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Research Article

Homomorphism between Rings

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Abstract. In this paper, using elementary algebra and analysis, we characterize and compute all ring homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ and from $\mathbb{Q}^n$ to $\mathbb{Q}^m$. Also, we characterize and compute all continuous ring homomorphism from $\mathbb{R}^n$ to $\mathbb{R}$.

Keywords. Homomorphism; Rings; Continuous function

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1. Introduction

Computing the number of ring homomorphism between two rings is a basic problem in abstract algebra. Solution to this problem exists for many classes of ring types [1, 3, 4, 6], but the solution to the general problem is still elusive, even for finite rings. Most of the existing results in this area pertains to finite rings of specific types. For example, Gallian and Van Buskrik [3] determined the number of ring homomorphism from $\mathbb{Z}_m$ to $\mathbb{Z}_n$, where m and n are natural numbers. Gallian and Jungreis [4] determined all ring homomorphisms between $\mathbb{Z}_m[i]$ into $\mathbb{Z}_n[i]$ where $i^2+1=0$ and also the number of ring homomorphisms between rings of the form $\mathbb{Z}_m[\rho]$ into $\mathbb{Z}_n[\rho]$ where $\rho^2+\rho+1=0$. Generalising the results of Gallian and Van Buskirk [3], Saleh and Yousef [6] computed the number of ring homomorphism from $\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}\times\cdots\times\mathbb{Z}_{m_r}$ to $\mathbb{Z}_{k_1}\times\mathbb{Z}_{k_2}\times\cdots\times\mathbb{Z}_{k_s}$. A result similar to that of Gallian and Jungreis [4], developed by Cameron [1], estimates the number of ring homomorphisms between the rings of the form $\mathbb{Z}_m[\rho]$ into $\mathbb{Z}_n[\rho]$ where $\rho^2+2=0$ and the number of ring homomorphisms from $\mathbb{Z}_m[\rho]$ into $\mathbb{Z}_n[\rho]$ where $\rho^2-2=0$. Given any two rings it is not easy to determine the number of distinct homomorphism between them. Hence the purpose of the paper is to characterize and compute all ring homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^m$, from $\mathbb{Q}^n$ to $\mathbb{Q}^m$ and from $\mathbb{R}^n$ to $\mathbb{R}$.

2. Notations and Basic Results

Most of the notations, functions and terms we mentioned in this paper can be find in Gallian [2] and Kestelman [5]. We use the following standard notations: $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real number and $\mathbb{C}$ the set of complex numbers. Also, for any ring R and for any natural number n, denote $\mathbb{R}^n$ for the ring of n-copies of $\mathbb{R}$. Let R_1 and R_2 be two rings. We denote $\mathrm{Hom}(R_1 : R_2)$ for the ring of all homomorphisms from R_1 to R_2 and $|\mathrm{Hom}(R_1 : R_2)|$ for the cardinality of $\mathrm{Hom}(R_1 : R_2)$.

Theorem 2.1. *Let R and S be two rings and $\phi : R \to S$ be a ring homomorphisms. If x is an idempotent element of R then $\phi(x)$ is an idempotent elements of S.*

Theorem 2.2. *The idempotent elements of an integral domain D are 0 and 1 (unity of D).*

Theorem 2.3. *Let R and $R_i (1 \le i \le n)$ are rings. Then*

$$|\mathrm{Hom}(R : \Pi_{i=1}^n R_i)| = |\Pi_{i=1}^n \mathrm{Hom}(R : R_i)|.$$

Proof. Let $f_i : R \to R_i$ $(1 \le i \le n)$ be a ring homomorphism. Define a function $f : R \to \Pi_{i=1}^n R_i$ by $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ for all $x \in R$. Then for $x, y \in R$,

$$\begin{aligned} f(x)f(y) &= (f_1(x), f_2(x), \dots, f_n(x))(f_1(y), f_2(y), \dots, f_n(y)) \\ &= (f_1(x)f_1(y), f_2(x)f_2(y), \dots, f_n(x)f_n(y)) \\ &= (f_1(xy), f_2(xy), \dots, f_n(xy)) \\ &= f(xy). \end{aligned}$$

Also,

$$\begin{aligned} f(x+y) &= (f_1(x+y), f_2(x+y), \dots, f_n(x+y)) \\ &= (f_1(x)+f_1(y), f_2(x)+f_2(y), \dots, f_n(x)+f_n(y)) \\ &= (f_1(x), f_2(x), \dots, f_n(x)) + (f_1(y), f_2(y), \dots, f_n(y)) \\ &= f(x)+f(y). \end{aligned}$$

Hence $f : R \to \Pi_{i=1}^n R_i$ is a ring homomorphism.

Conversely, let $h : R \to \Pi_{i=1}^n R_i$ be a ring homomorphism where $h(x) = (h_1(x), h_2(x), \dots, h_n(x))$ for all $x \in R$.

$$\begin{aligned} (h_1(x+y), h_2(x+y), \dots, h_n(x+y)) &= h(x+y) \\ &= h(x)+h(y) \qquad \text{(since } h \text{ is homomorphism)} \end{aligned}$$

$$= (h_1(x), h_2(x), \dots, h_n(x)) + (h_1(y), h_2(y), \dots, h_n(y))$$
$$= (h_1(x) + h_1(y), h_2(x) + h_2(y), \dots, h_n(x) + h_n(y)).$$

Also,

$$\begin{aligned}(h_1(xy), h_2(xy), \dots, h_n(xy)) &= h(xy)\\ &= h(x)h(y)\\ &= (h_1(x), h_2(x), \dots, h_n(x))(h_1(y), h_2(y), \dots, h_n(y))\\ &= (h_1(x)h_1(y), h_2(x)h_2(y), \dots, h_n(x)h_n(y)).\end{aligned}$$

Hence $h_i : R \to R_i (1 \le i \le n)$ is a ring homomorphism.

Now, we define a function $\psi : \mathrm{Hom}(R : \Pi_{i=1}^n R_i) \to \Pi_{i=1}^n \mathrm{Hom}(R : R_i)$ by $\psi(h) = (h_1, h_2, \dots, h_n)$ for all $h \in \mathrm{Hom}(R : \Pi_{i=1}^n R_i)$; where $h(x) = (h_1(x), h_2(x), \dots, h_n(x))$ for all $x \in R$. Now it is clear that ψ is a bijection and $|\mathrm{Hom}(R : \Pi_{i=1}^n R_i)| = |\Pi_{i=1}^n \mathrm{Hom}(R : R_i)|$. □

3. Main Results

First, we will characterize all ring homomorphisms and the number of ring homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}^m$.

Theorem 3.1. *The number of ring homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}$ is $n+1$.*

Proof. Let $\phi : \mathbb{Z}^n \to \mathbb{Z}$ be a ring homomorphism. For $1 \le i \le n$, denote e_i for the n-tuple whose ith component is 1 and 0's elsewhere. Since e_i is an idempotent and ϕ is a ring homomorphism $\phi(e_i)$ is an idempotent element in $\mathbb{Z}$ and hence $\phi(e_i) = 0$ or 1. Also, if $\phi(e_i) = \phi(e_j) = 1$ for some $i \ne j$, then we have a contradiction,

$$0 = \phi(0) = \phi(e_i e_j) = \phi(e_i)\phi(e_j) = 1.$$

Thus $\phi(e_i)$ assume the value 1 for at most one value of i.

If $\phi(e_i) = 0$ for all i $(1 \le i \le n)$, then for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$;

$$\phi(x) = \phi\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i \phi(e_i) = 0$$

and hence ϕ is the trivial ring homomorphism. If $\phi(e_k) = 1$ and $\phi(e_i) = 0$ for all $i \ne k$, then for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$;

$$\phi(x) = \phi\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i \phi(e_i) = x_k.$$

Also, it is clear that the map $\phi(x) = \phi(x_1, x_2, \dots, x_n) = x_k$ is a ring homomorphism. Hence there are n such non-trivial homomorphisms. So the number of ring homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}$ is $n+1$. □

Theorem 3.2. *The number of distinct ring homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ is $(n+1)^m$.*

Proof. The number of ring homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}$ is $n+1$. Hence from Theorem 2.3, the number of distinct ring homomorphisms from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ is $(n+1)^m$. □

Now, we will characterize all ring homomorphisms and the number of ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}^m$.

Theorem 3.3. *The number of distinct ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}$ is $n+1$.*

Proof. Let $\phi : \mathbb{Q}^n \to \mathbb{Q}$ be a ring homomorphism. For $1 \le i \le n$; denote e_i for the n-tuple whose i^{th} component is 1 and 0's elsewhere. Since e_i is an idempotent and ϕ is a ring homomorphism, $\phi(e_i)$ is an idempotent element in $\mathbb{Q}$ and hence $\phi(e_i) = 0$ or 1. Also, if $\phi(e_i) = \phi(e_j) = 1$ for some $i \ne j$, then we have a contradiction

$$0 = \phi(0) = \phi(e_i e_j) = \phi(e_i)\phi(e_j) = 1 \cdot 1 = 1.$$

Thus $\phi(e_i)$ assume the value 1 for at most one value of i.

Step 1: $\phi(ne_i) = n\phi(e_i)$ for all $n \in \mathbb{Z}$ and for all $1 \le i \le n$.
The argument is clear since ϕ is a ring homomorphism and $n \in \mathbb{Z}$.

Step 2: $\phi(re_i) = r\phi(e_i)$ for all $r \in \mathbb{Q}$ and for all $1 \le i \le n$.
Let $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Then $rq = p$ and hence $rqe_i = pe_i$. So

$$\begin{aligned} & \phi(rqe_i) = \phi(pe_i) \\ \Longrightarrow \quad & q\phi(re_i) = p\phi(e_i) \\ \Longrightarrow \quad & \phi(re_i) = \frac{p}{q}\phi(e_i) \\ \Longrightarrow \quad & \phi(re_i) = r\phi(e_i), \quad \text{for all } r \in \mathbb{Q} \text{ and for all } 1 \le i \le n. \end{aligned}$$

Step 3: Characterization of all ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}$.
If $\phi(e_i) = 0$ for all i $(1 \le i \le n)$, then for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{Q}^n$.

$$\phi(x) = \phi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} \phi(x_i e_i) = \sum_{i=1}^{n} x_i \phi(e_i) = 0.$$

Hence ϕ is the trivial ring homomorphism. If $\phi(e_k) = 1$ and $\phi(e_i) = 0$ for all $i \ne k$, then for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{Q}^n$;

$$\phi(x) = \phi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} \phi(x_i e_i) = \sum_{i=1}^{n} x_i \phi(e_i) = x_k \phi(e_k) = x_k.$$

Also, it is clear that the map $\phi(x) = \phi(x_1, x_2, \dots, x_n) = x_k$ is a ring homomorphism. Hence there are n such non-trivial ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}$. So the number of ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}$ is $n+1$. □

Theorem 3.4. *The number of distinct ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}^m$ is $(n+1)^m$.*

Proof. The number of ring homomorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}$ is $n+1$. Hence from Theorem 2.3, the number of distinct ring hommmorphisms from $\mathbb{Q}^n$ to $\mathbb{Q}^m$ is $(n+1)^m$. □

Now, we will characterize all ring homomorphisms and the number of ring homomorphisms $\mathbb{R}^n$ from $\mathbb{R}$.

Theorem 3.5. *The number of distinct ring homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}$ is $n+1$.*

Proof. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a ring homomorphism. For $1 \le i \le n$; denote e_i for the n-tuple whose ith component is 1 and 0's elsewhere. Since e_i is an idempotent and ϕ is a ring homomorphism, $\phi(e_i)$ is an idempotent element in $\mathbb{R}$ and hence $\phi(e_i) = 0$ or 1. Also, if $\phi(e_i) = \phi(e_j) = 1$ for some $i \neq j$, then we have a contradiction

$$0 = \phi(0) = \phi(e_i e_j) = \phi(e_i)\phi(e_j) = 1 \cdot 1 = 1.$$

Thus $\phi(e_i)$ assume the value 1 for at most one value of i.

Step 1: $\phi(ne_i) = n\phi(e_i)$ for all $n \in \mathbb{Z}$ and for all $1 \le i \le n$.
The argument is clear since ϕ is a ring homomorphism and $n \in \mathbb{Z}$.

Step 2: $\phi(re_i) = r\phi(e_i)$ for all $r \in \mathbb{Q}$ and for all $1 \le i \le n$.
Let $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Then $rq = p$ and hence $rqe_i = pe_i$. So

$$\begin{aligned} & \phi(rqe_i) = \phi(pe_i) \\ \Longrightarrow\quad & q\phi(re_i) = p\phi(e_i) \\ \Longrightarrow\quad & \phi(re_i) = \frac{p}{q}\phi(e_i) \\ \Longrightarrow\quad & \phi(re_i) = r\phi(e_i), \quad \text{for all } r \in \mathbb{Q} \text{ and for all } 1 \le i \le n. \end{aligned}$$

Step 3: For $a, b \in \mathbb{R}$ and $a < b$, $\phi(ae_i) \le \phi(be_i)$ for all $1 \le i \le n$.
Since $a, b \in \mathbb{R}$ and $a < b$, we have $b - a > 0$. So $b - a = t^2$ for some $t \in \mathbb{R}$. This implies that

$$\begin{aligned} (b-a)e_i &= t^2 e_i \\ &= t^2 e_i^2 \qquad (\text{since } e_i^2 = e_i) \\ &= (te_i) \cdot (te_i). \end{aligned}$$

Hence

$$\begin{aligned} \phi((b-a)e_i) &= \phi((te_i) \cdot (te_i)) = \phi(te_i)\phi(te_i) \\ &= [\phi(te_i)]^2 \ge 0. \end{aligned}$$

This gives

$$\begin{aligned} & \phi((b-a)e_i) = \phi(be_i) - \phi(ae_i) \ge 0 \\ \Longrightarrow\quad & \phi(ae_i) \le \phi(be_i) \qquad \text{for all } 1 \le i \le n. \end{aligned}$$

Step 4: $\phi : \mathbb{R}^n \to \mathbb{R}$ is continuous.
Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $\epsilon > 0$ be given. Choose a rational r such that $0 < r < \frac{\epsilon}{n}$. Then for any $y = (y_1, y_2, \dots, y_n) \in B(a; r)$, where $B(a; r)$ denote the open ball centered at a and radius r in $\mathbb{R}^n$, we have

$$|y_i - a_i| < r \qquad \text{for all } i \, (1 \le i \le n)$$

$$
\begin{aligned}
&\Longrightarrow \quad -r < y_i - a_i < r \\
&\Longrightarrow \quad \phi(-re_i) < \phi((y_i - a_i)e_i) < \phi(re_i) && \text{for all } i\,(1 \le i \le n) && \text{(by step 3)} \\
&\Longrightarrow \quad -r\phi(e_i) < \phi(y_ie_i - a_ie_i) < r\phi(e_i) && \text{for all } i\,(1 \le i \le n) && \text{(by step 2)} \\
&\Longrightarrow \quad -r\phi(e_i) < \phi(y_ie_i) - \phi(a_ie_i) < r\phi(e_i) && \text{for all } i\,(1 \le i \le n) \\
&\Longrightarrow \quad |\phi(y_ie_i) - \phi(a_ie_i)| < r\phi(e_i) && \text{for all } i\,(1 \le i \le n) \\
&\Longrightarrow \quad |\phi(y_ie_i) - \phi(a_ie_i)| < r && \text{for all } i\,(1 \le i \le n) && \text{(since } \phi(e_i) = 0 \text{ or } 1) \\
&\Longrightarrow \quad |\phi(y_ie_i) - \phi(a_ie_i)| < r < \frac{\epsilon}{n} && \text{for all } i\,(1 \le i \le n)
\end{aligned} \tag{3.1}
$$

Hence

$$
\begin{aligned}
|\phi(y) - \phi(a)| &= |\phi((y_1, y_2, \dots, y_n)) - \phi((a_1, a_2, \dots, a_n))| \\
&= \left| \phi\left(\sum_{i=1}^{n} y_ie_i \right) - \phi\left(\sum_{i=1}^{n} a_ie_i \right) \right| \\
&= \left| \left(\sum_{i=1}^{n} \phi(y_ie_i) \right) - \left(\sum_{i=1}^{n} \phi(a_ie_i) \right) \right| \\
&= \left| \left(\sum_{i=1}^{n} \phi(y_ie_i) \right) - \phi(a_ie_i) \right| \\
&\le \sum_{i=1}^{n} |\phi(y_ie_i) - \phi(a_ie_i)| && \text{(by triangle inequality of } \mathbb{R} \\
&\le \sum_{i=1}^{n} \frac{\epsilon}{n} = \epsilon && \text{(by (3.1))}
\end{aligned}
$$

Thus $\phi : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Step 5: $\phi(xe_i) = x\phi(e_i)$ for all $x \in \mathbb{R}$ for all i $(1 \le i \le n)$.

Let $x \in \mathbb{R}$ and $1 \le i \le n$. Then there exist a rational sequence (r_m) such that $r_m \to x$ in $\mathbb{R}$. Then $r_me_i \to xe_i$ in the space $\mathbb{R}^n$. Since ϕ is continuous at x, we have

$$
\begin{aligned}
\phi(xe_i) &= \lim_{m\to\infty} \phi(r_me_i) \\
&= \big(\lim_{m\to\infty} r_m\big)\phi(e_i) \\
&= x\phi(e_i).
\end{aligned}
$$

Step 6: Characterize all ring homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}$.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. If $\phi(e_i) = 0$ for all i $(1 \le i \le n)$, then

$$
\phi(x) = \phi\left(\sum_{i=1}^{n} x_ie_i \right) = \sum_{i=1}^{n} \phi(x_ie_i) = \sum_{i=1}^{n} x_i\phi(e_i) = 0
$$

and hence ϕ is the trivial ring homomorphism. Otherwise, if $\phi(e_k) = 1$ and $\phi(e_i) = 0$ for all $i \ne k$, then

$$
\phi(x) = \phi\left(\sum_{i=1}^{n} x_ie_i \right) = \sum_{i=1}^{n} \phi(x_ie_i) = \sum_{i=1}^{n} x_i\phi(e_i) = x_k\phi(e_k) = x_k\,.
$$

Clearly ϕ is a ring homomorphism. Hence non-trivial ring homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}$ has the form

$$\phi(x_1, x_2, \dots, x_n) = x_k, \quad 1 \le k \le n.$$

So the number of ring homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}$ is $n+1$.

4. Conclusion

We characterized and computed all ring homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^m$, from $\mathbb{Q}^n$ to $\mathbb{Q}^m$ and $\mathbb{R}^n$ to $\mathbb{R}$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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