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Asymmetric Hölder Spaces of Sign Sensitive Weighted Integrable Functions*

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Abstract. We consider the space $L(u,v)$ of 2π-periodic real-valued functions which are integrable with respect to a sign sensitive weight (u,v). With some necessary hypothesis for this weight, $L(u,v)$ is an asymmetric Banach space. After defining a convenient modulus of smoothness we introduce the corresponding space $Lip_\alpha(u,v)$ and its subspace $lip_\alpha(u,v)$ of Hölder (or Lipschitz) functions associated to this modulus. We prove these spaces are asymmetric Banach spaces too and use the result to study approximation problems.

1. Introduction

Consider the space $E = L(w)$ of 2π-periodic real-valued (classes of) functions which are integrable with respect to a weight $w \in C_{2\pi}$, $w \geq 0$ $a.e.$, and the normalized Lebesgue measure dx in $\mathbb{T} = [0, 2\pi)$. With the norm

$$\|f\|_w = \int |fw|(x)dx,$$

this is a classical Banach space. But there are practical problems in which the weights depend on the signs of the functions. Let us go in.

Suppose $w = (u,v)$, $u,v \in C_{2\pi}$, $u,v \geq 0$ $a.e.$ Set

$$\|f\|_{u,v} = \int (u(x)f^+(x) + v(x)f^-(x))dx, \tag{1.1}$$

where

$$f^+ = \frac{|f|+f}{2} \quad \text{and} \quad f^- = \frac{|f|-f}{2},$$

are the positive and negative parts of f. Then the functional $\rho = \|\cdot\|_{u,v}$ satisfies the following

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Properties.

(i) $\rho(f) = 0$ if and only if $f = 0$.

(ii) $\forall\, f, g, \rho(f+g) \leq \rho(f) + \rho(g)$. (1.2)

(iii) $\forall\, f\ \forall \lambda \in \mathbf{R}_+,\ \rho(\lambda f) = \lambda \rho(f)$.

But perhaps there exists f such that

$$\rho(-f) \neq \rho(f). \tag{1.3}$$

Set $E = L(u, v)$ to be the functions f for which $\|f\|_{u,v} < \infty$. Under the general conditions above we can only assert that E is a certain cone. But even in case of obtaining a vector space the functional ρ might not be a norm due to the asymmetric property (1.3).

Positive homogeneous and sub-additive functionals defined by properties (i)-(iii) were early considered in the Russian literature by M.G. Krein (see the comments and notes in [18]). They have been extended to introduce quasi-metric spaces or more general quasi-uniform topological spaces. In Künzi [19], the author claims that there was much progress in quasi-uniform spaces between the years 1966 and 1982. This time is just when different authors introduced the main steps in asymmetric approximation.

In fact, one of the first asymmetric approximation problems was presented by Moursund [22] in 1966. Since this was the origin of studying approximation problems with the asymmetric property (1.3), we briefly review the idea. Given a compact set $X \subset \mathbf{R}$ with at least $n+1$ points, $(\varphi_i)_{1\leq i\leq n}$ a Haar system on the convex hull of X, $p(x) = \sum_{i=1}^{n} a_i\ \varphi_i(x)$ a generalized polynomial and $f \in C(X)$, Moursund considered a general weight $W(x, y)$ defined on $X \times (-\infty, \infty)$ with values in $[-\infty, \infty]$ such that

(i) For each $x \in X$, $W\left(x, y\right)$ is a monotone non-decreasing function of y.

(ii) $\operatorname{sgn} W(x, y) = \operatorname{sgn} y$.

(iii) $\lim\limits_{|x|\to\infty} |W(x, y)| = \infty$.

Then p is said to approximate f if

$$\sup_x |W(x, p(x) - f(x))| < \infty$$

and p is called the best approximation to f if

$$\forall\, q \in P_n,\ \sup_x |W(x, p(x) - f(x))| \leq \sup_x |W(x, q(x) - f(x))|.$$

Examples included in this general setting are Chebyshev or uniform norm with $W(x, y) = y$; one-sided Chebyshev approximation with $W(x, y) = y$ for $y \leq 0$ or $+\infty$ for $y > 0$; uniform approximation and interpolation at $x_1, x_2, \ldots, x_m$, with $W(x, y) = +\infty$ if $x = x_1, x_2, \ldots, x_m$, and $y > 0$, $= -\infty$ if $x = x_1, x_2, \ldots, x_m$, and $y < 0$; y otherwise; an asymmetric approximation with $W(x, y) = y^2$ if $y > 0$, or y if $y \leq 0$.

The Moursund paper was followed by a series of papers of the same author and other mathematicians. We shall mention [7] that contains a long list of references. Independently by Krein and Nudel'man [18] in 1973, where they considered a functional ρ on $C[a,b]$ given by

$$\rho(f) = \sup(u(x)f^{+}(x) + v(x)f^{-}(x)), \tag{1.4}$$

$u, v, f \in C[a,b]$, u, v fixed and strictly positive. Since u acts on the positive part f^{+} of f and v on its negative part f^{-}, the pair (u,v) was called later a sign sensitive weight (see the survey paper [8] and the long list of references quoted there).

The uniqueness of the best polynomial approximation in this asymmetric uniform normed space can be obtained as a consequence of a generalized alternation Chebyshev theorem firstly considered also by Moursund. It is interesting to mention this general alternation theorem was independently stated and proved in [10] for clarifying an application of the Karush-Kuhn-Tucker theorem to a problem that arises in the petroleum industry [9]. A very complete paper dedicated to uniqueness of best uniform approximation by positive homogeneous functionals can be found in [23].

Although the main efforts concern to uniform approximation with constraints, sign sensitive weights also appear in dealing with weighted integration as introduced in (1.1). Early papers in that direction correspond to Babenko, see [3], for instance. Other references deal with the study of different inequalities in asymmetric norms by Kozko [16] and [17], or contribution to the existence of elements of best asymmetric approximation by Simonov [24], among others.

At present, motivated not only by its own theoretical interest and its applications to Approximation Theory, but also applications to Computer Sciences, there exists a large theory on abstract asymmetric spaces, and more general on quasi-metric spaces. In this general case a *quasi-metric* d is a function $d : X \times X \to \mathbf{R}_{+}$, such that together with the triangular inequality satisfies

$$d(x,y) = d(y,x) = 0 \text{ if and only if } x = y.$$

This general condition is important for applications in Computer Sciences. We remark that notations and definitions are not yet unanimous in the literature. Several references are [1], [2], [21] and [25], wherein the reader may find references or direct treatments on how to extend or interpret concepts and results such as theorems of Hahn-Banach type, duality theory and weak topologies, Chebyshev sets, fixed point theorems, compactness, applications of the theory, and others.

Although a general setting is important due to its applications, here we shall restrict ourselves to functionals ρ defined on linear spaces E that together with **properties** (i) and (iii) also satisfy

(ii′) $\exists\, \mu \geq 1\ \forall\, f, g, \rho(f+g) \leq \mu(\rho(f) + \rho(g))$, which is somewhat more general than (1.2). But that also are restricted by

(iv) $\exists\, \lambda \geq 1\ \forall\ f \in E,\ \rho(-f) \leq \lambda \rho(f).$ (1.5)

We shall call such a functional an *asymmetric norm* and the pair (E, ρ) an *asymmetric normed space.* For instance, under certain conditions the functionals given in (1.1) and (1.4) are asymmetric norms. Concepts as subspaces, Banach, Cauchy sequences, and so on, have the same interpretation in this setting than in the usual theory of normed spaces. The importance of considering this class of spaces (and not the most general asymmetric ones for which (1.5) is not assumed) is that many well known results from usual normed spaces can be easily extended with only small changes to these asymmetric normed spaces.

The goal of this paper is to prove the Hölder spaces of weighted integrable functions defined in next section by means of sign sensitive weights are asymmetric Banach spaces in the sense just defined above. This is accomplished in section 3. With this result in hands, we can directly characterize the convergent sequences in these spaces.

2. **Hölder spaces**

The literature on Hölder —or what is the same, Lipschitz— functions (even restricted to approximation problems) is very extensive. The early evolution of the subject can be followed in the survey paper [5].

Given a homogeneous Banach space H of measurable periodic functions as defined in Katznelson [15], $f \in H$, $\delta > 0$, $0 < \alpha < 1$, we define successively

$$\theta_\alpha(f, \delta)_H = \sup_{0<|t|\leq\delta\leq\pi} \frac{\|\Delta_t f\|_H}{|t|^\alpha},$$

where $(\Delta_t f)(x) = f_t(x) - f(x)$ and $f_t(x) = f(x + t)$,

$$\theta_\alpha(f)_H = \sup_{\delta>0} \theta_\alpha(f, \delta)_H = \theta_\alpha(f, \pi)_H,$$

$$Lip_\alpha(H) = \{f \in H : \theta_\alpha(f)_H < \infty\},$$

$$lip_\alpha(H) = \{f \in Lip_\alpha(H) : \theta_\alpha(f, \delta)_H \to 0 \text{ as } \delta \to 0\}.$$

With the norm

$$\|f\|_{Lip_\alpha(H)} = \|f\|_H + \theta_\alpha(f)_H,$$

or an equivalent one, we obtain that $Lip_\alpha(H)$ is a Banach space and $lip_\alpha(H)$ is a homogeneous Banach space.

Different Hölder approximation problems have been successfully studied in this context. See for instance [4], [5], [6], [14], [20], among others. But $L(w)$ is not a homogeneous Banach space because the translation of functions is not defined in this space. To remedy this situation, for any $f \in L(w)$ we have introduced in [13] the definition

$$(\Delta_t f)_w = \Delta_t(f w).$$

This kind of translation coincides with the typical one in case $w = 1$. Thus it is an extension of the case in $L^1_{2\pi}$.

If $f \in L(w)$, $\delta > 0$, and $0 < \alpha < 1$, define

$$\theta_\alpha(f,\delta)_w = \theta_\alpha(f,\delta)_{L(w)} = \sup_{0<|t|\le\delta\le\pi} \frac{\|(\Delta_t f)_w\|_{L^1_{2\pi}}}{|t|^\alpha}.$$

Observe that for each $\delta > 0$, the functional $\theta_\alpha(\cdot,\delta)_w$ is a seminorm on $L(w)$; for each fixed $f \in L(w)$, the function $\theta_\alpha(f,\cdot)_w$ is an increasing function; and $\theta_\alpha(f,\delta)_w = \theta_\alpha(fw,\delta)_{L^1_{2\pi}}$.

Further, following in a natural way the scheme introduced above for homogeneous spaces, define $\theta_\alpha(f)_w$, $Lip_\alpha(w)$, $lip_\alpha(w)$, and the norm $\|f\|_{Lip_\alpha(w)} = \|f\|_w + \theta_\alpha(f)_w$. In [13], we have studied several approximation problems in $lip_\alpha(w)$, under the natural assumption that w is a smooth function.

Now let us come back to the functional $\|\cdot\|_{u,v}$ defined in (1.1) and to the cone $L(u,v)$. To get a linear space one needs that $-f \in L(u,v)$ whenever $f \in L(u,v)$. That is $\|-f\|_{u,v} < \infty$. It is not hard to prove we reach this inference with the following not so restrictive assumption:

There exists a measurable function ω, and numbers A,B, such that

$$0 < A \le \omega \le B, \text{ and } u = \omega v \quad a.e. \tag{2.1}$$

In fact with this assumption, for every measurable function g,

$$\|g\|_u \le B\|g\|_v, \quad \|g\|_v \le (1/A)\|g\|_u.$$

Thus the Banach spaces $L^1(u)$ and $L^1(v)$ are boundedly equivalent. Moreover, we have proved in [13] that for every

$$f \in L(u,v), \; \min(1,A)\|f\|_v \le \|f\|_{u,v} \le \max(1,B)\|f\|_v.$$

With these inequalities in hand, it easily follows that

$$\forall\, f \in L(u,v), \|-f\|_{u,v} \le \max(1/A,B)\|f\|_{u,v}. \tag{2.2}$$

Then $(L(u,v),\|\cdot\|_{u,v})$ is an asymmetric Banach space.

From now on together with (2.1) and a fixed $0 < \alpha < 1$, we shall also suppose the stronger hypothesis $u,v,\omega \in lip_\alpha(C_{2\pi})$. These hypothesis of smoothness could be deleted in different proofs, but any Hölder space with rich properties always deals with relatively soft functions.

Observe that

$$(f^+)_t(x) = \frac{|f|+f}{2}(x+t) = \frac{|f|(x+t)+f(x+t)}{2} = (f_t)^+(x),$$

and similar equalities hold for the negative part of the function f. Thus without any confusion

$$f_t^+ = (f^+)_t = (f_t)^+ \quad \text{and} \quad f_t^- = (f^-)_t = (f_t)^-.$$

Define

$$\Delta_t(f)_{u,v} = (f_t^+u_t - f_t^-v_t) - (f^+u - f^-v) = f_t^+u_t - f_t^-v_t - f^+u + f^-v, \quad (2.3)$$

and

$$\theta_\alpha(f,\delta)_{u,v} = \sup_{0<|t|\le\delta\le\pi} \frac{\|\Delta_t(f)_{u,v}\|_1}{|t|^\alpha}. \quad (2.4)$$

Then, as before we did it, define in a natural way $\theta_\alpha(f)_{u,v}$, $Lip_\alpha(u,v)$, $lip_\alpha(u,v)$, and the functional

$$\|f\|_{Lip_\alpha(u,v)} = \|f\|_{u,v} + \theta_\alpha(f)_{u,v}.$$

If $u = v = w$, these definitions are consistent in the sense that

$$\Delta_t(f)_{u,v} = \Delta_t(f)_w,\ \theta_\alpha(f,\delta)_{u,v} = \theta_\alpha(f,\delta)_w,\ Lip_\alpha(u,v) = Lip_\alpha(w).$$

The main goal of this paper is to prove the pair $(Lip_\alpha(u,v), \|\cdot\|_{Lip_\alpha(u,v)})$ is an asymmetric Banach space. Further we shall use this result in studying approximation problems in $lip_\alpha(u,v)$.

3. The space $Lip_\alpha(u,v)$

We shall need some auxiliary results. For any real $t \neq 0$ and $\Delta_t(\cdot)_{u,v}$ as introduced in (2.3), define the functional $S_t : L(u,v) \to \mathbb{R}$, by

$$S_t(f) = \int_{\mathbb{T}} |\Delta_t(f)_{u,v}(x)| dx. \quad (3.1)$$

Also set

$$C_1 = \max\left\{B, \frac{1}{A}\right\},\ C_2 = \min\left\{A, \frac{1}{B}\right\} \text{ and } C_3 = \left(1 + \frac{1}{A^2}\right),$$

where A and B are the bounds introduced in (2.1) and the value of C_1 already appeared in (2.2).

Lemma 3.1. *For any $f, g \in L(u,v)$ and real γ, the functional S_t satisfies the inequalities*

(i) $S_t(\gamma f) \le |\gamma| C_1 S_t(f) + C_3|\gamma|\ \|f\|_{u,v} \|\omega_t - \omega\|_\infty.$

(ii) $S_t(f+g) \le \dfrac{(1+C_1)}{(1+C_2)}(S_t(f) + S_t(g)) + \dfrac{2C_3}{1+C_2}(\|f\|_{u,v} + \|g\|_{u,v})\|\omega_t - \omega\|_\infty.$

Proof. We begin by expressing the interval $\mathbb{T} = [0, 2\pi)$ as the union of the sets

$$A_1(h) = \{(h_t^+ > 0 \ \wedge \ h^- > 0) \vee (h_t^+ > 0 \ \wedge \ h = 0)\}$$
$$A_2(h) = \{(h_t^- > 0 \ \wedge \ h^- > 0) \vee (h_t^- > 0 \ \wedge \ h = 0)\}$$
$$A_3(h) = \{(h_t^+ > 0 \ \wedge \ h^+ > 0) \vee (h^+ > 0 \ \wedge \ h_t = 0)\}$$
$$A_4(h) = \{(h_t^- > 0 \ \wedge \ h^+ > 0) \vee (h^- > 0 \ \wedge \ h_t = 0)\}$$
$$A_5(h) = \{h = 0 \ \wedge \ h_t = 0\}$$

where $\wedge$ and $\vee$ have the usual logical meanings and h is any real measurable function on the given interval.

Of course (i) is true whenever $\gamma \geq 0$. While for $\gamma < 0$ one has

$$S_t(\gamma f) = |\gamma| S_t(-f).$$

Now we shall present a detailed estimation of $S_t(-f)$ for showing the procedure of proof. A development of (2.3) proves the expression

$$\Delta_t(-f)_{(u,v)} = -f_t^+ u_t \frac{1}{\omega_t} + f_t^- v_t \omega_t + f^+ u \frac{1}{\omega} - f^- v\omega, \tag{3.2}$$

while

$$S_t(-f) = \sum_{i=1}^{4} \int_{A_i(f)} |\Delta_t(-f)_{(u,v)}(x)| dx.$$

We shall use the restrictions of (3.2) to the sets $A_i(f)$, $1 \leq i \leq 4$, for estimating the corresponding integrals.

For every $x \in A_1(f)$,

$$|\Delta_t(-f)_{(u,v)}(x)| = f_t^+ u_t \frac{1}{\omega_t}(x) + f^- v\omega(x),$$

and

$$|\Delta_t(f)_{(u,v)}(x)| = f_t^+ u_t(x) + f^- v(x).$$

Thus

$$\begin{aligned}\int_{A_1(f)} |\Delta_t(-f)_{(u,v)}(x)| dx &\leq (1/A) \int_{A_1(f)} f_t^+ u_t(x) dx + B \int_{A_1(f)} f^- v(x) dx \\ &\leq C_1 \int_{A_1(f)} |\Delta_t(f)_{(u,v)}(x)| dx.\end{aligned}$$

A very similar proof gives

$$\int_{A_4(f)} |\Delta_t(-f)_{(u,v)}(x)| dx \leq C_1 \int_{A_4(f)} |\Delta_t(f)_{(u,v)}(x)| dx,$$

in the case $h_t^- > 0$ and $h^+ > 0$. While this inequality is trivial if $h^- > 0$ and $h_t = 0$.

On $A_2(f)$, for every x in the set,

$$\Delta_t(-f)_{(u,v)}(x) = f_t^- v_t \omega_t(x) - f^- v\omega(x),$$

and

$$\Delta_t(-f)_{(u,v)}(x) = |f_t^- v_t(x) - f^- v(x)|.$$

Thus, introducing the null term $f_t^- v_t \omega(x) - f_t^- v_t \omega(x)$,

$$\int_{A_2(f)} |\Delta_t(-f)_{(u,v)}(x)|dx$$
$$\leq \int_{A_2(f)} (|f_t^- v_t(\omega_t - \omega)| + |f_t^- v_t - f^- v|\omega)(x)dx$$
$$\leq C_1 \int_{A_2(f)} |\Delta_t(f)_{(u,v)}(x)|dx + \|f\|_\omega \|\omega_t - \omega\|_\infty.$$

For every $x \in A_3(f)$, at which $f_t^+ \neq 0$,

$$|\Delta_t(-f)_{(u,v)}(x)| = \left| f_t^+ u_t \frac{1}{\omega_t}(x) - f^+ u \frac{1}{\omega}(x) \right|$$

and

$$|\Delta_t(f)_{(u,v)}(x)| = |f_t^+ u_t(x) - f^+ u(x)|.$$

Introducing the null term $f_t^+ u_t \frac{1}{\omega}(x) - f_t^+ u_t \frac{1}{\omega}(x)$, one get

$$\int_{A_3(f)} |\Delta_t(-f)_{(u,v)}(x)|dx$$
$$\leq \int_{A_3(f)} \left(f_t^+ u_t(x) \left| \frac{w_t - w}{w_t w}(x) \right| + |\Delta_t(f)_{(u,v)}(x)| \frac{1}{w}(x) \right) dx.$$

From which

$$\int_{A_3(f)} |\Delta_t(-f)_{(u,v)}(x)|dx$$
$$\leq C_1 \int_{A_3(f)} |\Delta_t(f)_{(u,v)}(x)|dx + C_3 \|f\|_\omega \|\omega_t - \omega\|_\infty,$$

that is also true if $f_t^+ = 0$.

Then summing

$$S_t(-f) \leq C_1 S_t(f) + C_3 \|f\|_{L(u,v)} \|\omega_t - \omega\|_\infty,$$

and so we complete the proof of (i).

For the proof of part (ii), we must estimate the four integrals in the right hand of

$$\int_{\mathbb{T}} |\Delta_t(f+g)_{(u,v)}(x)|dx = \sum_{i=1}^{4} \int_{A_i(f+g)} |\Delta_t(f+g)_{(u,v)}(x)|dx.$$

To do it, we observe that for any two functions p and q, one has

$$(p+q)^+ = p^+ + q^+ - p^- - q^- + (p+q)^-,$$

and now we follow the same procedure showed in part (i). By this way we obtain

$$
\begin{aligned}
&(1+C_2)\int_{A_1(f+g)} |\Delta_t(f+g)_{(u,v)}(x)|dx \\
&\quad\le \int_{A_1(f+g)} |\Delta_t(f)_{(u,v)}(x)|dx + \int_{A_1(f+g)} |\Delta_t(g)_{(u,v)}(x)|dx \\
&\qquad + \int_{A_1(f+g)} |\Delta_t(-f)_{(u,v)}(x)|dx + \int_{A_1(f+g)} |\Delta_t(-g)_{(u,v)}(x)|dx \\
&\qquad + \int_{A_1(f+g)} |(f+g)_t^+ u_t \left(\frac{1}{\omega_t} - \frac{1}{\omega}\right)(x)|dx, \\
&(1+C_2)\int_{A_2(f+g)} |\Delta_t(f+g)_{(u,v)}(x)|dx \\
&\quad\le \int_{A_2(f+g)} |\Delta_t(f)_{(u,v)}(x)|dx + \int_{A_1(f+g)} |\Delta_t(g)_{(u,v)}(x)|dx \\
&\qquad + \int_{A_2(f+g)} |\Delta_t(-f)_{(u,v)}(x)|dx + \int_{A_2(f+g)} |\Delta_t(-g)_{(u,v)}(x)|dx \\
&\qquad + \int_{A_2(f+g)} |(f+g)_t^- v_t(w_t - w)(x)|dx, \\
&(1+C_2)\int_{A_3(f+g)} |\Delta_t(f+g)_{(u,v)}(x)|dx \\
&\quad\le \int_{\mathbb{A}_3(f+g)} |\Delta_t(f)_{(u,v)}(x)|dx + \int_{\mathbb{A}_3(f+g)} |\Delta_t(g)_{(u,v)}(x)|dx \\
&\qquad + \int_{\mathbb{A}_3(f+g)} |\Delta_t(-f)_{(u,v)}(x)|dx + \int_{\mathbb{A}_3(f+g)} |\Delta_t(-g)_{(u,v)}(x)|dx \\
&\qquad + \int_{\mathbb{A}_3(f+g)} |(f+g)_t^+ u_t \left(\frac{1}{\omega_t} - \frac{1}{\omega}\right)(x)|dx, \\
&(1+C_2)\int_{A_4(f+g)} |\Delta_t(f+g)_{(u,v)}(x)|dx \\
&\quad\le \int_{A_4(f+g)} |\Delta_t(f)_{(u,v)}(x)|dx + \int_{A_4(f+g)} |\Delta_t(g)_{(u,v)}(x)|dx \\
&\qquad + \int_{A_4(f+g)} |\Delta_t(-f)_{(u,v)}(x)|dx + \int_{A_4(f+g)} |\Delta_t(-g)_{(u,v)}(x)|dx \\
&\qquad + \int_{A_4(f+g)} |(f+g)_t^- v_t(\omega_t - \omega)(x)|dx.
\end{aligned}
$$

Summing these inequalities, we get (ii) of the lemma. □

Lemma 3.2. *For any $f, g \in L(u,v)$, real γ, $\delta > 0$, and $0 < \alpha < 1$,*

$$\theta_\alpha(\gamma f,\delta)_{u,v} \le |\gamma| C_1 \theta_\alpha(f,\delta)_{u,v} + C_3|\gamma|\|f\|_{u,v}\theta_\alpha(\omega,\delta)_{C_{2\pi}},$$

$$\theta_\alpha(f+g,\delta)_{u,v} \le \frac{(1+C_1)}{(1+C_2)}(\theta_\alpha(f,\delta)_{u,v} + \theta_\alpha(g,\delta)_{u,v}) + \frac{2C_3}{1+C_2}(\|f\|_{u,v} + \|g\|_{u,v})\theta_\alpha(\omega,\delta)_{C_{2\pi}}.$$

Proof. It easily follows from definition (2.4) and the lemma above. □

Taking sup for δ in these inequalities we also get the bounds

$$\theta_\alpha(\gamma f)_{u,v} \le |\gamma| C_1 \theta_\alpha(f)_{u,v} + C_3|\gamma|\|f\|_{u,v}\theta_\alpha(\omega)_{C_{2\pi}},$$

$$\theta_\alpha(f+g)_{u,v} \le \frac{(1+C_1)}{(1+C_2)}(\theta_\alpha(f)_{u,v} + \theta_\alpha(g)_{u,v}) + \frac{2C_3}{1+C_2}(\|f\|_{u,v} + \|g\|_{u,v})\theta_\alpha(\omega)_{C_{2\pi}}.$$

Theorem 3.3. *The pair $(\mathrm{Lip}_\alpha(u,v), \|\cdot\|_{\mathrm{Lip}_\alpha(u,v)})$ is an asymmetric Banach space and $\mathrm{lip}_\alpha(u,v)$ is a closed subspace.*

Proof.

$$\begin{aligned}\|\gamma f\|_{Lip_\alpha(u,v)} &= \|\gamma f\|_{u,v} + \theta_\alpha(\gamma f)_{u,v}\\ &\le C_1|\gamma|\|f\|_{u,v} + |\gamma| C_1\theta_\alpha(f)_{u,v} + C_3|\gamma|\|f\|_{u,v}\theta_\alpha(\omega)_{C_{2\pi}}\\ &= (C_1 + C_3\theta_\alpha(\omega)_{C_{2\pi}})|\gamma|\|f\|_{u,v} + |\gamma|C_1\theta_\alpha(f)_{u,v}\\ &\le C_4|\gamma|(\|f\|_{u,v} + \theta_\alpha(f)_{u,v}) = C_4|\gamma|\|f\|_{Lip_{(u,v)}(\mathbb{T})},\end{aligned}$$

where $C_4 = C_1 + C_3\theta_\alpha(\omega)_{C_{2\pi}}$.

$$\begin{aligned}\|f+g\|_{Lip_{(u,v)}(\mathbb{T})} &= \|f+g\|_{u,v} + \theta_\alpha(f+g)_{u,v} \le \|f\|_{u,v} + \|g\|_{u,v}\\ &\quad + \frac{(1+C_1)}{(1+C_2)}(\theta_\alpha(f)_{u,v} + \theta_\alpha(g)_{u,v})\\ &\quad + \frac{2C_3}{1+C_2}(\|f\|_{u,v} + \|g\|_{u,v})\theta_\alpha(\omega)_{C_{2\pi}}\\ &= \left(1 + \frac{2C_3}{1+C_2}\theta_\alpha(\omega)_{C_{2\pi}}\right)(\|f\|_{u,v} + \|g\|_{u,v})\\ &\quad + \frac{(1+C_1)}{(1+C_2)}(\theta_\alpha(f)_{u,v} + \theta_\alpha(g)_{u,v})\\ &\le C_5(\|f\|_{u,v} + \|g\|_{u,v} + \theta_\alpha(f)_{u,v} + \theta_\alpha(g)_{u,v})\\ &= C_5(\|f\|_{Lip_{(u,v)}(\mathbb{T})} + \|g\|_{Lip_{(u,v)}(\mathbb{T})}),\end{aligned}$$

where $C_5 = \max\left(1 + \frac{2C_3}{1+C_2}\theta_\alpha(w)_{C_{2\pi}}, \frac{1+C_1}{1+C_2}\right)$.

These inequalities prove $Lip_\alpha(u,v)$ is a linear space and $\|\cdot\|_{Lip_\alpha(u,v)}$ is an asymmetric norm on it. The proofs of the completeness of $Lip_\alpha(u,v)$ and $lip_\alpha(u,v)$ follow classical schemes and will be omitted. □

The knowledge that $Lip_\alpha(u,v)$ *is an asymmetric Banach space wherein* $lip_\alpha(u,v)$ *is a closed subspace, let us in a position of characterize the convergent sequences in this last subspace.*

Definition 3.4. A non-empty set G of functions in $Lip_\alpha(u,v)$ is called 0-*equicontinuous* (or *equilipchitzian*) if

$$\theta_\alpha(G,\delta) := \sup\{\theta_\alpha(g,\delta)_{u,v} : g \in G\} \to 0 \text{ as } \delta \to 0.$$

A sequence (f_n) in $Lip_\alpha(u,v)$ is called 0-equicontinuous if the set $\{f_n,\, n \in N\}$ is.

Remark 3.5. Of course any 0-equicontinuous set in $Lip_\alpha(u,v)$ lies in $lip_\alpha(u,v)$.

Theorem 3.6. *Let* (f_n) *be a sequence in* $lip_\alpha(u,v)$ *and* $f \in L(u,v)$. *Then the following conditions are equivalent:*

(i) $f \in lip_\alpha(u,v)$ *and* $\|f_n - f\|_{Lip_\alpha(u,v)} \to 0$.
(ii) (f_n) *is 0-equicontinuous and* $\|f_n - f\|_{u,v} \to 0$.

Proof. With minor modifications we can adapt the proofs sketched in [11] and completely developed in [12] to a similar result for linear metric spaces instead $L(u,v)$ and families of semi-norms instead $\theta_\alpha(\cdot,\delta)_{u,v}$, $\delta > 0$. □

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