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Research Article

A Note on the Double Total Graph $T_u(\Gamma(R))$ and $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$

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Abstract. Considering a commutative ring R with unity as the set of vertices and two vertices x and y are adjacent if and only if $u+(x+y) \in Z(R)$ for some $u \in U(R)$, the resulting graph $T_u(\Gamma(R))$ is known as the *double total graph*. In this paper we find the degree of any vertex in $T_u(\Gamma(R))$ for a weakly unit fusible ring R and domination number of $T_u(\Gamma(R))$ for any ring R. Also, we investigate the properties of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ and characterize R in terms of toroidal $T_u(\Gamma(R))$.

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1. Introduction

Graph theory is an important tool to characterize algebraic structures and there are various graphs associated with an algebraic structure. But among these, the one that characterizes a particular type of algebraic structure is of much interest. In this paper, we consider the double total graph for which the complete graphs are characterized by fusible rings. All rings considered here are finite commutative rings with unity. Many authors widely studied the *unit graph* [2] and the *total graph* [1]. In general the *total graph* and the *unit graph* are disconnected. The *double total graph* [6] is a connected graph containing the *unit graph* of the ring as a subgraph ([6, Proposition 3.12]).

For definitions, terminologies and results on graph theory readers are referred to [8].

Definition 1.1 ([4])**.** A nonzero element $a \in R$ is said to be unit fusible if it can be expressed as the sum of a zero divisor and a unit in R. A ring R is unit fusible if every non zero element of R is fusible.

Remark 1.2. It is well known that for the complete graph K_n, the genus $g(K_n) = \lceil (n-3)(n-4)/12 \rceil$ when $n \geq 3$ and for the complete bipartite graph $K_{n,m}$, the genus $g(K_{n,m}) = \lceil (n-2)(m-2)/4 \rceil$ when $n, m \geq 2$.

Remark 1.3. In a graph G, $g(G) = 1$ if G contains a K_5 or a $K_{3,3}$ but does not contain any of the graphs $K_8 - K_3$ or $K_8 - (2K_2 \cup P_3)$ or $K_8 - K_{2,3}$, where $G - H$ denotes edges of G minus edges of H.

Theorem 1.4 ([5])**.** If G is a split graph then G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.

It is known that $T_u(\Gamma(R))$ is a connected graph with $\mathrm{diam}(T_u(\Gamma(R))) \leq 2$ and every vertex in $Z(R)$ is adjacent to every vertex in $U(R)$. If $|Z(R)| \geq 2$, $|U(R)| \geq 2$ then $\mathrm{gr}(T_u(\Gamma(R))) = 3$ or 4. $T_u(\Gamma(R))$ is a complete graph if and only if R is a reduced unit fusible ring with $\mathrm{Char}(R) = 2$.

In Section 2 of this note, we find the degree of any vertex in $T_u(\Gamma(R))$ for a *weakly unit fusible* ([6, Definition 3.7]) ring R and the domination number $\gamma(T_u(\Gamma(R)))$. In Section 3, we find properties of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$. In Section 4, we characterize the rings R in terms of toroidal $T_u(\Gamma(R))$.

2. Properties of $T_u(\Gamma(R))$

This section is devoted to deduce some properties of $T_u(\Gamma(R))$.

Proposition 2.1. Let R be a local ring with $|R/M| = 2$, where M is the unique maximal ideal of R. Then $T_u(\Gamma(R))$ is a complete bipartite graph.

Proof. For every pair $x, y \in M$, x is not adjacent to y since R is a local ring. Also as $|R/M| = 2$, we have $R = M \cup (M + a) = M \cup (M + (-a))$ where $a \in R \setminus M$ is a fixed element of R. Then for every $u_1, u_2 \in R \setminus M$, we have $u_1 = m + a$ and $u_2 = m' - a$, where $m, m' \in M$. If $u_1 + u_2 \in R \setminus M$ then $m + m' \in R \setminus M$ which is a contradiction. So u_1 is not adjacent to u_2 for every $u_1, u_2 \in R \setminus M$. Hence $T_u(\Gamma(R))$ is a complete bipartite graph. $\square$

We find the degree of any vertex of $T_u(\Gamma(R))$ for a weakly unit fusible ring R in the following proposition.

Proposition 2.2. If R is weakly unit fusible and $v \in T_u(\Gamma(R))$ then $\deg(v) = |R| - |\mathrm{Nil}(R)|$ or $\deg(v) = |R| - (|\mathrm{Nil}(R)| + 1)$.

Proof. Let $\mathrm{Nil}(R) = \{n_1, \ldots, n_k\}$ be the set of distinct nilpotent elements of R. Let $x \in \mathrm{Nil}(R)$. Then x is adjacent to u, for all $u \in U(R)$ since $(-u) + (x + u) = x \in Z(R)$. Let $w \in Z(R) \setminus \mathrm{Nil}(R)$.

Then $x+w = x+(z+u)$, for some $z \in Z(R)$ since R is weakly unit fusible. Therefore, we can choose a unit $(-u)$ such that $(-u)+x+w = (-u)+x+z+u = x+z \in Z(R)$. Hence, $\deg(x) = |U(R)|+|Z(R)|-|\mathrm{Nil}(R)| = |R|-|\mathrm{Nil}(R)|$.

Let $x \in U(R)$. Then x is adjacent to z for all $z \in Z(R)$. Since R is a ring with unity and $\{n_1,\dots,n_k\}$ are the k distinct nilpotent elements of R. For $i = 1,2,\dots,k$, n_i+x being a unit leads to the existence of at least k units. Let $\{x,u_2,\dots,u_k\}$ be the set of k units of R. Now $n_i+x = u_i$ for all $i \le k$. Therefore, x is adjacent to all elements of $U(R)$ except $-u_1,\dots,-u_k$. If x is one of $-u_i$ then $\deg(x) = |Z(R)|+(|U(R)|-|\mathrm{Nil}(R)|)-1 = |R|-(|\mathrm{Nil}(R)|+1)$. If x is not any one of $-u_i$ then $\deg(x) = |Z(R)|+(|U(R)|-|\mathrm{Nil}(R)|) = |R|-|\mathrm{Nil}(R)|$.

Finally, let $x \in Z(R)\setminus \mathrm{Nil}(R)$. Since there are k nilpotent elements, there exists a set consisting of at least k elements say $\{x,w_2,\dots,w_k\}$ in $Z(R)\setminus\mathrm{Nil}(R)$. Then x is adjacent to all elements of $Z(R)\setminus\mathrm{Nil}(R)$ except $-w_1,\dots,-w_k$ elements of $Z(R)\setminus\mathrm{Nil}(R)$. If x is one of $-w_i$ then $\deg(x) = |Z(R)|+(|U(R)|-|\mathrm{Nil}(R)|)-1 = |R|-(|\mathrm{Nil}(R)|+1)$ and if x is not any one of $-u_i$ then $\deg(x) = |Z(R)|+(|U(R)|-|\mathrm{Nil}(R)|) = |R|-|\mathrm{Nil}(R)|$.

Hence the result follows. □

In the next three propositions we note the domination number of $T_u(\Gamma(R))$.

Proposition 2.3. Let R be a finite commutative ring with $|\mathrm{Nil}(R)| \ge 2$ and $|U(R)| \ge 2$. Then $\gamma(T_u(\Gamma(R))) = 2$.

Proof. Since $|\mathrm{Nil}(R)| \ge 2$ and $|U(R)| \ge 2$, we have $\{1,0\}$ as a dominating set.
Hence, $\gamma(T_u(\Gamma(R))) = 2$. □

The following propositions hold clearly.

Proposition 2.4. If $\mathrm{Nil}(R) = \{0\}$ then $\gamma(T_u(\Gamma(R))) = 1$.

Proposition 2.5. If $U(R) = \{1\}$ then $\gamma(T_u(\Gamma(R))) = 1$.

3. Properties of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$

In this section, we prove some properties of the double total graph of $\mathbb{Z}_n \times \mathbb{Z}_m$. For the properties of the ring $\mathbb{Z}_n \times \mathbb{Z}_m$ readers are referred to [3].

Proposition 3.1. $\mathbb{Z}_n \times \mathbb{Z}_m$ is weakly unit fusible.

Proof. It is enough to show that $(a,b) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)\setminus\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)$ can be written as the sum of a unit and a zero divisor. Let $n = p_1^{\alpha_1} \times \dots \times p_k^{\alpha_k}$ and $m = q_1^{\beta_1} \times \dots \times q_l^{\beta_l}$.
Case 1: If neither a nor b is nilpotent, then $(a,b) = (sp_1^{\alpha_1} \times \dots \times p_{i-1}^{\alpha_{i-1}} \times p_{i+1}^{\alpha_{i+1}} \times \dots \times p_k^{\alpha_k}, rq_1^{\beta_1} \times \dots \times q_{j-1}^{\beta_{j-1}} \times q_{j+1}^{\beta_{j+1}} \times \dots \times q_l^{\beta_l})$, where $s < p_i$ and $r < q_j$. Now $(a,b) = (p_i,q_j)+(a,b)-(p_i,q_j)$. We have (p_i,q_j) is a zero divisor since p_i and q_j are zero divisors, and $(a,b)-(p_i,q_j)$ is a unit since $(a-p_i,n) = 1$ and $(b-q_j,m) = 1$.
Case 2: If one of a or b, say b is nilpotent, then $(a,b) = (sp_1^{\alpha_1} \times \dots p_{i-1}^{\alpha_{i-1}} \times p_{i+1}^{\alpha_{i+1}} \times \dots \times p_k^{\alpha_k}, b)$, where $b \in \mathrm{Nil}(\mathbb{Z}_m)$. Now $(a,b) = (p_i,u)+(a,b)-(p_i,u)$, where $u \in U(\mathbb{Z}_m)$. Since $p_i \in Z(Z_n)$ therefore

$(p_i,u) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)$, and since $b+(-u) \in U(Z_m)$ and $(a-p_i,n)=1$, therefore $(a,b)-(p_i,u)$ is a unit.
Hence the proposition. □

In the following proposition, we find when is $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ a regular graph.

Proposition 3.2. If $n = 2^i$, $m = 2^j$, $i,j \in \mathbb{N}$, then $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is a $nm - \phi(n)\phi(m)$ regular graph.

Proof. $U(\mathbb{Z}_n \times \mathbb{Z}_m)$ and $\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)$ are two independent sets having the same number of elements. We claim that the vertices in $W(\mathbb{Z}_n \times \mathbb{Z}_m) = Z(\mathbb{Z}_n \times \mathbb{Z}_m) \setminus \mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)$ forms a complete bipartite graph with each partite set having $|W(\mathbb{Z}_n \times \mathbb{Z}_m)|/2$ number of vertices. Let $w \in W(\mathbb{Z}_n \times \mathbb{Z}_m)$ then $w = (u,n)$ or $w = (n,u)$ where $u \in U(\mathbb{Z}_n)$ and $z \in \mathrm{Nil}(\mathbb{Z}_m)$ or $n \in \mathrm{Nil}(\mathbb{Z}_n)$ and $u \in U(\mathbb{Z}_m)$. We claim that $S = \{(z,u)|z \in \mathrm{Nil}(\mathbb{Z}_n), u \in U(\mathbb{Z}_m)\}$ and $T = \{(u,z)|u \in U(\mathbb{Z}_n), z \in \mathrm{Nil}(\mathbb{Z}_m)\}$ are the two partite sets. Let o_i, e_j for $i,j \in \mathbb{N}$ represent odd and even numbers respectively. Let $w_1,w_2 \in S$ then $w_1 = (z_1,u_1), w_2 = (z_2,u_2)$. Now $w_1+w_2 = (z_1+z_2,u_1+u_2) = (e_1,e_2)$. This implies that $u+w_1+w_2 = (o_1,o_2)+(e_3,e_4) = (o_3,o_4) \notin Z(\mathbb{Z}_n \times \mathbb{Z}_m)$. Therefore, w_1 and w_2 are not adjacent. Similarly, if $v_1,v_2 \in T$ then $v_1 = (u_1,z_1), v_2 = (u_2,z_2)$. Now $v_1+v_2 = (u_1+u_2,z_1+z_2) = (e_5,e_6)$. This implies that $u+v_1+v_2 = (o_5,o_6)+(e_5,e_6) = (o_7,o_8) \notin Z(\mathbb{Z}_n \times \mathbb{Z}_m)$. Therefore, v_1 and v_2 are not adjacent.

Now, let $w = (z_1,u_1) \in S$ and $v = (u_2,z_2) \in T$. Then $w+v = (z_1+u_2,u_1+z_2) = (o_{11},o_{12})$. This implies that $u+w+v = (o_{11},o_{12})+(o_{13},o_{14}) = (e_7,e_8) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)$. Therefore, w is adjacent to v. Hence, the claim. Now, if $u \in U(\mathbb{Z}_n \times \mathbb{Z}_m)$ then u is adjacent to $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| = nm - \phi(n)\phi(m)$ number of vertices. If $z \in \mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)$ then z is adjacent to any $w \in W(\mathbb{Z}_n \times \mathbb{Z}_m)$ since $z+w = (z_1,z_2)+(w_1,w_2) = (e_9,o_{15})$ or (o_{16},e_{10}). So z is adjacent to $|U(\mathbb{Z}_n \times \mathbb{Z}_m)|+|W(\mathbb{Z}_n \times \mathbb{Z}_m)| = \phi(n)\phi(m)+(nm-\phi(n)\phi(m))-2^{i-1}2^{j-1} = nm-\phi(n)\phi(m)$ number of vertices. If $w \in W(\mathbb{Z}_n \times \mathbb{Z}_m)$ then $w \in S$ or T, say w is in S. Now w is adjacent to $|U(\mathbb{Z}_n \times \mathbb{Z}_m)|+|\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)|+|T| = \phi(n)\phi(m)+2^{i-1}2^{j-1}+((nm-\phi(n)\phi(m))-2^{i-1}2^{j-1})/2 = nm/2+\phi(n)\phi(m) = \frac{3}{4}2^i2^j = nm-\phi(n)\phi(m)$ number of vertices. Similarly, if w is in T, $\deg(w) = nm-\phi(n)\phi(m)$. Hence the result follows. □

We find the clique number $\omega(T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)))$ for any two prime numbers n and m as follows.

Proposition 3.3. Let p and q be any two prime numbers. If $q \neq 2$, then $w(T_u(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_q))) = q+1$. If $q = 2$, then $w(T_u(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = 4$.
And, $w(T_u(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))) = (pq+1)/2$ for $p \neq 2$, $q \neq 2$.

Proof. Here, $\mathrm{Nil}(\mathbb{Z}_2 \times \mathbb{Z}_q) = \{(0,0)\}$ and $|U(\mathbb{Z}_2 \times \mathbb{Z}_q)| = q-1$ and $(0,0)$ is the only element having self additive inverse. Let $U(\mathbb{Z}_2 \times \mathbb{Z}_q) = A \cup B$ where B contains all the additive inverses of the elements of A. Here $Z_2 \times Z_q$ is unit fusible. Therefore, A and B form two complete subgraphs of order $(q-1)/2$ each. Also, $|Z(\mathbb{Z}_2 \times \mathbb{Z}_q)| = 2q-(q-1) = q+1$. Let $W(\mathbb{Z}_2 \times \mathbb{Z}_q) = Z(\mathbb{Z}_2 \times \mathbb{Z}_q) \setminus \{(0,0)\}$. Let $\{C,D\}$ be a partition of $W(\mathbb{Z}_2 \times \mathbb{Z}_q)$ such that D contains all the additive inverses of the elements of C. So C and D form two complete subgraphs of order $(q+1)/2$ each as $Z_2 \times Z_q$ is unit fusible. And $(0,0)$ is adjacent to all other elements of $\mathbb{Z}_2 \times \mathbb{Z}_q$. Let $(0,0) \in D$. Then $w(T_u(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_q))) = |A|+|C|+1 = (q-1)/2+(2q-(q-1))/2+1 = q+1$.

If $q = 2$, then $T_u(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) \cong K_4$, and therefore $w(T_u(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = 4$.
Let $p \neq 2$, $q \neq 2$. Then $W(\mathbb{Z}_p \times \mathbb{Z}_q)$ forms two complete subgraphs of order $(pq-(p-1)(q-1)-1)/2$ each. Since $(0,0)$ is adjacent to all other elements of $\mathbb{Z}_p \times \mathbb{Z}_q$, $w(T_u(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))) = (p-1)(q-1)/2 + (pq-(p-1)(q-1)-1)/2 + 1 = (pq+1)/2$. □

Now we compute the minimum degree $\delta(T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)))$ and the maximum degree $\triangle(T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)))$ in the following two propositions.

Proposition 3.4. Let $G = T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$, then

$$\delta(G) = \begin{cases} nm - |\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)|, & n = 2^i \text{ and } m = 2^j,\ i \geq 2,\ j \geq 2, \\ nm - (|\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)| + 1), & n \neq 2^i,\ m \neq 2^j,\ i \geq 2,\ j \geq 2. \end{cases}$$

Proof. If $n = 2^i$, $m = 2^j$, $i \geq 2$ and $j \geq 2$, then $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is $nm - \phi(n)\phi(m)$ regular graph. When $n \neq 2^m$, $|U(\mathbb{Z}_n \times \mathbb{Z}_m)| > |\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)|$. Then by Proposition 2.2 there exist $x \in \mathbb{Z}_n \times \mathbb{Z}_m$ such that $x \in U(\mathbb{Z}_n \times \mathbb{Z}_m)$ and x is not any one of $-u_i$. Therefore, $\deg(x) = n - |\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)| - 1$. Hence the proposition. □

Proposition 3.5. Let $G = T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$. Then $\triangle(G) = nm - |\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)|$.

Proof. Let x be a vertex in G. If $n = 2^i$, $m = 2^j$ for $i \geq 2$ and $j \geq 2$ then G is $nm - \phi(n)\phi(m)$ regular graph. Otherwise, $x \in \mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)$ has degree $nm - \phi(n)\phi(m)$ by Proposition 2.2. Therefore, $\triangle(G) = nm - |\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m)|$. □

In the succeeding four propositions, we find when is $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ Eulerian, Hamiltonian, planar or split for any two positive integers $n, m \geq 2$.

Proposition 3.6. $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is Eulerian if and only if $n = 2^i$, $m = 2^j$, for $i, j \in \mathbb{N}$ and one of i or j is greater than 1.

Proof. If $i = 1 = j$, then $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ contains K_4 as its subgraph and therefore the graph is not Eulerian. If one of i or j is greater than 1 then the degree of each vertex of the graph is even since the graph is $nm - \phi(n)\phi(m)$ regular.

If one of n or m is not equal to 2^i, then the graph is not regular. We have $\triangle(G) = \delta(G) + 1$. So either $\triangle(G)$ or $\delta(G)$ is odd. Then there exists at least one vertex having an odd degree. Therefore, the graph is not Eulerian. □

Proposition 3.7. $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is Hamiltonian.

Proof. $\mathbb{Z}_n \times \mathbb{Z}_m$ is weakly unit fusible. Let $n = {p_1}^{\alpha_1} \times \ldots \times {p_k}^{\alpha_k}$, $m = {q_1}^{\beta_1} \times \ldots \times {q_l}^{\beta_l}$ and $r = p_1 \times p_2 \times \ldots \times p_k$, $s = q_1 \times q_2 \times \ldots \times q_l$ where $p_i's$ and $q_j's$ are distinct primes for $1 \leq i \leq k$ and $1 \leq j \leq k$. Suppose $n = 2^i$ *and* $m = 2^j$ for some $i, j \in \mathbb{N}$. Then $\delta(T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))) = nm/2$. Otherwise, $\delta(T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))) = nm - \mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m) - 1$. Now $nm - \mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m) - 1 - nm/2 = nm((rs-2)/rs) - 1 > 0$. Therefore, $\delta(T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))) > nm/2$. Hence, $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is Hamiltonian. □

Proposition 3.8. $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is planar if and only if $n = 2$ and $m = 2$.

Proof. It is easy to see $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is planar for $n = 2$ and $m = 2$. We consider the following cases:

Case 1: If $n \geq 3$ and $m \geq 3$ then $|U(\mathbb{Z}_n \times \mathbb{Z}_m)| \geq 4$ and so $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| \geq 3$. Therefore, $K_{3,3}$ is a subgraph of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Case 2: If $n = 2$ and $m \geq 7$ then $|U(\mathbb{Z}_n \times \mathbb{Z}_m)| \geq 3$ and $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| \geq 3$. Therefore, $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ contains $K_{3,3}$ as a subgraph.

Case 3: If $n = 2$ and $m = 3$ then $U(\mathbb{Z}_n \times \mathbb{Z}_m) = \{(1,1),(1,2)\}$ but $(0,0)$ is adjacent to every element of $\mathbb{Z}_n \times \mathbb{Z}_m$. Therefore $K_{3,3}$ is a subgraph of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Case 4: If $n = 2$ and $m = 4$ then $U(\mathbb{Z}_n \times \mathbb{Z}_m) = \{(1,1),(1,3)\}$ and $\mathrm{Nil}(\mathbb{Z}_n \times \mathbb{Z}_m) = \{(0,0),(0,2)\}$. Therefore, $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ contains $K_{4,4}$ as its subgraph.

Case 5: If $n = 2$ and $m = 5$, then $|U(\mathbb{Z}_n \times \mathbb{Z}_m)| = 3$. Therefore, $K_{3,3}$ is a subgraph of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Case 6: If $n = 2$ and $m = 6$ then $U(\mathbb{Z}_n \times \mathbb{Z}_m) = \{(1,1),(1,5)\}$ but $(0,0)$ is adjacent to all the vertices of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$. Therefore, we see that $K_{3,3}$ is a subgraph of $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Thus, $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is non planar if $n \neq 2$ *or* $m \neq 2$.

Hence the proposition. □

Proposition 3.9. $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is split if and only if $n = 2$ and $m = 2$.

Proof. If $n = 2$ and $m = 2$, then $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)) \cong K_4$ and hence $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is split.

Conversely, suppose one of n or m is not equal to 2.

Case 1: Let $m = 2$, $n \neq 2$. Then, the zero divisor $(1,0)$ has additive inverse $(-1,0)$ with $(1,0) \neq (-1,0)$. Also, the unit $(1,1)$ has additive inverse $(-1,-1)$ and $(1,1) \neq (-1,-1)$. Hence, we get an induced subgraph C_4 in $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$. Therefore, the graph $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is not split.

Case 2: Let $m \geq 3$ and $n \geq 3$. We know that $(1,1)$ and $(0,0)$ are adjacent. Since $|U(\mathbb{Z}_n \times \mathbb{Z}_m)| \geq 2$, and $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| \geq 2$, the vertex set $\{(1,1),(-1,-1),(1,0),(-1,0)\}$ induces a C_4 in $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$.

Hence, the graph $T_u(\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m))$ is not split. □

4. Toroidal $T_u(\Gamma(R))$

In this section we find the non-isomorphic rings R for which $T_u(\Gamma(R))$ is toroidal.

Proposition 4.1. $T_u(\Gamma(R))$ is toroidal if and only if R is isomorphic to one of the following rings:

$$\mathbb{Z}_7,\ \mathbb{Z}_8,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_4[x]/(2x,x^2),\ \mathbb{Z}_4[x]/(2x,x^2-2),\ \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}.$$

Proof. Let R be a finite commutative ring with unity. By [7, Theorem 2.11(2)], [2, Theorem 5.14] and [6, Proposition 4.1], it is enough to consider the following rings as $G(R)$ is a spanning subgraph of $T_u(\Gamma(R))$.

$\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l} \times \mathbb{Z}_3$ where $l \geq 2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l} \times \mathbb{Z}_4$, where $l \geq 2$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x-x^2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$.

Case 1: $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Here, $U(\mathbb{Z}_3 \times \mathbb{Z}_3) = \{(1,1),(1,2),(2,1),(2,2)\}$, therefore $|U(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 4$ and $|Z(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 5$. This shows that, $T_u(\Gamma(R))$ contains a $K_{4,5}$ and hence $g(T_u(\Gamma(R))) \geq 2$.

Case 2: $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. As $T_u(\Gamma(\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2))$ is complete graph for all l so $g(T_u(\Gamma(\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l}))) \geq 2$ if $l \geq 3$.

Case 3(i): $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $g(T_u(\Gamma(R))) = 1$ since it is a subgraph of K_6.

Case 3(ii): $R = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l} \times \mathbb{Z}_3$, where $l \geq 2$. Then, $T_u(\Gamma(R))$ contains a $K_{|S|,|T|}$ where $S = \{(1,\ldots,1,1),(1,\ldots,1,2),(0,\ldots,0,0)\}$ and $T = Z(R) \setminus \{(0,\ldots,0,0)\}$ so there exist at least one $K_{3,9}$ therefore $g(T_u(\Gamma(\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l} \times \mathbb{Z}_3))) \geq 2$, for $l \geq 2$.

Case 4(i): $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $g(T_u(\Gamma(R))) = 1$ as shown in the Figure 1.

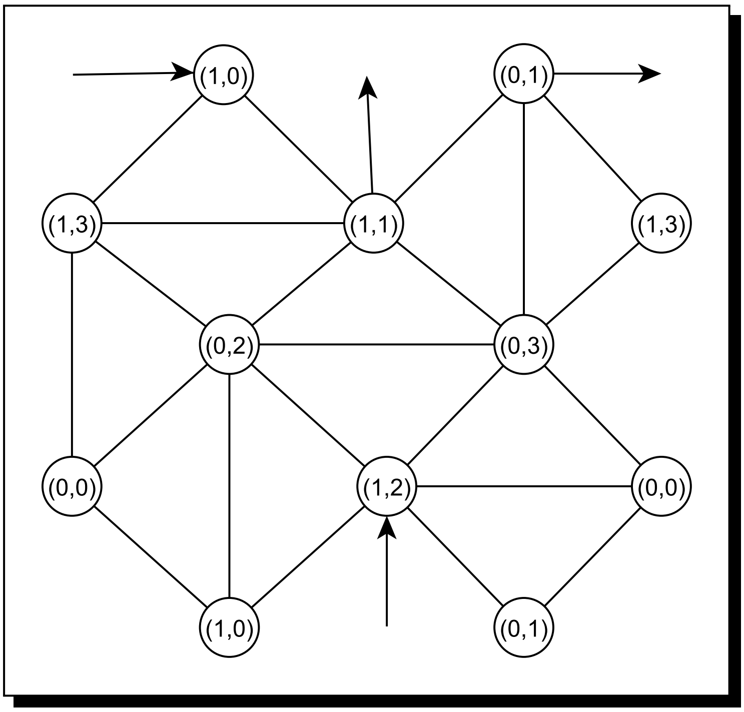

Figure 1

Case 4(ii): $R = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l} \times \mathbb{Z}_4$, where $l \geq 2$. Then $|U(R)| = 2$ and $|\mathrm{Nil}(R)| = 2$. Let $r = |U(R)| + |\mathrm{Nil}(R)|$ and $s = |Z(R)| - |\mathrm{Nil}(R)|$. So $T_u(\Gamma(R))$ contains $K_{r,s}$, as $r \geq 4$, $s \geq 12$ so $K_{4,12}$ is a subgraph of $T_u(\Gamma(R))$, therefore $g(T_u(\Gamma(R))) \geq 5$.

Case 5(i): $R = \mathbb{Z}_2 \times \mathbb{F}_4$. Then $T_u(\Gamma(R)) \cong K_8$ and hence $g(T_u(\Gamma(R))) = 2$.

Case 5(ii): $R = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{l} \times \mathbb{F}_4$, where $l \geq 2$. Then $|U(R)| = 3$ and $|Z(R)| \geq 13$, hence $K_{3,13}$ is a subgraph of $T_u(\Gamma(R))$. Therefore, $g(T_u(\Gamma(R))) \geq 3$.

Case 6(i): $R = \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. As $T_u(\Gamma(R)) \cong T_u(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4))$, $g(T_u(\Gamma(R))) = 1$.

Case 6(ii): $R = \underbrace{\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2}_{l}\times\frac{\mathbb{Z}_2[x]}{(x^2)}$, where $l\geq 2$. Then, $U(R)=\{(1,\dots,1,1),(1,\dots,1,1+x)\}$ and $\mathrm{Nil}(R)=\{(0,\dots,0,0),(0,\dots,0,x)\}$ and therefore, $K_{4,|Z(R)|}$ is a subgraph of $T_u(\Gamma(R))$. So, $g(T_u(\Gamma(R)))\geq 5$.

Case 7: $R=\mathbb{Z}_7$. Then $g(T_u(\Gamma(R)))=1$ since $g(K_7)=1$.

Case 8: $R=\mathbb{Z}_8$. Then $T_u(\Gamma(R))\cong K_{4,4}$ and $g(T_u(\Gamma(R)))=1$.

Case 9: $R=\mathbb{Z}_2[x]/(x^3)$ or $R=\mathbb{Z}_4[x]/(2x-x^2)$ or $R=\mathbb{Z}_2[x,y]/(x,y)^2$. Then $Z(R)$ and $U(R)$ are two independent sets having four elements each. Therefore, $T_u(\Gamma(R))\cong K_{4,4}$ and so $g(T_u(\Gamma(R)))=1$.

Case 10: $R=\mathbb{Z}_2\times\mathbb{Z}_5$. Then $U(R)=\{(1,1),(1,2),(1,3),(1,4)\}$ and so $k_{4,6}$ is a subgraph of $T_u(\Gamma(R))$. Therefore, $g(T_u(\Gamma(R)))\geq 2$.

Case 11: $R=\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$. Then $U(R)=\{(1,1,1),(1,1,2),(1,2,1),(1,2,2)\}$. Therefore, $K_{4,14}$ is a subgraph of $T_u(\Gamma(R))$. So, $g(T_u(\Gamma(R)))\geq 6$.

Case 12: $R=\mathbb{Z}_3\times\mathbb{Z}_4$. Then $U(R)=\{(1,1),(1,3),(2,1),(2,3)\}$. Therefore, $K_{4,8}$ is a subgraph of $T_u(\Gamma(R))$. So, $g(T_u(\Gamma(R)))\geq 3$.

Case 13: $R=\mathbb{Z}_3\times\mathbb{Z}_2[x]/(x^2)$. Then $U(R)=\{(1,1),(1,1+x),(2,1),(2,1+x)\}$. Therefore, $T_u(\Gamma(R))$ contains $K_{4,8}$ as its subgraph. So, $g(T_u(\Gamma(R)))\geq 3$.

Hence the proposition. □

5. Conclusion

By considering the double total graph, the associated ring can be characterized upto fusible ring, unit fusible ring, etc. The characterization of the various types of generalized unit fusible rings are possible by associating the double total graph. The computation of the graph parameters such as split, Eulerian, Hamiltonian, matching, etc. for the double total graph of any ring may further be considered, which will lead to the characterization of rings.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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