(a^*xa)| \leq \omega(a^*a)^{1/2} \omega(a^*x^2a)^{1/2} \leq \mu\omega(a^*a), \quad \forall\, \omega \in \mathscr{P}_c(\mathfrak{A}),\, a \in \mathfrak{A}.$$

(ii)⇒ (i): Assume now that there exists $\gamma_x > 0$ such that

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \quad \forall\, \omega \in \mathscr{P}_c(\mathfrak{A}),\, a \in \mathfrak{A}$$

and define

$$\tilde{\gamma} := \sup\{|\omega(a^*xa)| : \omega \in \mathscr{P}_c(\mathfrak{A}),\, a \in \mathfrak{A}, \omega(a^*a) = 1\}.$$

Then, for an arbitrary $\omega' \in \mathscr{P}_c(\mathfrak{A})$, we get,

$$\omega'(\tilde{\gamma} e \pm x) = \tilde{\gamma}\omega'(e) \pm \omega'(x) = \omega'(e)(\tilde{\gamma} \pm \omega'(u^*xu)) \geq 0,$$

where $u = \frac{e}{\omega'(e)^{1/2}}$.

Hence, $\omega'(\tilde{\gamma} e \pm x) \geq 0$, for every $\omega' \in \mathscr{P}_c(\mathfrak{A})$. Then, by (P), $-\tilde{\gamma} e \leq x \leq \tilde{\gamma} e$; i.e. x is order bounded.

(i)⇒ (iii): The GNS representation π_ω is (τ, τ_w)-continuous, hence, by Proposition 3.12, if $x = x^* \in \mathfrak{A}$, $\pi_\omega(x)$ is bounded and $\|\overline{\pi(x)}\| \leq \|x\|_b$. Thus,

$$\begin{aligned} |\omega(b^*xa)| &= |\langle \pi_\omega(x)\lambda_\omega(a) | \lambda_\omega(b)\rangle| \leq \|\pi_\omega(x)\|\, \|\lambda_\omega(a)\| \|\lambda_\omega(b)\| \\ &\leq \|x\|_b\, \omega(a^*a)^{1/2} \omega(b^*b)^{1/2} \end{aligned}$$

(iii)$\Rightarrow$ (ii) is obvious.

(iii)$\Rightarrow$ (iv): Let $\pi \in \mathrm{Rep}_c(\mathfrak{A})$ and $\xi \in \mathcal{D}_\pi$. Put $\omega_\xi(y) := \langle \pi(y)\xi|\xi\rangle$, $y \in \mathfrak{A}$. Then $\omega_\xi \in \mathcal{P}_c(\mathfrak{A})$. Hence by (iii), $|\omega_\xi(x)| \leq \gamma_x \omega_\xi(e)$. Or, in other terms, $|\langle \pi(x)\xi|\xi\rangle \leq \gamma_x \|\xi\|^2$. This, in turn easily implies that $|\langle \pi(x)\xi|\eta\rangle| \leq \gamma_x \|\xi\|\|\eta\|$, for every $\xi, \eta \in \mathcal{D}_\pi$. Hence $\pi(x)$ is bounded and $\|\overline{\pi(x)}\| \leq \gamma_x$.

(iv)$\Rightarrow$(i): Put $\gamma_x := \sup\{\|\overline{\pi(x)}\|, \pi \in \mathrm{Rep}_c(\mathfrak{A})\}$. Then

$$|\langle \pi(x)\xi|\xi\rangle \leq \|\pi(x)\xi\| \leq \gamma_x \|\xi\|^2, \quad \forall\, \xi \in \mathcal{D}_\pi.$$

Hence, $-\gamma_x I_{\mathcal{D}_\pi} \leq \pi(x) \leq \gamma_x I_{\mathcal{D}_\pi}$. In particular this holds for the GNS representation associated to every $\omega \in \mathcal{P}_c(\mathfrak{A})$. Therefore,

$$\omega(x + \gamma_x e) \geq 0 \ \text{ and } \ \omega(x - \gamma_x e) \leq 0, \quad \forall\, \omega \in \mathcal{P}_c(\mathfrak{A}).$$

By (P) it follows that $-\gamma_x e \leq x \leq \gamma_x e$. □

Let x be order bounded and define

$$\mathsf{q}(x) = \sup\{|\omega(b^* x a)|;\ \omega \in \mathcal{P}_c(\mathfrak{A}), a, b \in \mathfrak{A}; \omega(a^* a) = \omega(b^* b) = 1\}.$$

Lemma 3.14. $\mathsf{q}(x) = \|x\|_b$*, for every* $x = x^* \in \mathfrak{A}_b$.

Proof. The proof of Proposition 3.12 shows that for $x = x^*$,

$$\|x\|_b \leq \sup\{|\omega(a^* x a)| : \omega \in \mathcal{P}_c(\mathfrak{A}),\ a \in \mathfrak{A}, \omega(a^* a) = 1\}.$$

Hence, $\|x\|_b \leq \mathsf{q}(x)$, for every $x = x^* \in \mathfrak{A}_b$. For any $\gamma > 0$ such that $-\gamma e \leq x \leq \gamma e$, we have, by the proof of Theorem 3.13, $\mathsf{q}(x) \leq \gamma$; whence the statement follows. □

Since q extends $\|\cdot\|_b$, we adopt the notation $\|\cdot\|_b$ for both. It is easy to see that $\|\cdot\|_b$ is a norm on $\mathfrak{A}_b$ such that, for every $x, y \in \mathfrak{A}_b$,

(i) $\|x^*\|_b = \|x\|_b$;
(ii) $\|xy\|_b \leq \|x\|_b \, \|y\|_b$.

Moreover, for every $x \in \mathfrak{A}_b$,

$$\|x\|_b = \sup\{\|\overline{\pi(x)}\|,\ \pi \in \mathrm{Rep}_c(\mathfrak{A})\}. \tag{3.1}$$

Proposition 3.15. *Let* $\mathfrak{A}$ *be a* FR**-algebra. Then* $\|\cdot\|_b$ *is* C^**-norm on* $\mathfrak{A}_b$.

Proof. This follows easily from (3.1). □

The family of functionals $\mathcal{P}_c(\mathfrak{A})$ may be used to define on $\mathfrak{A}$ some more topologies. In what follows we will use the *strong** topology τ_{s^*}, defined by the family of seminorms

$$x \in \mathfrak{A} \to \max\{\omega(x^* x)^{1/2}, \omega(x x^*)^{1/2}\}, \quad \omega \in \mathcal{P}_c(\mathfrak{A}).$$

Proposition 3.16. $\mathfrak{A}$ *be a* FR**-algebra with unit e and assume that* $\mathfrak{A}$ *is* τ_{s^*}*-complete. Then* $\mathfrak{A}_b$ *is a C*-algebra with norm* $\|\cdot\|_b$.

Proof. Since $\|\cdot\|_b$ is a C*-norm on $\mathfrak{A}_b$, we need only to prove the completeness of $\mathfrak{A}_b$.

Let $\{x_n\}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_b$. Then $\{x_n^*\}$ is Cauchy too. Hence, for every $\omega \in \mathscr{P}_c(\mathfrak{A})$ and $a \in \mathfrak{A}$ we have

$$\omega(a^*(x_n^* - x_m^*)(x_n - x_m)a) \to 0, \text{ as } n, m \to \infty$$

and

$$\omega(a^*(x_n - x_m)(x_n^* - x_m^*)a) \to 0, \text{ as } n, m \to \infty.$$

Therefore, $\{x_n\}$ is Cauchy also with respect to τ_{s^*}. Then, there exists $x \in \mathfrak{A}$ such that $x_n \xrightarrow{\tau_{s^*}} x$. Since

$$\omega(a^*x^*xa) = \lim_{n\to\infty} \omega(a^*x_n^*x_na) \leq \limsup_{n\to\infty} \|x_n\|_b^2\, \omega(a^*a)$$

and $\limsup_{n\to\infty} \|x_n\|_b^2 < \infty$ (by the boundedness of the sequence $\{\|x_n\|_b\}$), we conclude that x is order bounded. Finally, by the Cauchy condition, for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that, for every $n, m > n_\epsilon$, $\|x_n - x_m\|_b < \epsilon$. This implies that

$$\omega(a^*(x_n^* - x_m^*)(x_n - x_m)a) < \epsilon\omega(a^*a), \quad \forall\, \varphi \in \mathscr{M},\, a \in \mathfrak{A}.$$

Then if we fix $n > n_\epsilon$ and let $m \to \infty$, we obtain

$$\omega(a^*(x_n^* - x^*)(x_n - x)a) \leq \epsilon\omega(a^*a), \quad \forall\, \varphi \in \mathscr{M},\, a \in \mathfrak{A}.$$

This, in turn, implies that $\|x_n - x\|_b \leq \epsilon$, for $n \geq n_\epsilon$. □

3.3. *Spectral boundedness*

Once one has at hand the algebra $\mathfrak{A}_b$ of bounded elements of a topological *-algebra $\mathfrak{A}$, it is natural to use it for a coherent definition of spectrum and investigate on the relationship between the order boundedness of an element $x \in \mathfrak{A}$ and the boundedness of its spectrum. But for making this meaningful one has to suppose that $\mathfrak{A}_b$ is large enough to avoid trivial situations. Thus, in this section we will consider only an FR*-algebra $\mathfrak{A}$ satisfying the following condition

(A) $\mathfrak{A}_b$ is a C*-algebra, τ_{s^*}-dense in $\mathfrak{A}$.

Definition 3.17. Let $\mathfrak{A}$ be an FR*-algebra $\mathfrak{A}$, with unit e, and satisfying (A). The *resolvent* $\rho_\circ(x)$ of x is defined by

$$\rho_\circ(x) = \{\lambda \in \mathbb{C} : (x - \lambda e)^{-1} \text{exists in } \mathfrak{A}_b\}.$$

The *spectrum* of x is defined as $\sigma_\circ(x) := \mathbb{C} \setminus \rho_\circ(x)$.

In similar way as in [9] it can be proved that: (a) $\rho_\circ(x)$ is an open subset of the complex plane; (b) the map $\lambda \in \rho_\circ(x) \mapsto (x - \lambda e)^{-1} \in \mathfrak{A}_b$ is analytic in each connected component of $\rho_\circ(x)$; (c) $\sigma_\circ(x)$ is nonempty.

As usual, we define the *spectral radius* of $x \in \mathfrak{A}$ by

$$r_\circ(x) := \sup\{|\lambda| : \lambda \in \sigma_\circ(x)\}.$$

Theorem 3.18. *Let $\mathfrak{A}$ be an* FR**-algebra $\mathfrak{A}$, with unit e, and satisfying* (A). *Then $r_\circ(x) < \infty$ if and only if $x \in \mathfrak{A}_b$.*

Proof. If $x \in \mathfrak{A}_b$, then $\sigma_\circ(x)$ coincides with the spectrum of x as an element of the C*-algebra $\mathfrak{A}_b$ and so $\sigma_\circ(x)$ is compact. Conversely, assume that $r_\circ(x) < \infty$. Then the function $\lambda \mapsto (x - \lambda e)^{-1}$ is $\|\cdot\|_b$-analytic in the region $|\lambda| > r_\circ(x)$. Therefore it has there a $\|\cdot\|_b$-convergent Laurent expansion

$$(x - \lambda e)^{-1} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}, \quad |\lambda| > r^{\mathscr{M}}(x),$$

with $a_k \in \mathfrak{A}_b$ for each $k \in \mathbb{N}$. As usual

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{(x - \lambda e)^{-1}}{\lambda^{-k+1}} d\lambda, \quad k \in \mathbb{N},$$

where $\gamma := \{\lambda \in \mathbb{C} : |\lambda| = R : R > r_\circ(x)\}$ and the integral on the r.h.s. is meant to converge with respect to $\|\cdot\|_b$.

Using the previous integral representation and the continuity, for every $\omega \in \mathscr{P}_c(\mathfrak{A})$ and $b, b' \in \mathfrak{A}$, we have

$$\omega(b'^* x a_k b) = \omega(b'^* a_{k+1} b).$$

This implies that $x a_k = a_{k+1}$.

In particular,

$$\begin{aligned}\omega(b'^*(x a_1) b) &= \frac{1}{2\pi i} \int_\gamma \omega(b'^* x (x - \lambda e)^{-1} b) d\lambda \\ &= -\,\omega(b'^* x b).\end{aligned}$$

Hence $x a_1 = -x$. Thus finally $x = -a_2 \in \mathfrak{A}_b$. □

Corollary 3.19. *Let $\mathfrak{A}$ be an* FR**-algebra $\mathfrak{A}$, with unit e, and satisfying* (A). *Then*

$$\begin{cases} r_\circ(x) \le \|x\|_b & \text{if } x \in \mathfrak{A}_b \\ r_\circ(x) = +\infty & \text{if } x \notin \mathfrak{A}_b. \end{cases}$$

3.4. *Concluding remark*

As we have seen the notion of order boundedness for elements of a topological *-algebra $\mathfrak{A}$ has plenty of interesting consequences on the structure of $\mathfrak{A}$, at least if $\mathfrak{A}$ has sufficiently many representations. There are however several questions that remain unsolved. The first one concerns the *size* of the algebra $\mathfrak{A}_b$ of order bounded elements, since the density of $\mathfrak{A}_b$ in $\mathfrak{A}$ cannot be deduced from the set-up presented in this paper and probably requires tighter assumptions on the topology τ of $\mathfrak{A}$. The second one is about the notion of *left τ-bounded* element given in the Introduction: as shown in [3] for the case of locally convex partial *-algebras, this definition leads to reasonable results on the spectral behavior. It is not difficult to see that left τ-boundedness implies, in the case of topological *-algebras, Allan boundedness; thus it is really worth investigating it in detail. We leave both these questions to future papers.

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