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Research Article

Some Properties of a Generalized Integral Operator

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Abstract. The object of the present paper is to derive some properties of holomorphic functions in the open unit disc which are defined by using a new generalized integral operator by applying a lemma due to Miller and Mocanu. Also we mention some interesting consequences of our main results.

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1. Introduction and Preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathscr{A}(p,n)$ denote the family of functions of the form

$$f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad p,n \in \mathbb{N},$$

which are holomorphic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. In particular, we set $\mathscr{A}(p,1) = \mathscr{A}_p$, $\mathscr{A}(1,n) = \mathscr{A}(n)$ and $\mathscr{A}(1,1) = \mathscr{A}$, which are well-known families of holomorphic functions in $\mathbb{U}$.

Many operators have been used from the beginning of the study of holomorphic functions. Introducing new families of holomorphic functions was the most common trend that involves operators. An integral operator was one such operator which has attracted many researchers. Historically investigations of integral operators begun five decades ago, around the year 1965, which can be seen from the papers of Libera [12] and Bernardi [5]. Later Sălăgean [18], Kumar and Shukla [11], Bhoosnurmath and Swamy [6] and Noor and Noor [15] have studied certain types of integral operators. Generally interest was shown to find the properties of an integral operator that maps certain subfamily of $\mathscr{A}$ into (or onto) itself. For more details about the properties of integral operators, one can refer [1], [4], [2], [7], [8] and [22].

Inspired by the recent trends on integral operators, we define the following new generalized integral operator.

Definition 1.1. Let $f \in \mathscr{A}(p,n)$. The new generalized integral operator

$$J^m_{p,\alpha,\beta} : \mathscr{A}(p,n) \to \mathscr{A}(p,n)$$

is defined by the following infinite series

$$J^m_{p,\alpha,\beta} f(z) = z^p + \sum_{j=p+n}^{\infty} \left(\frac{\alpha + p\beta}{\alpha + j\beta}\right)^m a_j z^j, \tag{1.1}$$

where $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$.

It follows from (1.1) that (see [20])

$$J^0_{p,\alpha,\beta} = f(z), \quad J^1_{p,0,1} f(z) = p\int_0^z \frac{f(t)}{t} dt \tag{1.2}$$

and

$$(\alpha + p\beta) J^m_{p,\alpha,\beta} f(z) = \alpha J^{m+1}_{p,\alpha,\beta} f(z) + \beta z (J^{m+1}_{p,\alpha,\beta} f(z))'. \tag{1.3}$$

The operator $J^m_{p,\alpha,\beta}$ on $f \in \mathscr{A}(p,1) = \mathscr{A}_p$ was introduced by Swamy in [20], the operator $J^m_{1,\alpha,\beta} = J_{\alpha,\beta}(\alpha)$ on $f \in \mathscr{A}(1,1) = \mathscr{A}$, $\alpha > -p$ was considered by Swamy in [21], $J^m_{p,\alpha,1} = J^m_p(\alpha)$ on $f \in \mathscr{A}_p$, $\alpha > -p$ was considered by Aouf *et al.* in [3] the operator $J^m_{p,l+p-p\beta,\beta} = L^m_p(l,\beta)$ on $f \in \mathscr{A}_p$, $l > -p$, $\beta > 0$ was due to Aouf *et al.* in [3]. Also, we have $J^m_{p,p-p\beta,\beta} = \mathscr{A} M^m_{p,\beta}$ on $f \in \mathscr{A}_p$ was mentioned by Aouf *et al.* in [3] and $J_{p,0,\beta} = J^m_p$ on $f \in \mathscr{A}_p$ was due to Aouf *et al.* in [3], the operator $J^m_{p,1-\beta,\beta} = P^m(\beta)$ was investigated by Patel in [17], the operator $J^m_{p,1,1} = L^m_p$ on $f \in \mathscr{A}_p$ was due to Patel and Sahoo in [16] and also by Shams *et al.* in [19] and the operator $J^m_{1,1,1} = L^m$ on $f \in \mathscr{A}$ was introduced in [9] and [10]. We note that the operator $J^m_p(\alpha)$ was studied for $\alpha \geq 0$ and the operator $L^m_p(l,\beta)$ for $l \geq 0$.

The main object of this paper is to define a function $\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)$ by

$$\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z) = \delta J^{m+1}_{p,\alpha,\beta} f(z) + \gamma J^m_{p,\alpha,\beta} f(z), \quad z \in \mathbb{U}, \tag{1.4}$$

where $f \in \mathscr{A}(p,n)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\delta \in \mathbb{C}$, $\gamma \in \mathbb{C}$ such that $\delta + \gamma \in \mathbb{R}, \Re(\gamma) \geq 0$, $\beta > 0$ and $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$ and to present some interesting properties of function $\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)$. We note that:

1. If $\delta = 1 - \gamma$, $\gamma \in \mathbb{C}$, $\Re(\gamma) \geq 0$ in (1.4), then for $f \in \mathscr{A}(p,n)$, we obtain

$$M_{p,\alpha,\beta}(m+1,\gamma;z) = (1-\gamma) J^{m+1}_{p,\alpha,\beta} f(z) + \gamma J^m_{p,\alpha,\beta} f(z), \quad z \in \mathbb{U}, \tag{1.5}$$

where $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\beta > 0$ and $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$.

2. If $\alpha = l + p - p\beta$, $l > -p$, $\beta > 0$ in (1.4), then for $f \in \mathscr{A}(p,n)$, we have

$$N_{p,l,\beta}(m+1,\delta,\gamma;z) = \delta L_p^{m+1}(l,\beta)f(z) + \gamma L_p^m(l,\beta)f(z), \quad z \in \mathbb{U}, \tag{1.6}$$

where $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\delta \in \mathbb{C}$, $\gamma \in \mathbb{C}$ such that $\delta + \gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$.

3. If $\alpha = p - p\beta$, $\beta > 0$ in (1.4), then for $f \in \mathscr{A}(p,n)$ we have

$$T_{p,\beta}(m+1,\delta,\gamma;z) = \delta \mathscr{A}D_{p,\beta}^{m+1}f(z) + \gamma \mathscr{A}D_{p,\beta}^m f(z), \quad z \in \mathbb{U}, \tag{1.7}$$

where $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\delta \in \mathbb{C}$, $\gamma \in \mathbb{C}$ such that $\delta + \gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$.

4. If $\alpha = 0$ in (1.4), then for $f \in \mathscr{A}(p,n)$ we have

$$Q_p(m+1,\delta,\gamma;z) = \delta D_p^{m+1}f(z) + \gamma D_p^m f(z), \quad z \in \mathbb{U}, \tag{1.8}$$

where $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\delta \in \mathbb{C}$, $\gamma \in \mathbb{C}$ such that $\delta + \gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$.

5. If $\alpha = m = 0$ in (1.4), then for $f \in \mathscr{A}(p,n)$ we have from (1.2)

$$S(p,\gamma;z) = \delta p \int_0^z \left(\frac{f(t)}{t}\right) dt + \gamma f(z), \quad z \in \mathbb{U}, \tag{1.9}$$

where $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\delta \in \mathbb{C}$, $\gamma \in \mathbb{C}$ such that $\delta + \gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$.

In order to prove our main results, we will make use of the following lemma.

Lemma 1.1 ([14], [13])**.** *Let $\Psi(x,y)$ be a complex function defined by*

$$\Psi : \Theta \to \mathbb{C}, \quad \Theta \subset \mathbb{C} \times \mathbb{C}$$

and let $x = x_1 + ix_2$, $y = y_1 + iy_2$. Suppose that $\Psi(x,y)$ satisfies

1. *$\Psi(x,y)$ is continuous in Θ,*
2. *$(1,0) \in \Theta$ and $\Re\{\Psi(1,0)\} > 0$,*
3. *$\Re\{\Psi(ix_2,y_1)\} \leq 0$ for all $(ix_2,y_1) \in \Theta$ such that $y_1 \leq -\frac{1}{2}n(1+x_2^2)$, $n \in \mathbb{N}$.*

Let $p(z) = 1 + c_n z^n + \cdots$ is holomorphic in $\mathbb{U}$ such that $(p(z), zp'(z)) \in \Theta$ for all $z \in \mathbb{U}$. If $\Re\{\Psi(p(z),zp'(z))\} > 0$, $z \in \mathbb{U}$. Then $\Re(p(z)) > 0$ in $\mathbb{U}$.

2. A Set of Main results

Theorem 2.1. *Let $f \in \mathscr{A}(p,n)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\delta,\gamma \in \mathbb{C}$ with $\delta + \gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$, $\rho < (\delta + \gamma)$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)$ be as defined by* (1.4)*. If*

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{z^p}\right\} > \rho, \quad z \in \mathbb{U}, \tag{2.1}$$

then

$$\Re\left\{\frac{J_{p,\alpha,\beta}^{m+1}f(z)}{z^p}\right\} > \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{2(\delta+\gamma)(\alpha+p\beta)+n\beta\Re(\gamma)}, \quad z \in \mathbb{U}. \tag{2.2}$$

Proof. Let $\tau = \dfrac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{2(\delta+\gamma)(\alpha+p\beta)+n\beta\Re(\gamma)}$. Define the function $p(z)$ by

$$\frac{J_{p,\alpha,\beta}^{m+1}f(z)}{z^p} = \tau + (1-\tau)p(z). \tag{2.3}$$

We see that $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ is holomorphic in $\mathbb{U}$. Making use of the identity (1.3), we find from (2.3) that

$$\frac{J^m_{p,\alpha,\beta} f(z)}{z^p} = \tau + (1-\tau)p(z) + \frac{\beta(1-\tau)}{(\alpha+p\beta)} z p'(z) \tag{2.4}$$

It follows from that (1.4), (2.3) and (2.4) that

$$\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{z^p} = (\delta+\gamma)\tau - \rho + (\delta+\gamma)(1-\tau)p(z) + \frac{\beta\gamma(1-\tau)}{(\alpha+p\beta)} z p'(z). \tag{2.5}$$

From (2.1) and (2.5), we obtain

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{z^p} - \rho\right\} = \Re\left\{(\delta+\gamma)\tau - \rho + (\delta+\gamma)(1-\tau)p(z) + \frac{\beta\gamma(1-\tau)}{(\alpha+p\beta)} z p'(z)\right\}$$
$$> 0.$$

If we define $\Psi(x,y)$ by

$$\Psi(x,y) = (\delta+\gamma)\tau - \rho + (\delta+\gamma)(1-\tau)x + \frac{\beta\gamma(1-\tau)}{(\alpha+p\beta)} y$$

with $p(z) = x = x_1 + ix_2$ and $zp'(z) = y = y_1 + iy_2$, then we have

(i) $\Psi(x,y)$ is continuous in Θ,

(ii) $(1,0) \in \Theta$ and $\Re\{\Psi(1,0)\} = (\delta+\gamma) - \rho > 0$,

(iii) for all $(ix_2, y_1) \in \Theta$ such that $y_1 \le -\frac{1}{2}n(1+x_2^2)$,

$$\Re\{\Psi(ix_2,y_1)\} = (\delta+\gamma)\tau - \rho + \frac{\beta(1-\tau)\Re(\gamma)}{\alpha+p\beta} y_1 \le (\delta+\gamma)\tau - \rho - \frac{\beta(1-\tau)\Re(\gamma)}{2(\alpha+p\beta)} n(1+x_2^2) \le 0.$$

Therefore, the function $\Psi(x,y)$ satisfies all the conditions of Lemma 1.1. Thus we have $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$ which yields (2.2). This proves our theorem. □

Theorem 2.2. *Let $f \in \mathscr{A}(p,n)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\delta,\gamma \in \mathbb{C}$ with $\delta+\gamma \in \mathbb{R}$, $\Re(\gamma) \ge 0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$, $\rho > (\delta+\gamma)$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)$ be as defined by* (1.4)*. If*

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{z^p}\right\} < \rho, \quad z \in \mathbb{U},$$

then

$$\Re\left\{\frac{J^{m+1}_{p,\alpha,\beta} f(z)}{z^p}\right\} < \frac{2(\alpha+p\beta)\rho + \beta n \Re(\gamma)}{2(\delta+\gamma)(\alpha+p\beta) + \beta n \Re(\gamma)}, \quad z \in \mathbb{U}. \tag{2.6}$$

Proof. Let $\tau = \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{2(\delta+\gamma)(\alpha+p\beta) + n\beta\Re(\gamma)} > 1$. Define the function $p(z)$ by

$$\frac{J^{m+1}_{p,\alpha,\beta} f(z)}{z^p} = \tau + (1-\tau)p(z).$$

We observe that $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$. Following the proof of Theorem 2.1, we get

$$\Re\left\{\rho - \frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{z^p}\right\} = \Re\left\{\rho - (\delta+\gamma)\tau - (\delta+\gamma)(1-\tau)p(z) - \frac{\beta\gamma(1-\tau)}{\alpha+p\beta} z p'(z)\right\}$$
$$> 0.$$

Let

$$\Psi(x,y) = \rho - (\delta+\gamma)\tau - (\delta+\gamma)(1-\tau)x - \frac{\beta\gamma(1-\tau)}{\alpha+p\beta}y \tag{2.7}$$

with $p(z) = x = x_1 + ix_2$ and $zp'(z) = y = y_1 + iy_2$. Then it follows from (2.7) that

(i) $\Psi(x,y)$ is continuous in Θ,

(ii) $(1,0) \in \Theta$ and $\Re\{\Psi(1,0)\} = \rho - (\delta+\gamma) > 0$,

(iii) for all $(ix_2, y_1) \in \Theta$ such that $y_1 \le -\frac{1}{2}n(1+x_2^2)$,

$$\Re\{\Psi(ix_2,y_1)\} = \rho - (\delta+\gamma)\tau - \frac{\beta(1-\tau)\Re(\gamma)}{\alpha+p\beta}y_1 \le \rho - (\delta+\gamma)\tau + \frac{\beta(1-\tau)\Re(\gamma)}{2(\alpha+p\beta)}n(1+x_2^2) \le 0.$$

Therefore, the function $\Psi(x,y)$ satisfies all the conditions of Lemma 1.1. Thus we have $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$ which yields (2.6). This completes the proof of Theorem 2.2. □

Using the techniques of Theorem 2.1 and Theorem 2.2, we have the following results.

Theorem 2.3. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)$ be as given by* (1.4)*. Then for $\delta,\gamma \in \mathbb{C}$ with $\delta+\gamma \in \mathbb{R}$, $\Re(\gamma) \ge 0$ and $\rho < (\delta+\gamma)$, we have*

$$\Re\left\{\frac{(J^{m+1}_{p,\alpha,\beta}f(z))'}{z^p}\right\} > \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{2(\delta+\gamma)(\alpha+p\beta) + n\beta\Re(\gamma)}, \quad z \in \mathbb{U},$$

whenever

$$\Re\left\{\frac{\mathfrak{J}'_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{pz^{p-1}}\right\} > \rho, \quad z \in \mathbb{U}.$$

Theorem 2.4. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\beta > 0, \alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\delta,\gamma;z)$ be given by* (1.4)*. Then for $\delta,\gamma \in \mathbb{C}$ with $\delta+\gamma \in \mathbb{R}$, $\Re(\gamma) \ge 0$ and $\rho > (\delta+\gamma)$, we have*

$$\Re\left\{\frac{(J^{m+1}_{p,\alpha,\beta}f(z))'}{z^p}\right\} < \frac{2(\alpha+p\beta)\rho + \beta n\Re(\gamma)}{2(\delta+\gamma)(\alpha+p\beta) + \beta n\Re(\gamma)}, \quad z \in \mathbb{U},$$

whenever

$$\Re\left\{\frac{\mathfrak{J}'_{p,\alpha,\beta}(m+1,\delta,\gamma;z)}{pz^{p-1}}\right\} < \rho, \quad z \in \mathbb{U}.$$

Remark 1. Putting (i) $\alpha = l + p - p\beta$, $l > -p$, $\beta > 0$, (ii) $\alpha = p - p\beta$, $\beta > 0$ and (iii) $\alpha = 0$, in Theorems 2.1, 2.2, 2.3 and 2.4, we obtain corresponding results for the functions $N_{p,l,\beta}(m+1,\delta,\gamma;z)$, $T_{p,\beta}(m+1,\delta,\gamma;z)$, $Q_p(m+1,\delta,\gamma;z)$ which are defined by (1.6), (1.7) and (1.8), respectively.

3. Corollaries and Consequences

Theorem 2.1 would yield the following corollary when $\delta = 1-\gamma$ in Theorem 2.1.

Corollary 3.1. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\gamma \in \mathbb{C}$ with $\Re(\gamma) \geq 0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$, $\rho < 1$ and $M_{p,\alpha,\beta}(m+1,\gamma;z)$ be as in* (1.5)*. If*

$$\Re\left\{\frac{M_{p,\alpha,\beta}(m+1,\gamma;z)}{z^p}\right\} > \rho, \quad z \in \mathbb{U},$$

then

$$\Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\} > \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{2(\alpha+p\beta)+n\beta\Re(\gamma)}, \quad z \in \mathbb{U}.$$

We conclude the below result by taking $\delta = \bar{\gamma}$ in Theorem 2.1.

Corollary 3.2. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\gamma \in \mathbb{C}$ such that $\Re(\gamma) \geq 0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$, $\rho < 2\Re(\gamma)$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)$ be as given by* (1.4)*. If*

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{z^p}\right\} > \rho, \quad z \in \mathbb{U},$$

then

$$\Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\} > \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{4[(\alpha+p\beta)+n\beta]\Re(\gamma)}, \quad z \in \mathbb{U}.$$

Further, if

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{z^p}\right\} > \frac{3}{2}\Re(\gamma),\ z \in \mathbb{U},\ \text{then}\ \Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\} > \frac{3(\alpha+p\beta)+n\beta}{4(\alpha+p\beta)+n\beta}, \quad z \in \mathbb{U}.$$

Corollary 3.3 asserts immediate consequence of Theorem 2.2 when $\delta = 1-\gamma$.

Corollary 3.3. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\gamma \in \mathbb{C}$ such that $\Re(\gamma) \geq 0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$, $\rho > 1$ and $M_{p,\alpha,\beta}(m+1,\gamma;z)$ be as in* (1.5)*. If*

$$\Re\left\{\frac{M_{p,\alpha,\beta}(m+1,\gamma;z)}{z^p}\right\} < \rho, \quad z \in \mathbb{U},$$

then

$$\Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\} < \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{2(\alpha+p\beta)+n\beta\Re(\gamma)}, \quad z \in \mathbb{U}.$$

Corollary 3.4 asserts an another interesting consequence of Theorem 2.2 when we take $\delta = \bar{\gamma}$.

Corollary 3.4. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\gamma \in \mathbb{C}$ such that $\Re(\gamma) > 0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$, $\rho > 2\Re(\gamma)$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)$ be as in* (1.4)*. If*

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{z^p}\right\} < \rho, \quad z \in \mathbb{U}$$

then

$$\Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\} < \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{4[(\alpha+p\beta)+n\beta]\Re(\gamma)}, \quad z \in \mathbb{U}.$$

Further, if

$$\Re\left\{\frac{\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{z^p}\right\}<\frac{3}{2}\Re(\gamma),\ z\in\mathbb{U},\ \textit{then}\ \Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\}<\frac{3(\alpha+p\beta)+n\beta}{4(\alpha+p\beta)+n\beta},\ z\in\mathbb{U}.$$

Setting $\delta=1-\gamma$ in Theorem 2.3, we have

Corollary 3.5. *Let $f\in\mathscr{A}(p,n)$, $p,n,\in\mathbb{N}$, $m\in\mathbb{N}_0$, $\beta>0$, $\alpha\in\mathbb{R}$ such that $\alpha+p\beta>0$ and $M_{p,\alpha,\beta}(m+1,\gamma;z)$ be as in* (1.5)*. Then for $\gamma\in\mathbb{C}$, $\Re(\gamma)\ge 0$ and $\rho<1$, we have*

$$\Re\left\{\frac{\left(J^{m+1}_{p,\alpha,\beta}f(z)\right)'}{z^p}\right\}>\frac{2(\alpha+p\beta)\rho+n\beta\Re(\gamma)}{2(\alpha+p\beta)+n\beta\Re(\gamma)},\quad z\in\mathbb{U},$$

whenever

$$\Re\left\{\frac{M'_{p,\alpha,\beta}(m+1,\gamma;z)}{pz^{p-1}}\right\}>\rho,\quad z\in\mathbb{U}.$$

Allowing $\delta=\bar{\gamma}$ in Theorem 2.3, we arrive the following

Corollary 3.6. *Let $f\in\mathscr{A}(p,n)$, $p,n,\in\mathbb{N}$, $m\in\mathbb{N}_0$, $\beta>0$, $\alpha\in\mathbb{R}$ such that $\alpha+p\beta>0$ and $\mathfrak{J}_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)$ be as in* (1.4)*. Then for $\gamma\in\mathbb{C}$ with $\Re(\gamma)>0$ and $\rho<2\Re(\gamma)$, we have*

$$\Re\left\{\frac{\left(J^{m+1}_{p,\alpha,\beta}f(z)\right)'}{z^p}\right\}>\frac{2(\alpha+p\beta)\rho+n\beta\Re(\gamma)}{4[(\alpha+p\beta)+n\beta]\Re(\gamma)},\quad z\in\mathbb{U},$$

whenever

$$\Re\left\{\frac{\mathfrak{J}'_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{pz^{p-1}}\right\}>\rho,\quad z\in\mathbb{U}.$$

Further, if

$$\Re\left\{\frac{\mathfrak{J}'_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{z^p}\right\}>\frac{3}{2}\Re(\gamma),\ z\in\mathbb{U},\ \textit{then}\ \Re\left\{\frac{J^{m+1}_{p,\alpha,\beta}f(z)}{z^p}\right\}>\frac{3(\alpha+p\beta)+n\beta}{4(\alpha+p\beta)+n\beta},\ z\in\mathbb{U}.$$

We conclude the following result by taking $\delta=1-\gamma$ in Theorem 2.4.

Corollary 3.7. *Let $f\in\mathscr{A}(p,n)$, $p,n,\in\mathbb{N}$, $m\in\mathbb{N}_0$, $\beta>0$, $\alpha\in\mathbb{R}$ such that $\alpha+p\beta>0$ and $M_{p,\alpha,\beta}(m+1,\gamma;z)$ be as in* (1.5)*. Then for $\gamma\in\mathbb{C}$, $\Re(\gamma)\ge 0$ and $\rho>1$, we have*

$$\Re\left\{\frac{\left(J^{m+1}_{p,\alpha,\beta}f(z)\right)'}{z^p}\right\}<\frac{2(\alpha+p\beta)\rho+n\beta\Re(\gamma)}{2(\alpha+p\beta)+n\beta\Re(\gamma)},\quad z\in\mathbb{U},$$

whenever

$$\Re\left\{\frac{M'_{p,\alpha,\beta}(m+1,\gamma;z)}{pz^{p-1}}\right\}<\rho,\quad z\in\mathbb{U}.$$

Putting $\delta=\bar{\gamma}$ in Theorem 2.4, we obtain

Corollary 3.8. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $m \in \mathbb{N}_0$, $\beta > 0$, $\alpha \in \mathbb{R}$ such that $\alpha + p\beta > 0$ and $\mathfrak{J}'_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)$ be as in (1.4). Then for $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $\rho > 2\Re(\gamma)$, we have*

$$\Re\left\{\frac{\left(J^{m+1}_{p,\alpha,\beta}f(z)\right)'}{z^p}\right\} < \frac{2(\alpha+p\beta)\rho + n\beta\Re(\gamma)}{4[(\alpha+p\beta)+n\beta]\Re(\gamma)}, \quad z \in \mathbb{U},$$

whenever

$$\Re\left\{\frac{\mathfrak{J}'_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{pz^{p-1}}\right\} < \rho, \quad z \in \mathbb{U}.$$

Further, if

$$\Re\left\{\frac{\mathfrak{J}'_{p,\alpha,\beta}(m+1,\bar{\gamma},\gamma;z)}{z^p}\right\} < \frac{3}{2}\Re(\gamma),\ z \in \mathbb{U},\ \text{then}\ \Re\left\{\frac{(J^{m+1}_{p,\alpha,\beta}f(z))'}{z^p}\right\} < \frac{3(\alpha+p\beta)+n\beta}{4(\alpha+p\beta)+n\beta},\ z \in \mathbb{U}.$$

If we put $m = 0$ and $\alpha = 0$ in Theorem 2.1 and Theorem 2.3, then we obtain

Corollary 3.9. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $\delta,\gamma \in \mathbb{C}$ such that $\delta+\gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$, $\rho < (\delta+\gamma)$, and $S_p(\delta,\gamma;z)$ be as in (1.9). Then we have*

$$\Re\left\{\frac{p}{z^p}\int_0^z \frac{f(t)}{t}dt\right\} > \frac{2p\rho + n\Re(\gamma)}{2(\delta+\gamma)p + n\Re(\gamma)},\ z \in \mathbb{U},\ \text{whenever}\ \Re\left\{\frac{S_p(\delta,\gamma;z)}{z^p}\right\} > \rho,\ z \in \mathbb{U}$$

and

$$\Re\left\{\frac{f(z)}{z^p}\right\} > \frac{2p\rho + n\Re(\gamma)}{2(\delta+\gamma)p + n\Re(\gamma)},\ z \in \mathbb{U},\ \text{whenever}\ \Re\left\{\frac{S'_p(\delta,\gamma;z)}{z^p}\right\} < \rho,\ z \in \mathbb{U}.$$

If we put $m = 0$ and $\alpha = 0$ in Theorem 2.2 and Theorem 2.4, then we obtain

Corollary 3.10. *Let $f \in \mathscr{A}(p,n)$, $p,n,\in \mathbb{N}$, $\delta,\gamma \in \mathbb{C}$ such that $\delta+\gamma \in \mathbb{R}$, $\Re(\gamma) \geq 0$, $\rho > (\delta+\gamma)$, and $S_p(\delta,\gamma;z)$ be as in (1.9). Then we have*

(i) $$\Re\left\{\frac{p}{z^p}\int_0^z \frac{f(t)}{t}dt\right\} < \frac{2p\rho + n\Re(\gamma)}{2(\delta+\gamma)p + n\Re(\gamma)},\ z \in \mathbb{U},\ \text{whenever}\ \Re\left\{\frac{S_p(\delta,\gamma;z)}{z^p}\right\} < \rho,\ z \in \mathbb{U}$$

and

(ii) $$\Re\left\{\frac{f(z)}{z^p}\right\} < \frac{2p\rho + n\Re(\gamma)}{2(\delta+\gamma)p + n\Re(\gamma)},\ z \in \mathbb{U},\ \text{whenever}\ \Re\left\{\frac{S'_p(\delta,\gamma;z)}{z^p}\right\} < \rho,\ z \in \mathbb{U}.$$

Conclusion

We have introduced a special holomorphic function in the unit disc defined by using a generalized integral operator. We have then derived certain inequalities of this special holomorphic function by using a lemma of Miller and Mocanu. Further by specializing the parameters, several consequences of our main results are indicated.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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