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Commutativity Conditions in Algebras with C^*-equalities

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Abstract. Departing from Fuglede-Putnam-Rosenblum's theorem, we examine several commutativity conditions in involutive algebras with C^*-equalities. Among questions considered are Ogasawara's theorem on operator algebras and Radjavi-Rosenthal's result on an algebra of normal operators. In the frame of C^*-algebras, conditions of apparently different natures turn out to be equivalent. Also, remarks are made about Hirshfeld-Zelazko's problem.

I. **Introduction**

C. Lepage [19] is the first to consider commutativity conditions in the frame of general Banach algebras. But there are earlier results in C^*-algebras ([13], [21], [26], [25]). Fuglede-Putnam-Rosenblum's theorem (cf. [28], Theorem 12.16) seems to be the first one, both concerning the content and the technique used; the latter meaning Rosenblum's proof of Fuglede-Putnam' theorem. That proof makes use of Liouville's theorem on bounded holomorphic functions while, before, authors were accustomed to harmonic functions. Still, one has to notice that Liouville's theorem has been already used by R. Arens, in even the more general context of topological algebras [2].

Using Fuglede's theorem, H. Radjavi and P. Rosenthal show that if a set of normal operators is a vector space then it is made of pairwise commuting elements [26].

In another point of view, T. Ogasawara [21] showed that if, in a C^*-algebra, the square map is monotonic then the algebra is commutative.

The aim of this paper is to reexamine commutativity conditions, among which the preceding, in involutive algebras where a C^*-equality is involved (C^*-algebras, locally C^*-algebras, C^*-bornological algebras, ...).

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Section III is devoted to Fuglede-Putnam-Rosenblum's theorem in C^*-algebras. It is extended to Hausdorff locally C^*-convex algebras (Proposition III.3) and to C^*-bornological algebras (Remark III.4). So the completion condition is not necessary.

Radjavi-Rsenthal's result is the subject matter of Section IV. Modulo an additional condition, that is the algebra structure, we state it for hypo-normal operators (Proposition IV.2); so also for sub-normal and quasi-normal ones.

In operator algebras, Ogasawara considered a commutativity condition in relation with the order, that is the monotonicity of the square map [21]. The result of that author is still valid in locally C^*-convex algebras (Proposition V.4) and in C^*-bornological algebras (Proposition V.6). This is the content of Section V.

In Section VI, we consider different commutativity conditions in the frame of C^*-algebras. All of them imply commutativity (not only modulo the Jacobson radical). So they appear to be equivalent (Proposition VI.3) though being of diferent natures (topological, spectral, algebraic, ...).

Finally (Section VII), few remarks are made on Hirschfeld-Zelazko's problem. In C^*-algebras, the answer is positive (Proposition VI.3). In the same frame, the answer to the involutive version of that problem is negative (Counter-example VI.2). It appears then that in involutive algebras, Hirschfeld-Zelazko's problem can not be reducd to its involutive version. These are partial answers, but the problem remains open.

II. **Preliminaries**

In a unital algebra E (real or complex) the set of invertible elements is denoted by $G(E)$. For a complex algebra, the spectrum of an element x is $Sp_E(x) = \{z \in \mathbb{C} : x - ze \notin G(E)\}$. The spectral radius of x is $\rho(x) = \sup\{|z| : z \in Sp_E(x)\}$. A C^*-algebra is a complex Banach algebra $(E, \|\cdot\|)$ endowed with an involution * such that $\|xx^*\| = \|x\|^2$, for every $x \in E$. In the unital commutative case, it is known (Gelfand-Naimark theorem) that $(E, \|\cdot\|)$ is isometrically isomorphic to an algebra $C(K)$ of continuous complex functions on a compact space K. An element $x \in E$ is said to be positive if it is self-adjoint (that is $x^* = x$) and $Spx \subset \mathbb{R}_+$. Then the set E_+ of positive elements in E is a convex cone which is also stable by multiplication. Recall also that xx^* is positive, for every $x \in E$. All the needed notions and results, on C^*-algebras, can be found in [4].

Let (E, τ) be a locally convex algebra (*l.c.a.*), with a separately continuous multiplication, whose topology τ is given by a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms. If it happens that, for every λ,

$$p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y); \quad \forall\ x, y \in E,$$

then (E, τ) is named a locally m-convex algebra (*l.m.c.a.*; cf. [12], [20]).

Following the terminology of ([12, pp. 101–102]), if E is an involutive algebra and p a vector space seminorm on E, we say that p is a C^*-seminorm if $p(x^*x) = [p(x)]^2$, for every x. An involutive topological algebra whose topology is defined by a (saturated) family of C^*-seminorms is called a C^*-convex algebra. A complete C^*-convex algebra is called a locally C^*-algebra (by A. Inoue). A Fréchet C^*-convex algebra is a metrizable C^*-convex algebra, that is equivalently a metrizable locally C^*-algebra, or also a Fréchet locally C^*-algebra.

Let E be a complex algebra which is the union of subalgebras E_i, each one being a C^*-algebra $(E_i, \|x\|_i)$, such that (E_i, f_{ji}), $i \leq j$, is an inductive system where f_{ji} is the canonical injection of E_i into E_j. Endowed with the inductive limit bornology $\mathscr{B}$ of the subalgebras E_i, it is said to be a C^*-bornological algebra [24] (C^*-$b.a.$, in short). We write $(E, \mathscr{B}) = \varinjlim(E_i, \|x\|_i)$.

III. **Fuglede-Putnam-Rosenblum's commutativity result**

The result alluded to is the following, where $\mathscr{B}(H)$ is the algebra of bounded operators on a Hilbert space H.

Theorem III.1. *Let M, N and T be elements of $\mathscr{B}(H)$. Suppose that M and N are normal. If $MT = TN$, then $M^*T = TN^*$.*

According to a comment of W. Rudin ([28, p. 381]), Fuglede showed the result with $M = N$ and Putnam with $M \neq N$. Then Rosenblum gave a proof that Halmos ([14, p. 304]) qualifies as "... a breathtakingly elegant and simple proof". The two first authors make use of the spectral resolution of normal operators, while the third relies on Liouville's theorem on bounded holomorphic functions.

Apart from standard techniques in $\mathscr{B}(H)$, actually in any C^*-algebra, the crucial argument in Rosenblum's proof is Liouville's theorem. This makes it possible to extend Theorem III.1, stated above, to some non normed algebras. One also observes that the completion is a redundant hypothesis. The proof goes along the lines of that given in [28].

Remark III.2. The previous theorem is, up to my knowledge, the seminal commutativity result. It must be the ancestor of the later ones. As examples, we can mention.

(1) The crutial argument in the proof, that is the use of Liouville's theorem, must be at the origin of the now famous inequality of C. Le Page i.e.,

(LP) $\quad \|ab\| \leq \|ba\|, \quad \forall\, a, b \in E.$

Le Page has shown, using the same argument as Rosenblum, that a unital Banach algebra satifying (LP) is necessarily commutative ([19, p. 235, Proposition 2]). Thus the so called Le Page argument should be Rosenblum-Le Page one.

(2) The condition $MT = TN$, led Putnam to introduce bilateral algebras (cf. [15]), that is

$$(\forall\, x, y \in E)(\exists\, u \in E) : xy = yu$$

and

$$(\forall\, x, y \in E)(\exists\, v \in E) : xy = vx\,.$$

(3) Some authors went in another direction [26]: What is about if a subalgebra of $\mathscr{B}(H)$ is entirely made of normal operators? See Section IV.

Now, here is an extension of the theorem mentioned above. It appears that completeness is not necessary.

Proposition III.3. *Let $(E, (|\cdot|_i)_i)$ be a Hausdorff locally C^*-convex algebra. Consider x, y and z in E, with x and y normal. If $xz = zy$, then $x^*z = zy^*$.*

Proof. Recall first that, by a result of Sebestièn ([29, Theorem 2]). Every C^*-seminorn is automatically submultiplicative. Then, calculations being made in the locally C^*-algebra $\widehat{E}$ (the completion of E), the proof goes along the lines of that given in ([28, p. 300, Theorem 12.6]) in $\mathscr{B}(H)$. For any s in E, put $v = s - s^*$ and

$$q = \exp(v) = e^v = \Sigma \frac{1}{n!} v^n, \text{ in } \widehat{E}.$$

One has $v^* = -v$ and $q^* = e^{-v} = q^{-1}$. Hence $q^*q = e$ and $qq^* = e$. Then

$$|q|_i^2 = |q^*q|_i = 1, \quad \forall\, i.$$

Whence

$$|e^{s-s^*}|_i = 1, \quad \forall s \in E, \forall\, i.$$

Now if $xz = zy$, then one shows by induction that $x^k z = zy^k$, for every positive integer k. Hence $e^x z = ze^y$. Whence $z = e^x z e^{-y}$. The link with x^* and y^* is as follows

$$e^{x^*} z e^{-y^*} = e^{x - x^*} e^x z e^{-y} e^{y - y^*}.$$

So

$$|e^{x^*} z e^{-y^*}|_i \leq |z|_i, \quad \forall\, i.$$

Finally, put

$$f(\lambda) = e^{\lambda x^*} z e^{-\lambda y^*}, \quad \lambda \in \mathbb{C}.$$

The hypotheses in the theorem are also satisfied by $\overline{\lambda} x$ and $\overline{\lambda} y$, for every $\lambda \in \mathbb{C}$. Hence

$$|f(\lambda)|_i \leq |z|_i, \quad \forall\, i, \ \forall\, \lambda.$$

One then concludes, using Liouville's theorem. ☐

Remark III.4. The previous proposition applies of course to any C^*-normed algebra. Actually, this is the case for any union of C^*-algebras E_i, where the set of indices (indexes) is a net. Indeed, one has just to consider a C^*-algebra E_j containing the elements x, y, z, x^* and y^*. So, Fuglede-Putnam-Rosenblum's theorem is also valid for C^*-b-a's.

IV. Radjavi-Rosenthal's commutativity result

We have first to recall some definitions. Let $\mathscr{B}(H)$ be the algebra of bounded operators on a Hilbert space H. **(1)** An operator T is said to be quasi-normal if it commutes with $T^*\ T$, that is $T(T^*T) = (T^*T)T$. Every normal operator is quasi-normal; the converse is not true. **(2)** An operator T is subnormal if it has a normal extension, that is there exists a Hilbert space K and a normal operator $S \in \mathscr{B}(K)$ such that H is a subspace of K, H is stable by S and the restriction of S to H is equal to T. **(3)** An operator T is hyponormal if $T^*\ T \geq T\ T^*$. Every quasi-normal operator is subnormal, and every subnormal operator is hyponormal. None of the inversion implications is true (cf. [14]).

In [14], Halmos recalls that if an algebra E, of operators, is closed under the formation of adjoints and consists of normal elements, then it is commutative. Actually, this fact is entirely algebraic, so it is worthwhile to state it in its general context; the proof being also short ([14, p. 104]).

Proposition IV.1. *Let E be an involutive algebra. If a subalgebra F, of E, is selfadjoint and made of normal elements, then it is commutative.*

Proof. If $a+ib$ and $c+id$ are in F, with a, b, c and d hermitian then the latter are also in F, due to selfadjointness. So this is also the case for $a+ic$, $a+id$, $b+ic$ and $b+id$. Finally, the normality of these elements yields the commutativity. □

Remark IV.2. Let $E \subset \mathscr{B}(H)$ be a collection of hypo-normal operators. If it is selfadjoint, then it is made of normal operators. Indeed, one already has $T^*\ T \geq T\ T^*$, by definition. Now, by hypothesis, one also has $(T^*)^*\ T^* \geq T^*(T^*)^*$ i.e., $T\ T^* \geq T\ {}^*T$. So $T^*\ T = T\ T^*$.

Actually, if a collection of normal operators is a vector space, then every pair in that collection is commuting. The proof uses a calculation trick and Fuglede's theorem. The result is due to H. Radjavi and P. Rosenthal [26].

Proposition IV.3 ([26]). *Let E a vector subspace, of $\mathscr{B}(H)$, made of normal operators. Then every pair, of E, is commuting.*

Hints. If S and T are in E, then $S+T$ and $S+iT$ are also in E. Hence

$$U = (S+T)^*(S+T) - (S+T)(S+T)^* = 0$$

and

$$V = (S+iT)^*(S+iT) - (S+iT)(S+iT)^* = 0.$$

Thus $U+iV=0$. But $U+iV = 2(S^*T - ST^*)$. Whence $S^*T = ST^*$. One concludes by Fuglede's theorem.

We extend the above mentioned result to hypo-normal operators modulo an additional condition, that is the algebra structure of the considered collection.

Proposition IV.4. *A subalgebra E, of $\mathscr{B}(H)$, entirely made of hypo-normal operators is commutative.*

Proof. For any hypo-normal operator A, one has $\|A\| = \rho(A)$ (cf. [14]). So, this is also the case for the elements of the $\|\cdot\|$-closure $\overline{E}$ of E, which is a Banach algebra. The latter is then without quasi-nilpotent elements. Hence, it is semi-simple. One concludes by a result of ([22, Théorème II.4]), for $\overline{E}$ satisfies the Le Page inequality. It is even a uniform algebra. $\square$

Question IV.5. *Is the vector space structure sufficient, as in Radjavi-Rosenthal's theorem?*

Remark IV.6. As in Rosenblum's proof, the one of the previous result avoids the spectral resolution of normal operators, used by Fuglede and Putnam.

Modulo an other extra condition, Radjavi-Rosenthal's result can be stated in a more general setting.

Proposition IV.7. *Let $(E, \|\cdot\|)$ be a C^*-normed Q-algebra. Then any subalgebra F, of E, entirely made of hypo-normal elements is necessarily commutative.*

Proof. Consider the C^*-algebra $(\widehat{E}, \|\cdot\|)$, the completion of $(E, \|\cdot\|)$. Since $(E, \|\cdot\|)$ is a Q-algebra, an element which is hypo-normal in E is also so in $\widehat{E}$. Now, for any element in F, one has $\|x\| = \rho(x)$. This remains true for the $\|\cdot\|$-closure $\overline{F}$ of F. The latter is then without quasi-nilpotent elements. Hence it is semisimple. One then concludes by a result of ([22, p. 10, Théorème II.4]), for $\overline{E}$ satisfies the Le Page inequality. $\square$

Remark IV.8. The proof given in ([14, p. 110]), to show that for a hypo-normal operator A, one has $\|A\| = \rho(A)$ uses the inner product $\langle\ ,\ \rangle$ on the Hilbert space H. If A quasi-normal, one can give an alternative proof. Indeed, by definition, A commutes with A^*A. Hence

$$(A^*A)(A^*A) = A^*(AA^*A) = A^*(A^*AA) = (A^*)^2(A)^2.$$

One then shows by induction that

$$(A^*A)^n = (A^*)^n(A)^n, \quad \forall\ n \in \mathbb{N}.$$

Whence

$$\rho(A^*A) \leq \rho(A^*)\rho(A) = \rho^2(A).$$

But

$$\rho(A^*A) = \|A^*A\| = \|A\|^2.$$

Thus $\|A\| \leq \rho(A)$. So $\|A\| = \rho(A)$, since one already has $\rho(A) \leq \|A\|$.

V. **Ogasawara's commutativity condition**

Ogasawara showed ([21, (b) of the Theorem]) that a C^*-algebra is commutative whenever it satisfies the following condition

$$\text{(Og)} \quad 0 \le x \le y \quad \Longrightarrow \quad 0 \le x^2 \le y^2.$$

The idea could reasonbly have been suggested by the Gelfand-Naimark theorem, that is a unital commutative C^*-algebra is isometrically isomorphic to an algebra $C(K)$ of complex continuous functions on a compact space K. Also, here is a counter-example, due to my colleague R. Choukri, showing that the condition (Og) is not satisfied in non commutative C^*-algebras.

Counter-example V.1. Take the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B - A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

One has

$$SpA = \{0, 2\} \subset \mathbb{R}_+, SpB = \left\{ \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2} \right\} \subset \mathbb{R}_+ \text{ and } Sp(B-A) = \{0, 1\} \subset \mathbb{R}_+.$$

Hence $0 \le A \le B$. One also has

$$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \quad \text{and} \quad B^2 - A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}.$$

So

$$Sp(B^2 - A^2) = \left(\frac{3+\sqrt{13}}{2}, \frac{3-\sqrt{13}}{2} \right), \quad \text{with } \frac{3-\sqrt{13}}{2} < 0.$$

Ogasawara gives two totally different proofs. The first uses the spectral resolution of positive operators and a result of Sherman. The second is mostly algebraic and uses only well known facts in C^*-algebras. It is based on a clever handling of inequalities. We give just hints about the second proof. A detailed one is given in ([21, pp. 308–309]).

Theorem V.2 ([21])**.** *Let $(E, \|\cdot\|)$ be a C^*-algebra. If it satisfies*

$$\text{(Og)} \quad 0 \le x \le y \quad \Longrightarrow \quad 0 \le x^2 \le y^2,$$

then it is commutative.

Hints. One can argue only with positive elements. Now, for $a \ge 0$ and $b \ge 0$, one has $a + b \ge a - b$ and $a + b \ge b - a$. Hence, by the condition (Og), one has $(a + b)^2 \ge (b - a)^2$. Whence $ab + ba \ge 0$. One also can write $ab = c + id$, with $c \ge 0$ and $d^* = d$. Then $ab + ba = c$. Thus $ab = c = ba$, if $d = 0$. So, the task is to prove that $d = 0$. Suppose that $d \ne 0$. After short but cleaver calculations, one obtains $c^2 \ge d^2$. One also proves that $(c^2 - d^2)^2 \ge (cd + dc)^2$. Now, let $\alpha > 0$ be the largest positive number such that $c^2 \ge \alpha d^2$. This number α is also the greatest one satisfying $(c^2 - d^2)^2 \ge \alpha(cd + dc)^2$. Finally, one obtains a contradiction, showing that $c^2 \ge (1 + \alpha^2)^{\frac{1}{2}} d^2$.

Remark V.3. The first step in the proof of the previous theorem is that $(a+b)^2 \geq (b-a)^2$, for all positive elements a and b in E. This inequality is implemented by Ogasawara's condition (Og). Counter-example V.1 shows that the condition (Og) is not stisfied in non commutative C^*-algebras. The referee, using the same counter-example, obtains that the implemented inequality above is not also satisfied. Indeed, taking again

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{in } \mathbb{M}_2(\mathbb{C})$$

one has

$$C = (A+B)^2 - (A-B)^2 = \begin{pmatrix} 8 & 10 \\ 10 & 12 \end{pmatrix}$$

with $spC = \{10 + \sqrt{104}, 10 - \sqrt{104}\}$.

The proof given in the C^*-algebra case can be reproduced in the frame of locally C^*-algebras. The ingredients needed are still true. They are collected and very well presented in ([12, pp. 125–130]).

Proposition V.4. *Let $(E, (|\cdot|_i)_i)$ be a locally C^*-algebra. If*

$$0 \leq a \leq b \quad \Longrightarrow \quad 0 \leq a^2 \leq b^2, \quad \forall\ a, b,$$

then E is commutative.

Remark V.5. In order to give a shorter proof, one is tempted to use the Arens-Michael decomposition (according to Mallios' terminology) of a locally C^*-algebra. Due to Apostol's result [1], every factor $(E_i, \|\cdot\|_i)$ is already a C^*-algebra; so the canonical map $\pi_i : E \to E_i$ is onto. Thus, one may use Ogasawara's result. But I could not go further in this direction.

Another class of algebras strongly related to C^*-algebras is the one of 'algèbres bornologiques stellaires' introduced by H. Hogbé-Nlend [18]. To point out the importance of the C^*-equality, I named them [24] C^*-bornological algebras (C^*-b-a, in short). Let $(E, \mathscr{B})$ be a bornological algebra, that is $\mathscr{B}$ is a bound structure such that the algebra operations are bounded. Endow it with an involution. Then $(E, \mathscr{B})$ is a C^*-b-a if and only if it is a bornological inductive limit of C^*-algebras $(E_i, \|\cdot\|_i)$ such that the canonical maps $f_{ji} : E_i \to E_j$, $i \leq j$, are one-to-one ([18, p. 46]).

Proposition V.6. *Let $(E, \mathscr{B}) = \varinjlim (E_i, \|\cdot\|_i)$ be a C^*-b-a. If*

$$0 \leq x \leq y \quad \Longrightarrow \quad 0 \leq x^2 \leq y^2, \ \ \forall\ x, y \in E$$

then E is commutative.

Proof. It is reduced to the C^*-algebra case, for [24] (see also [23])

$$Sp_E x = Sp_{Ej} x, \quad \forall\ j \in I(x), \ \text{where } I(x) = \{i \in I : x \in E_i\}. \qquad \square$$

VI. **Various conditions in C^*-algebras**

It is known that a non unital Banach algebra satisfying the Le Page inequality (LP) is commutative modulo its Jacobson radical ([22, Théorème II.4]). But a C^*-algebra is always semisimple. Hence the algebra itself is commutative. Now before considering the conditions alluded to in the heading of this section, we first observe that the Le Page inequality can be relaxed in the frame of C^*-algebras.

Proposition VI.1. *Let $(E, \|\cdot\|)$ be a C^*-algebra (unital or not). If*

$$\|ab\| \leq \|ab\|, \quad \forall\ a \in E_+,\ \forall\ b \in E$$

then E is commutative.

Proof. It is known (cf. [4]) that a C^*-algebra E is generated by $E^2 = \{xy : x, y \in E\}$. So it is sufficient to show that each product xy commutes with every positive element p. Now, for any complex number $\lambda = \alpha + i\beta$, put $f(\lambda) = e^{-\lambda p} xy e^{\lambda p}$. As in [22], calculations are conducted in the unitization E_1 of E. One has

$$\|f(\lambda)\| \leq \|e^{i(-\beta)p}\| \, \|e^{i\beta p}\| \, \|e^{-\alpha p} xy e^{\alpha p}\|.$$

It is known that there is an $M > 0$ such that $\|e^{i(-\beta)p}\| \leq M$ and $\|e^{i\beta p}\| \leq M$. Also, $e^{-\alpha p}$ is a positive element. Then

$$\|f(\lambda)\| \leq M^2 \|xy\|, \quad \forall\ \lambda.$$

The proof is finished by the argument of Rosenblum-Le Page. □

Remark VI.2. If ρ is submultiplicative in a Banach algebra, then the latter is commutative modulo its Jacobson radical ([22, Théorème II.4]). If it is a C^*-algebra, then it is commutative. Actually, one observes that

$$\|x\|^2 = \|x^* x\| = \rho(x^* x) \leq \rho(x^*)\rho(x) = \rho^2(x).$$

Hence $\|\cdot\| \leq \rho(\cdot)$. So $\|\cdot\| = \rho(\cdot)$.

In the sequel, we will need the following definitions. Recall that an algebra E is said to be right-sided (or right-lateral) if

$$(\forall\ x, y \in E)\ (\exists\, u \in E) : xy = yu.$$

Left-sidedness and two-sidedness (bilaterality) are then selfexplanatory.

Now, it is worthwhile to put together apparently different but however equivalent conditions. Every one of them implies commutativity.

Proposition VI.3. *Let $(E, \|\cdot\|)$ be C^*-algebra (unital or not). The following conditions are equivalent*

(C_1) $\|ab\| \leq \alpha\|ab\|$, $\forall\ a \in E$, $\forall\ b \in E$; *where* $\alpha \in \mathbb{R}_+^*$.
(C_2) $\|ab\| \leq \alpha\|ab\|$, $\forall\ a \in E_+$, $\forall\ b \in E$; *where* $\alpha \in \mathbb{R}_+^*$.
(C_3) $0 \leq x \leq y \Longrightarrow 0 \leq x^2 \leq y^2$, $\forall\ x, y \in E$.
(C_4) *E is without nilpotent elements.*

(C_5) *E is without quasi-nilpotent elements.*
(C_6) $\rho(ab) \leq \alpha\rho(a)\rho(b)$, $\forall\ a, b \in E$; *where* $\alpha \in \mathbb{R}_+^*$.
(C_7) *E is unilateral (left or right).*
(C_8) *E is bilateral.*
(C_9) $H(E)$ *is a real subalgebra of E; where* $H = \{x \in E : x^* = x\}$.
(C_{10}) *E is commutative.*

Proof. Each one of these conditions implies commutativity.

(C_1) In any Banach algebra, $E^2 = \{xy : x, y \in E\}$ is contained in the center of E ([22, Théorème II.1]). But here E is generated by E^2, for E is a C^*-algebra.
(C_2) Proposition VI.1.
(C_3) Ogasawara's theorem [21].
(C_4) Cf. [4]
(C_5) E is without nilpotent elements (cf. (C_4)).
(C_6) Any complex Banach algebra satisfying this inequality is commutative modulo its Jacobson rdical [16]; and every C^*-algebra is semisimple.
(C_7) Any unilateral Banach algebra is commutative modulo its Jacobson rdical [5]. But a C^*-algebra is semisimple.
(C_8) A bilateral algebra is unilateral (cf. (C_7); See also [5]).
(C_9) For arbitrary hermitian elements h and k, one must have $(hk)^* = hk$. Hence $kh = hk$. □

Remark VI.4. It appears that conditions of different natures (topological, spectral, algebraic, ...) are, in fact equivalent. This is certainly due to the very nice setting we are working in. But still, what are the very specific properties which are decisive? As a speculation, I suggest a deep geomerical aspect, that is the shape of the unit ball or the sphere.

Remark VI.5. In the previous proposition, we have taken the simplest expressions for (C_1), (C_2) and (C_6), just to point out the phenomenon. For more general formulations, see [8], [9], [10], for example.

VII. **The involutive Hirschfeld-Zelazko's problem**

In 1968 Hirschfeld and Zelazko stated [16] the following problem

(H-Z). Is a Banach algebra $(E, \|\cdot\|)$ commutative, whenever the norm $\|\cdot\|$ and the spectral radius $\rho(\cdot)$ are equivalent on every commutative subalgebra?

As far as I know, this problem is still open. So one can state its analogue in the involutive case.

(*-H-Z) Is an involutive Banach algebra $(E, \|\cdot\|)$ commutative, whenever the norm $\|\cdot\|$ and the spectral radius $\rho(\cdot)$ are equivalent on every involutive commutative subalgebra?

We consider the above mentioned problems in the frame of C^*-algebras. The answer for the first (Hirschfeld-Zelazko's problem) is positive.

Proposition VII.2. *Let $(E, \|\cdot\|)$ be a C^*-algebra (unital or not). If the norm $\|\cdot\|$ and the spectral radius $\rho(\cdot)$ are equivalent on every commutative subalgebra, then E is commutative.*

Proof. Given $x \in E$, consider the maximal (closed) commutative subalgebra F, of E, that contains x. It is known that $\rho_E(x) = \rho_F(x)$. But, on F, the norm $\|\cdot\|$ and the spectral radius $\rho(\cdot)$ are equivalent. So E is without quasi-nilpotent elements, hence also without nilpotent elements. Whence the commutativity of E (cf. [4]). $\square$

The answer for the second problem is negative.

Counter-example VII.3. Let $(E, \|\cdot\|)$ be a non commutative C^*-algebra and F an involutive commutative subalgebra of E. Taking its closure, if need be, we can suppose it closed. So, actually, it is a sub-C^*-algebra of E. Since it is commutative, it is made of normal elements. Thus, on F, one even has $\|\cdot\| = \rho(\cdot)$.

Remark VII.4. This counter-example shows that, in involutive algebras, Hirschfeld-Zelazko's problem can not be reduced to its involutive version.

VIII. **Supplement**

The referee has kindly brought to attention four more papers on commutativity conditions in C^*-algebras. Here, they are. Notice that the characterizations inthere can be added in Proposition VI.3.

(1) M.J. Crabb, J. Duncan and C.M. Mcgregor exhibit a condition involving the norm and both addition and multiplication (Theorem 2, Characterizations of commutativity for C^*-algebras, *Glasgow Math. J.* **15**(1974), 172–175).

Theorem 1. *Let $(A, \|\cdot\|)$ be a C^*-algebra. It is commutative if and only if*

$$\|a+b\| \le 1 + \|ab\|$$

for all self-adjoint elements $a, b \in A$ with $\|a\| = \|b\| = 1$.

(2) G. Ji and J. Tomiyama came with a characterization using the exponential function (Corollary 1, On Characterizations of commutativity of C^*-algebras, *Proc. Amer. Math. Soc.* **131**(12) (2003), 3845–3849.

Corollary 1. *Let $(A, \|\cdot\|)$ be a C^*-algebra. The following are equivalent*

(i) *A is commutative;*
(ii) $\exp(a+b) = \exp(a)\exp(b)$ *for every pair of positive elements $a, b \in A$.*

(3) W. Wu has obtained the same result as in (2) (Corollary 2, An order characterization of commutativity for C^*-algebras, *Proc. Amer. Math. Soc.* **129**(4) (2000), 983–987).

(4) M. Joita displays a condition where appear the involution and the absolute value (2.5 Theorem, On the Cauchy-Shwarz inequality in a C^*-algebra, *Math. Reports* **3**(3) (53), 243–246).

Corollary 2. *Let $(A, \|\cdot\|)$ be a C^*-algebra. The A is commutative if and only if*

$$x^*y + y^*x \leq |x|\ |y| + |y||x|$$

for every $x, y \in A$.

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