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Some Varieties of Quasigroups, Loops and their Parastrophes

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Abstract. The role of the parastrophes in the theory of quasigroups and loops is well known. It is our approach to investigate remarkable classes of loops and quasigroups and to relate them to their parastrophes. Some consequences for code loops are presented.

1. Introduction

Denote by $\mathfrak{Q}_l$ the variety of all quasigroups which have a left identity element 1_l and by $\mathfrak{Q}_l^{\mathrm{LF}}$ its subvariety consisting of all quasigroups $Q \in \mathfrak{Q}_l$ satisfying the identity

(LF) $\quad x(yz) = (xy)(e(x)z)$,

where $xe(x) = x$ for all $x \in Q$.

In the variety $\mathfrak{Q}_l$ one has a convenient homomorphism theory, so it is not necessary to consider the general theory of congruence relations which is inevitable in the variety of all quasigroups (see [10], [1, Chapter IV.9], [3, p. 55f.]). More precisely, for $Q, R \in \mathfrak{Q}_l$ and a homomorphism $\eta : Q \to R$ one defines the *left kernel* by

$$\ker_l \eta = \{x \in Q \mid \eta(x) = 1_l\}.$$

It follows from [10, Section 3, p. 104] that $\ker_l \eta$ is a normal subquasigroup of Q and that $\eta(Q)$ is isomorphic to the factor quasigroup $Q/\ker_l \eta$, consisting of the cosets $(\ker_l \eta)x, x \in Q$ with the multiplication $(\ker_l \eta)x \cdot (\ker_l \eta)y = (\ker_l \eta)(xy)$.

In our note we make use of the well known fact that in an arbitrary LF-quasigroup Q the mapping $e : Q \to Q$ arising from the identity (LF) is an endomorphism of Q (see [2, p. 108], [23], [20, Proposition 1.1]). If in addition Q

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belongs to $\mathfrak{Q}_l$ we use the existence of the left kernel $\ker_l e$ to study the structure of Q. Note that $\ker_l e$ coincides with the left nucleus

$$Nuc_l(Q) = \{u \in Q \mid u(xy) = (ux)y \text{ for all } \in Q\}.$$

Having shown in [20, Proposition 1.3] that every LF-quasigroup is isotopic to an LF-quasigroup with a left identity element our interest is directed mostly to the variety $\mathfrak{Q}_l^{\mathrm{LF}}$. By [2, p. 108] every LF-quasigroup is isotopic to an LM-loop (see also [20, Proposition 1.4]). There are various answers to the question what LF-quasigroups are isotopic to Moufang loops ([8], [28], [9]). We would like to add that for a Moufang loop M the (left) parastrophe is isotopic to M.

Our approach gives another way to establish the relation between LF-quasigroups and Moufang loops. In recent publications (see [6]) one can find the parastrophe approach applied to Moufang loops in general.

A smooth LF-quasigroup Q is an LF-quasigroup defined on a connected manifold Q such that the algebraic operations are smooth mappings. Smooth LF-quasigroups define the transsymmetric spaces, a class of reductive spaces (see [22], [21, (A.4.3), p. 206]) which contains the Lie groups, the symmetric spaces (see [16]) and the generalized symmetric spaces (see [27], [15], [13]). A comprehensive list of references on this theme can be found in [24].

In Section 3 we give examples showing what smooth LF-quasigroups can be obtained as parastrophes of smooth loops (Theorem 4.5).

In the closing section we describe properties of parastrophes of Code loops. Similar results hold for Chein loops (see [5]).

2. Parastrophes and isotopisms

Given a quasigroup $Q_\circ = (Q, \circ, \backslash_\circ, /^\circ)$ we consider on the set Q the multiplication defined by

$$a * b = a \backslash_\circ b. \tag{2.1}$$

It is easy to see that in the magma $Q_{(*)} = (Q, *)$ all equations

$$a * x(a,b) = b, \quad y(a,b) * a = b \tag{2.2}$$

have unique solutions $x(a,b), y(a,b)$. Defining for $a, b \in Q$ operations

$$a \backslash_* b = x(a,b), \quad b /^* a = y(a,b) \tag{2.3}$$

one obtains a quasigroup $Q_* = (Q, *, \backslash_*, /^*)$. This quasigroup is called the *left parastrophe* of $Q_\circ$ and is denoted by $\mathfrak{P}(Q_\circ)$ (see [19, p. 43]). In what follows we will call it briefly a parastrophe. Directly from the definitions one obtains the

Proposition 2.1. *For every quasigroup $Q_\circ$ the following statements are true:*

(i) $\mathfrak{P}(\mathfrak{P}(Q_\circ)) = Q_\circ$.

(ii) *If $Q_\circ$ has a left neutral element 1_l, then 1_l is a left neutral element in $\mathfrak{P}(Q_\circ)$, too.*

(iii) *If $Q_\circ$ has a left neutral element and $\mathfrak{P}(Q_\circ)$ is commutative, then $\mathfrak{P}(Q_\circ)$ is a loop.*

In the usual way one defines the multiplication groups of the quasigroup $Q_\circ$. Define left and right translations of $Q_\circ$ as

$$L_a^\circ : Q_\circ \to Q_\circ, \quad x \mapsto a \circ x, \qquad R_b^\circ : Q_\circ \to Q_\circ, \quad x \mapsto x \circ b. \tag{2.4}$$

Then one obtains the groups of left, right and two sided multiplications of $Q_\circ$ as the subgroups of the symmetric group of the set Q generated by the respective sets of translations:

$$\mathrm{LMult}(Q_\circ) = \langle L_a \mid a \in Q\rangle, \quad \mathrm{RMult}(Q_\circ) = \langle R_b^\circ \mid b \in Q\rangle, \tag{2.5}$$

$$\mathrm{Mult}(Q_\circ) = \langle \mathrm{LMult}(Q_\circ), \mathrm{RMult}(Q_\circ)\rangle. \tag{2.6}$$

Proposition 2.2. *For every quasigroup $Q_\circ$ the equality* $\mathrm{LMult}(Q_\circ) = \mathrm{LMult}(\mathfrak{P}(Q_\circ))$ *holds.*

Proof. Denoting the multiplication in $\mathfrak{P}(Q_\circ)$ by $*$ one has $L_a^* = (L_a^\circ)^{-1}$. Hence the permutation group $\mathrm{LMult}(\mathfrak{P}(Q_\circ))$ is generated by the mappings $(L_a^\circ)^{-1}$, $a \in Q$. □

Proposition 2.3. *If Q is a quasigroup satisfying the left Bol identity, then $\mathfrak{P}(Q)$ satisfies this identity, too.*

Proof. The Bol identity is equivalent to $L_x L_y L_x = L_c$ where $c = x(yx)$. Hence the propositon follows from $(L_x L_y L_x)^{-1} = L_x^{-1} L_y^{-1} L_x^{-1}$. □

Proposition 2.4. *Let $Q_\circ$ be a quasigroup with a left neutral element 1_l and let $N_\circ$ be a normal subquasigroup of $Q_\circ$. Then*

(i) *$\mathfrak{P}(N_\circ) \triangleleft \mathfrak{P}(Q_\circ)$,*
(ii) *$\mathfrak{P}(Q_\circ/N_\circ) \cong \mathfrak{P}(Q_\circ)/\mathfrak{P}(N_\circ)$.*

Proof. Let φ be an epimorphism from $Q_\circ$ onto some quasigroup X with a left neutral element 1_l. Then the mapping $q \mapsto \varphi(q)$ gives us a homomorpism φ^* from the parastrophe $Q_* = \mathfrak{P}(Q_\circ)$ onto $\mathfrak{P}(X)$. By construction the quasigroups $Q_\circ$ and Q_* have the same left neutral element 1_l, and the left kernel of φ^* is the parastrophe of the left kernel of φ. Putting $X = Q_\circ/N_\circ$ the assertion follows from our remark on the left kernels in the introduction. □

Let $Q_\circ$ be a quasigroup and let $T = (f, g, h)$ be a triplet of bijectons on the set Q. With the definition

$$a \circ_T b = h^{-1}(f(a) \circ g(b)) \tag{2.7}$$

one obtains a quasigroup $Q_{\circ_T}$. This quasigroup is called an *isotope* of $Q_\circ$ and T an *isotopism* (see [3, Chapter III]).

One says that a loop Q has the left inverse property if there is a mapping $I : Q \to Q$ such that $I(x)(xy) = y$ for all $x, y \in Q$. Then one has for all $a, b \in Q$

$$a * b = a \backslash_\circ b = I(a) \circ b. \tag{2.8}$$

Consider the *isotopism* $T(I) = (I, id_Q, id_Q)$. Hence we have the following

Proposition 2.5. *For a loop $Q_\circ$ with the inverse property the quasigroups $\mathfrak{P}(Q_\circ)$ and $Q_{\circ_{T(I)}}$ coincide.*

Proposition 2.6. *A quasigroup Q_* with left neutral element 1_l is a parastrophe $\mathfrak{P}(Q_\circ)$ of a loop $Q_\circ$ if and only if $x * x = 1_l$ for all $x \in Q$. In this case 1_l is also the neutral element of $Q_\circ$.*

Proof. Assume that $Q_* = \mathfrak{P}(Q_\circ)$ for some loop $Q_\circ$. Then by Proposition 2.1 (ii) for all $x \in Q$ one has

$$x * x = x \backslash_\circ x = 1_{Q_\circ} = 1_l. \tag{2.9}$$

Conversely, if $x * x = 1_l$ for all $x \in Q$, then 1_l is left neutral element of the quasigroup $\mathfrak{P}(Q_*)$, again by Proposition 2.1(ii). Denoting the multiplication in $\mathfrak{P}(Q_*)$ by $\circ$ one obtains

$$x \circ 1_l = x \backslash_* (x * x) = x \tag{2.10}$$

for all $x \in Q$. Hence 1_l is the two-sided neutral element of $\mathfrak{P}(Q_*)$. □

3. LF-quasigroups

Definition 3.1. For a quasigroup Q_* the mapping

$$e^* : Q \to Q, \quad x \to x \backslash^* x$$

is called the *deviation of* Q_* [17]. The quasigroup Q_* is called an *LF-quasigroup* if the identity

$$x * (y * z) = (x * y) * (e^*(x) * z)$$

holds in Q_*.

Definition 3.2. A quasigroup in which the deviation is an endomorphism will be called a quasigroup with endomorphic deviation [17]. Note that every loop is a quasigroup with endomorphic deviation.

The following proposition is well known ([2, p. 108]).

Proposition 3.3. *An LF-quasigroup is a quasigroup with endomorphic deviation.*

A quasigroup Q_* is called *left square-distributive* (see [26]) if the identity

$$(x * x) * (t * u) = (x * t) * (x * u) \tag{3.1}$$

holds. One has the following well known result (see [2, p. 67]).

Proposition 3.4. *A loop satisfies the left square distributive identity if and only if it is a commutative Moufang loop.*

The validity of the following theorem was noted in [26]. Later square distributive quasigroups played a great role as parastrophes of LF-quasigroups in [12] – where they are called left semimedial – and in [28].

Theorem 3.5. *A quasigroup $(Q,\cdot,\backslash,/)$ is an LF-quasigroup if and only if the parastrophe $\mathfrak{P}(Q)$ satisfies the left square distributive identity.*

Proof. In order to show that $\mathfrak{P}(Q)$ satisfies identity (3.1) we denote the multiplication in $\mathfrak{P}(Q)$ by $*$. One has

$$x(yz) = (xy)(e(x)z), \quad e(x) = x\backslash x \tag{3.2}$$

and

$$a\backslash b = a * b. \tag{3.3}$$

Setting $t = xy$ and $w = (x\backslash x)z$ gives us

$$y = x\backslash t = x * t, \quad z = (x * x) * w. \tag{3.4}$$

From (3.2) we obtain

$$yz = x\backslash((xy)((x\backslash x)z) \tag{3.5}$$

which implies

$$(x * t)((x * x) * w) = x * (tw) \tag{3.6}$$

We substitute $u = tw$. Then $w = t\backslash u = t * u$. Inserting this in (3.6) yields

$$(x * x) * (t * u) = (x * t) * (x * u). \tag{3.7}$$

Thus we have shown that the quasigroup $\mathfrak{P}(Q)$ is left square distributive. To prove the converse it is sufficient to observe that by Proposition 2.1 (i) the quasigroups $\mathfrak{P}(\mathfrak{P}(Q))$ and Q coincide. □

Using Propostion 2.6, Lemma 3.5 and Proposition 3.4 one immediately obtains the following

Theorem 3.6. *The parastrophe $\mathfrak{P}(Q)$ of an LF-quasigroups Q is a loop if and only if $\mathfrak{P}(Q)$ is a commutative Moufang loop.*

Corollary 3.7. *The left parastrophe $Q_* = \mathfrak{P}(Q)$ of a commutative Moufang loop Q is an LF-quasigroups with left neutral element 1_l such that the identity $x * x = 1_l$ holds. Moreover, in this case the left Bol identity holds in Q_*.*

Let G be a group which is not of exponent 2. Since G is an LF-quasigroup, it follows from Theorem 3.5 that the parastrophe $\mathfrak{P}(G)$ is a quasigroup, satisying the left square distributive identity, but $\mathfrak{P}(G)$ is not a loop by Propositon 2.6. So we have shown the

Corollary 3.8. *There are square distributive quasigroups which are not loops.*

In contrast to this corollary it was shown in [14] that the validity of one of the Moufang identities in a quasigroup H implies that H is a Moufang loop.

We now use the fact that according to Proposition 3.3 in an LF-quasigroup Q the deviation e, given by $e(x) = x\backslash x$ is an endomorphism.

Remark 3.9. We would like to emphasize that a quasigroup (not a loop) with the endomorphic deviation is not necessarily an LF-quasigroup.

This is because that the property of $e(x)$ being an endomorphism in a quasigroup is equivalent to the fact that $x \mapsto x^2$ is the endomorphism in its left parastrophe. Such a situation happens in a commutative diassociative loop, which is not necessarily a Moufang loop (see [11]).

We define

$$N_i(Q) = \ker_l e^i, \quad C_i(Q) = e^i(Q).$$

Furthermore we call

$$N_\infty(Q) = \bigcup_{i=1}^{\infty} N_i(Q)$$

the *hypernucleus* of Q and

$$C_\infty(Q) = \bigcap_{i=1}^{\infty} C_i(Q)$$

the *hyperimage* of e.

Proposition 3.10. *In an* LF-quasigroup Q *with a left neutral element* 1_l *the following statements are true:*

(i) $C_1(Q) \geq C_2(Q) \geq \cdots \geq C_i(Q) \geq C_{i+1}(Q) \geq \cdots \geq C_\infty(Q)$ *is a chain of subquasigroups.*

(ii) $N_1(Q) \leq N_2(Q) \leq \cdots \leq N_i(Q) \leq N_{i+1}(Q) \leq \cdots \leq N_\infty(Q)$ *is a chain of normal subquasigroups of* Q.

(iii) $N_{i+1}(Q)/N_i(Q) = Nuc_l(Q/N_i(Q))$.

Proof. Statement (i) and the inclusions in statement (ii) are obvious. The remainder of the Proposition follows from remarks made in the introduction. □

Remark 3.11. In [20, Corrolary 3.4] we have shown that in a finite LF-quasigroup Q with the left neutral element 1_l one has a Fitting decomposition where $N_\infty(Q)$ is normal, $Q = N_\infty(Q)C_\infty(Q)$ and $N_\infty(Q) \cap C_\infty(Q) = 1_l$.

4. Decompositions of parastrophes of abelian groups, in particular abelian Lie groups

In this section we describe some peculiarities of parastrophes of connected abelian Lie groups.

Example 4.1. In the parastrophe $Q = \mathfrak{P}(\mathbb{Z})$ one has $e(x) = x + x$. It follows that the mapping e is injective but not surjective. One has

$$N_i(Q) = 0 \text{ for all } i, \quad C_\infty(Q) = \bigcap_{i=1}^{\infty} e^i(Q) = 0.$$

The same is true for every torsionfree abelian group which is not divisible by 2.

Example 4.2. We consider the Prüfer group C_{2^∞} which, additively written, is given by a sequence $x_1, \ldots, x_i, x_{i+1}, \ldots$ of generators and the relations $2x_1 = 0$, $2x_i = x_{i-1}$, $i > 1$. Put $\mu = (x \mapsto 2x) : C_{2^\infty} \to C_{2^\infty}$. Then μ is surjective and $\ker_l \mu = \langle x_1 \rangle_+$.

By Theorem 3.6 the parastrophe $Q_* = \mathfrak{P}(C_{2^\infty})$ is an LF-quasigroup. For a subset S of the set Q we denote by $\langle S \rangle_+$ the subgroup generated by S in the abelian group C_{2^∞}. By what we have shown before, in the quasigroup Q_* the following statements are hold:

(a) $e^*(x) = 2x = x + x$.
(b) $N_1(Q) = \ker_l e^* = \langle x_1 \rangle_+$.
(c) $e^*(Q) = 2Q = Q$.

For the series $(N_i)_{i=1}^{\infty}$ and $(C_i)_{i=1}^{\infty}$ described in Proposition 3.3 one obtains

(d) $N_i(Q) = \langle x_i \rangle_+$.
(e) $N_\infty(Q) = \bigcup_{i=1}^{\infty} N_i(Q) = Q$.
(f) $C_\infty(Q) = Q$.

Example 4.3. Example 4.2 shows that for a compact connected abelian Lie groups T the hypernucleus $N_\infty(\mathfrak{P}(T))$ is a dense countable subquasigroup of $\mathfrak{P}(T)$. To see this it is enough to consider the 1-dimensional torus $\mathbb{R}/\mathbb{Z}$. But this group contains a dense subgroup isomorphic to C_{2^∞}.

Example 4.4. For the additive group $\mathbb{R}_+$ and $Q_* = \mathfrak{P}(\mathbb{R}_+)$ one easily sees that the following statements are true

(a) $N_1(Q_*) = \ker_l e^* = 0$, hence $N_\infty(Q_*) = 0$.
(b) e^* is an automorphism of Q_*, hence $C_\infty(Q_*) = Q_*$.

The statements under (a) and (b) hold as well for any vector group $\mathbb{R}^n$.

Using the examples above we obtain

Theorem 4.5. *Let G be a connected analytic commutative Moufang loop. Then the following propositions hold*

(a) $G = (\mathbb{R}/\mathbb{Z})^m \times \mathbb{R}^n$ *is an abelian Lie group.*
(b) $\mathfrak{P}(G) = \mathfrak{P}((\mathbb{R}/\mathbb{Z})^m) \times \mathfrak{P}(\mathbb{R}^n)$.
(c) *The closure of the hypernucleus of* $\mathfrak{P}(G)$ *coincides with* $\mathfrak{P}((\mathbb{R}/\mathbb{Z})^m)$.
(d) e^* *is surjective but for* $m > 0$ *it is not an automorphism.*

Proof. Proposition (a) follows from [18], [25], while (b) is obvious. The remaining propositions are consequences of the examples in this section. □

5. Nuclear LF-quasigroups

We call an LF-quasigroup Q with a left neutral element *nuclear* if its nuclear series

$$N_1(Q) \le N_2(Q) \le \cdots \le N_i(Q) \le N_{i+1}(Q) \le \cdots \le N_\infty(Q)$$

becomes stationary for some index $i_0 \in \mathbb{N}$. In this case we call the smallest number $\nu = \nu(Q) \in \mathbb{N}$ such that $N_\nu(Q) = N_{\nu+1}(Q)$ the *nuclear length* of Q. Let $C_{2^n} = \mathbb{Z}/(2^n\mathbb{Z})$ be the cyclic group of order 2^n. As in Example 4.2 of Section 3 one sees that the LF-quasigroup $\mathfrak{P}(C_{2^n})$ is nuclear of length n. It is clear that $\mathfrak{P}(C_{2^n})$ is isotopic to C_{2^n} In [28, Theorem 2.7, p. 219] V. Shcherbacov has shown that every nuclear LF-quasigroup is isotopic to group. We give a simple proof of this result.

Theorem 5.1. *Let Q be an LF-quasigroup with a left neutral element 1_l. Then $N_\infty(Q)$ is isotopic to a group.*

Proof. To shorten the notation we put $N_n = N_n(Q)$. As mentioned already in the introduction we know that $N_1 = \ker_l e = Nuc_l(Q) \triangleleft Q$ and all N_{i+1}/N_i are groups.

First we want to show that N_2 is isotopic to a group. On Q we consider the isotopism $T = (R_{1_l}^{-1}, id, id)$. The multiplication of Q_T is given by

$$x \circ y = (x/1_l)y. \tag{5.1}$$

With the notation $A = x \circ (y \circ z)$ and $B = (x \circ y) \circ z$ we have to show that $A = B$ holds for $x, y, z \in N_2$.

Choose $a, b, c \in N_2$ and put $w = b/c$. Then

$$a(wc) = (aw)(e(a)c), \tag{5.2}$$

hence

$$\begin{aligned} a(b/c) = aw &= (a(wc))/(e(a)c) \\ &= (a((b/c)c)/(e(a)c) = (ab)/(e(a)c). \end{aligned} \tag{5.3}$$

It follows that

$$\begin{aligned} A = x \circ (y \circ z) &= (x/1_l)((y/1_l)z) = ((x/1_l)(y/1_l))((e(x/1_l)z) \\ &= ((x/1_l)y)/(e(x/1_l)1_l)((e(x)/e(1_l))z). \end{aligned} \tag{5.4}$$

Now use the substitution $t = (x/1_l)y$ and observe that $e(1_l) = 1_l$, $e(x)/1_l = e(x)$ since N_1 is a group with neutral element 1_l and $e(x) \in N_1$ to obtain

$$A = (t/e(x))(e(x)z) = t(e(t/e(x))z) = t(e(t)z) \tag{5.5}$$

because of $e(t/e(x)) = e(t)/e^2(x), e^2(x) = 1_l$ and $e(t)/1_l = e(t)$. On the other hand

$$\begin{aligned} B = (x \circ y) \circ z &= (((x/1_l)y)/1_l)z = (t/1_l)z = (t/1_l)(1_l z) \\ &= t((e(t)/e(1_l))z) = t(e(t)z). \end{aligned} \tag{5.6}$$

Thus the theorem is proved for $n = 2$.

In the general case one obtains

$$A = B = t(e(t)(e^2(t)(\dots(e^{n-1}(t)z)\dots) \text{ for all } t, z \in N_n \tag{5.7}$$

in a similar way as for $n = 2$ using the following facts:

- The LF-identity,
- e is an endomorphism,
- $e^n(t) = 1_l$ for all $t \in N_n$,
- $e^{n-1}(t)$ lies in N_1 for all $t \in N_n$,
- N_1 is the greatest subgroup of Q ([20, Proposition 2.1]).

Since all N_n are isotopic to groups it follows that N_∞ is isotopic to a group too. □

6. Code loops

For an elementary abelian 2-group G the parastrophe $\mathfrak{P}(G)$ has the remarkable property that it is equal to G. This property characterizes the elementary abelian 2-groups. A class of loops which found great interest some years ago are the code loops (see [4], [7]). A code loop is a Moufang loop M of order 2^{r+1} in which the comutator subloop is a cyclic group C of order 2 such that M/C is an elementary abelian 2-group of rank r. If a 2-group is a code loop, it called special 2-group.

Theorem 6.1. *For the parastrophe P of a code loop the following statements hold:*

(1) *P is quasigroup with a left unit element,*
(2) *The commutator subloop C of P is a cyclic group order* 2 *and coincides with the Frattini subquasigroup of P,*
(3) *P/C is an elementary abelian* 2*-group,*
(4) *P satisfies the left Bol condition,*
(5) *P is not an LF*-quasigroup.

Proof. Statement (1) is obtained from Proposition 2.1, (ii), statements (3) and (4) follow from Proposition 2.4. Statement (3) comes from Proposition 2.3 and (5) is a consequence of Theorem 3.6. □

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