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Research Article

On Solving Minimization Problem and Common Fixed Point Problem Over Geodesic Spaces With Curvature Bounded Above

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Abstract. In this paper, we introduce a new modified proximal point algorithm for solving minimization problems and common fixed point problem in CAT(1) spaces. We prove strong and Δ-convergence theorems under some mild conditions. Further, an application on convex minimization and common fixed point problem over CAT(κ) spaces with the bounded positive real number κ are presented. Our results extend and improve the corresponding recent results in the literature.

Keywords. Minimization problem; Fixed point problem; Iteration process; Proximal point algorithm

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1. Introduction

Let E be a uniformly convex Banach space, C be a non-empty closed convex set and $C \subseteq E$. In this article, N denotes the set of all positive integers and $F(T) := \{x : Tx = x\}$. A self-mapping T in C is called non-expansive if $\|Tx - Ty\| \leq \|x - y\|\ \forall\ x, y \in C$ and $\forall\ n \in N$.

The iteration process for approximating fixed points were studied by many authors as follows. In this kind of iteration, we choose $x_0 \in X$ arbitrarily and $\{x_n\}_{n=0}^{\infty}$ was introduced iteratively by the following successive iteration method:

$$x_{n+1} = Tx_n, \quad \forall\ n \geq 0. \tag{1.1}$$

We called the iteration method (1.1) as Picard iteration.

The iterative scheme of $\{x_n\}_{n=0}^{\infty}$ was given by

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \quad \forall\ n \geq 0, \tag{1.2}$$

where $\lambda \in (0,1)$. We called the iteration method (1.2) as Krasnoselskij iteration.

In 1953, Mann introduced the well-known iteration process, called Mann iteration, which start from $x_0 \in E$ and defined the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall\ n \geq 0, \tag{1.3}$$

where the sequence $\{\alpha_n\}$ is in $(0,1)$.

In 1974, Ishikawa introduced the iteration as follows: the sequences $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \in N, \end{cases} \tag{1.4}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. This iteration reduces to the iteration in (1.3) when $\beta_n = 0$, $\forall\ n \in N$.

On the other hand, the initials of the term CAT are in honor of E. Cartan, A. D. Alexanderov and V. A. Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function. A CAT(κ) space is a geodesic metric space which no geodesic triangle is fatter than the corresponding comparison triangle in a model space with constant curvature κ, for $\kappa \in R$. It is a generalization of a simply-connected Riemannian manifold with sectional curvature $\leq \kappa$.

Kirk ([23,24]) first studied the theory of fixed point in CAT(κ) spaces. Later on, many authors generalized the notion of CAT(κ) given in [23,24], mainly focusing on CAT(0) spaces (see e.g., [1,8–10,12,22,26,38,40,43]). The results of a CAT(0) space can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see in [7]). Although, CAT(κ) spaces for $\kappa > 0$, were studied by some authors (see e.g., [13,17,35,39,44]).

Furthermore, let (X,d) be a geodesic metric space and f be a proper and convex function from the set X to $(-\infty,\infty]$. Some major problems in optimization is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

The set of minimizers of f was denoted by $\arg\min_{y \in X} f(y)$. In 1970, Martine [29] first introduced the effective tool for solving this problem which is the proximal point algorithm (for short term, PPA). Later in 1976, Rockafellar [37] found that the PPA converges to the solution of the convex problem in Hilbert spaces.

Let f be a proper, convex, and lower semi-continuous function on a Hilbert space H which attains its minimum. The PPA is defined by $x_1 \in H$ and

$$x_{n+1} = \arg\min_{y\in H}\left[f(y) + \frac{1}{2\lambda_n}\parallel y - x_n \parallel^2\right]$$

for each $n \in N$, where $\lambda_n > 0$ for all $n \in N$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\sum\limits_{n=1}^{\infty} \lambda_n = \alpha$. However, as shown by Guler [15], the PPA does not necessarily converges strongly in general. In 2000, Kamimura-Takahashi [18] combined the PPA with Halpern's algorithm [16] so that the strong convergence is guaranteed (see also [6,28,49,50]).

In 2013, Bacak [5] introduced the PPA in a CAT(0) space (X,d) as follows: $x_1 \in X$ and

$$x_{n+1} = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right]$$

for each $n \in N$, where $\lambda_n > 0$ for all $n \in N$. Based on the concept of the Fejer monotonicity, it was shown that, if f has a minimizer and $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer (see also [5]). In 2014, Bacak [3] employed a split version of the PPA for minimizing a sum of convex functions in complete CAT(0) spaces. Another interesting results can also be found in [4,5,14].

In 2015, Cholamjiak *et al.* [11] introduce modified proximal point algorithm involving fixed point iterates of nonexpansive mappings in CAT(0) spaces as follows:

$$\begin{cases} z_n = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right], \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1-\alpha_n)T_1 \oplus \alpha_n T_2 y_n \end{cases} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in the interval $[0,1]$. Another interesting results can also be found in [25,31,32,34,42,45].

In 2017, Kimura and Kohsaka [21] introduced the asymptotic behaviour of the sequences generated by the proximal point algorithm for a convex function in geodesic spaces with curvature bounded above. Also, they introduced the proximal point algorithm $\{x_n\}$ in a CAT(1) space X as follows:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = \arg\min_{y\in X}\left[g(y) + \frac{1}{\lambda_n}\tan(d(y, x_n))\sin(d(y, x_n))\right] \end{cases} \tag{1.6}$$

for each $n \in N$, where $\lambda_n > 0$ for all $n \in N$. Based on the concept of the Fejér monotonicity, it was shown that, if f has a minimizer and $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer (see also [5]). Recently, in 2014, Bačák [3] employed a split version of the PPA for minimizing a sum of convex functions in complete CAT(0) spaces. Another interesting results can also be found in [4,5,14].

In 2018, Pakkaranang *et al.* [33] introduced the proximal point algorithm for a convex function and nonexpansive mapping in CAT(1) spaces X as follows:

$$\begin{cases} x_1 \in X, \\ w_n = \operatorname*{argmin}_{y \in X} \left[g(y) + \frac{1}{\lambda_n} \tan(d(y,x_n)) \sin(d(y,x_n)) \right], \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T w_n \end{cases} \tag{1.7}$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a real sequences in the interval $[0,1]$. They proved Δ-convergence theorem under some mild conditions.

Recently, many convergence results by the PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [14, 27, 36, 48]. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds (see in [2, 41, 46, 47]).

Motivated and inspired by (1.4), (1.6), (1.5) and (1.7), we introduce a new modified proximal point algorithm by using Ishikawa as follows. Let (X,d) be an admissible complete CAT(1) space and $g :\to (-\infty,\infty)$ bbe a proper lower semi-continuous. Suppose that $T,S : K \to K$ are two non-expansive mappings such that $\Omega \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $[a_1,a_2]$ for some $a_1,a_2 \in (0,1)$ and $\{\lambda_n\}$ be a sequence such that, for each $n \geq 1$, $\lambda_n \geq \lambda \geq 0$ for some λ.

$$\begin{cases} z_n = \operatorname*{argmin}_{y \in X} \left[g(y) + \frac{1}{\lambda_n} \tan(d(y,x_n)) \sin(d(y,x_n)) \right], \\ y_n = (1-\beta_n) x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1-\alpha_n) x_n \oplus \alpha_n T_2 y_n \end{cases} \tag{1.8}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real appropriate sequences in the interval $[0,1]$.

2. Preliminaries

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map γ from a closed interval $[0,l] \subset R$ to X such that $\gamma(0) = x$, $\gamma(l) = y$, and $\rho(\gamma(t),\gamma(t')) = |t-t'|$ for all $t,t' \in [0,l]$. In particular, γ is an isometry and $d(x,y) = l$. The image $\gamma([0,l])$ of γ is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$d(x,z) = (1-\alpha)d(x,y) \quad \text{and} \quad d(y,z) = \alpha d(x,y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space (X,ρ) is said to be a geodesic space (*D-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x,y \in X$ (for $x,y \in X$ with $d(x,y) < D$). A subset K of X is said to be convex if K includes every geodesic segment joining any two of its points. The set K is said to be *bounded* if

$$\operatorname{diam}(K) := \sup\{d(x,y) : x, y \in K\} < \infty.$$

Now, we introduce the model spaces M_κ^n, for more details on these spaces the reader is referred to [7]. Let $n \in N$. We denote by E^n the metric space R^n endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in R^n, that is,

$$(x|y) = x_1y_1 + \ldots + x_ny_n \quad \text{where } x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n).$$

Let S^n denote the *n-dimensional sphere* defined by

$$S^n = \{x = x_1, \ldots, x_{n+1} \in R^{n+1} : (\cdot|\cdot) = 1\},$$

with metric $d_{S^n} = \arccos(x|y)$, $x, y \in S^n$.

Let $E^{n,1}$ denote the vector space R^{n+1} endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \ldots, u_{n+1})$ and $v = (v_1, \ldots, v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i\,.$$

Let H^n denote the *hyperbolic n-space* defined by

$$H^n = \{u = (u_1, u_2, \ldots, u_{n+1}) \in E^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 1\}$$

with metric d_{H^n} such that

$$\cosh d_{H^n}(x,y) = -\langle x|y\rangle, \quad x, y \in H^n.$$

Definition 2.1. Given $\kappa \in R$, we denote by M_κ^n the following metric spaces:

(1) if $\kappa = 0$ then M_0^n is the Euclidean space E^n;

(2) if $\kappa > 0$ then M_κ^n is obtained from the spherical space S^n by multiplying the distance function by the constant $1/\sqrt{\kappa}$;

(3) if $\kappa < 0$ then M_κ^n is obtained from the hyperbolic space H^n by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\Delta(x,y,z)$ in a geodesic space (X,d) consists of three points x, y, z in X (the *vertices* of Δ) and three geodesic segments between each pair of vertices (the *edges* of Δ). A comparison triangle for a geodesic triangle $\Delta(x,y,z)$ in (X,d) is a triangle $\Delta(\overline{x},\overline{y},\overline{z})$ in M_κ^2 such that

$$d(x,y) = d_{M_\kappa^2}(\overline{x},\overline{y}),\ d(x,z) = d_{M_\kappa^2}(\overline{x},\overline{z}) \ \text{ and } \ \rho(z,x) = d_{M_\kappa^2}(\overline{z},\overline{x}).$$

If $\kappa \le 0$ then such a comparison triangle always exists in M_κ^2. If $\kappa > 0$ then such a triangle exists whenever $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a comparison point for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\overline{x},\overline{p})$.

A geodesic triangle $\Delta(x,y,z)$ in X is said to satisfy the CAT(κ) inequality if for any $p,q \in \Delta(x,y,z)$ and for their comparison points $\overline{p},\overline{q} \in \Delta(\overline{x},\overline{y},\overline{z})$, one has

$$d(p,q) \le d_{M_\kappa^2}(\overline{p},\overline{q}).$$

Definition 2.2. If $\kappa \le 0$, then X is called a CAT(κ) *space* if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality. If $\kappa > 0$, then X is called a CAT(κ) *space* if and only if X is D_κ-geodesic and any geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Definition 2.3. A mapping $T : X \to X$ is said to be:

(1) *nonexpansive* if $d(Tx,Ty) \le d(x,y)$ for any $x,y \in X$.

(2) *demi-compact* if, for any sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, $\{x_n\}$ has a convergent subsequence.

Let (X,d) be a CAT(1) space such that $x,y,z \in X$ satisfy $d(x,y)+d(y,z)+d(z,x) < 2D_1$. Then, we have

$$\cos d(\alpha x \oplus (1-\alpha)y, z) \ge \alpha \cos d(x,z) + (1-\alpha)\cos d(y,z) \tag{2.1}$$

for all $\alpha \in [0,1]$.

Definition 2.4 ([30])**.** (1) An open set U in a geodesic metric space (X,d) is called a C_R-*domain* for any $R \in [0,2]$ if for any $x,y,z \in U$, any minimal geodesic $\gamma : [0,1] \to X$ between y and z for all $\alpha \in [0,1]$,

$$d^2(x,(1-\alpha)y \oplus \alpha z) \le (1-\alpha)d^2(x,y) + \alpha d^2(x,z) - \frac{R}{2}(1-\alpha)\alpha d^2(y,z). \tag{2.2}$$

(2) A geodesic metric space (X,d) is called R-*convex* for any $R \in [0,2]$ if X itself a C_R-*domain*.

(3) A geodesic metric space (X,d) is called *locally* R-*convex* for $R \in [0,2]$ if every point in X contained in a C_R-*domain*.

Definition 2.5. Let X be a CAT(1) space. A sequence $\{x_n\}$ in X is said to be $\triangle$-*convergent* to a point $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and denote $W_\triangle(x_n) := \cup\{A(\{u_n\})\}$.

Let $g : X \to (-\infty,\infty]$ be a function. The domain of g is the set,

$$\mathrm{Dom}(g) = \{x \in X : g(x) \in R\}.$$

The function g is said to be *proper* if Dom(g) is nonempty. The function g is said to be *lower semi-continuous* if the set $K = \{x \in X : g(x) \le \beta\}$ is closed in X for all $\beta \in R$.

A CAT(1) space X is said to be *admissible* if $d(v,v') < \frac{\pi}{2}$ for all $v,v' \in X$. In addition, the sequence $\{x_n\}$ in a CAT(1) space is said to be *spherically bounded* if

$$\inf_{y\in X} \limsup_{n\to\infty} d(y,x_n) < \frac{\pi}{2}.$$

Let $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. For all $\lambda > 0$, define the *resolvent* of g in admissible CAT(1) spaces as follows:

$$R_\lambda(x) = \arg\min_{y\in X} \left[g(y) + \frac{1}{\lambda} \tan d(y,x) \sin d(y,x) \right]$$

for all $x \in X$. The mapping R_λ is well define for all $\lambda > 0$. In particular, the set $F(R_\lambda)$ of fixed points of the resolvent associated with g coincides with the set $\arg\min_{y\in X} g(y)$ of minimizers of g.

Lemma 2.6. *Let $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function and (X,d) be a admissible complete CAT(1) space. If $\lambda > 0$, $\in X$ and $u \in \arg\min_X g$, then the following inequalities hold:*

$$\frac{\pi}{2}\left(\frac{1}{\cos^2 d(R_\lambda x,x)}+1\right)(\cos d(R_\lambda x,x)\cos d(u,R_\lambda x)-\cos d(u,x)) \geq \lambda(g(R_\lambda x)-g(u)) \tag{2.3}$$

and

$$\cos d(R_\lambda x,x)\cos d(u,R_\lambda x) \geq \cos d(u,x). \tag{2.4}$$

Lemma 2.7. *Let (X,d) be the admissible complete $CAT(1)$ space. If $g : X \to (-\infty,\infty]$ is a proper semi-continuous convex function, then g is $\Delta-$ lower semi-continuous.*

Lemma 2.8. *Let (X,d) be a complete CAT(1) space and $\{x_n\}$ be a spherical bounded sequence in X. If $d(d_n,\rho)$ is convergent for all $\rho \in W_\Delta(\{x_n\})$, then the sequence is Δ-convergent.*

Corollary 2.9. *Let C be a nonempty closed and convex subset of complete CAT(1) space (X,d). Let $T : C \to C$ be a nonepansive mapping. If $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and Δ-$\lim_{n\to\infty} x_n = \omega$, then $\omega \in C$ and $\omega = T\omega$.*

3. Main Results

In this section, we prove strong and Δ-convergence of modified Ishikawa proximal point algorithm for solving minimization problems and fixed point problems in $CAT(1)$ spaces as follows.

Lemma 3.1. *Let (X,d) be a admissible complete CAT(1) space and $g : X \to (-\infty,\infty)$ be proper lower semi-continuous. Suppose that T_1 and T_2 are two nonexpansive mappings, such that $\Omega = F(T_1)\cap F(T_2)\cap \arg\min_{x\in X} g(x)$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $[a_1,a_2]$ for some $a_1,a_2 \in (0,1)$ and $\{\lambda_n\}$ be a sequence such that, for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. Suppose that the sequence $\{x_n\}$ is generated by (1.8), for each $n \geq 1$. Then we have the following:*

1. *for all $q \in \Omega$, $\lim_{n\to\infty} d(x_n,q)$ exists;*
2. $\lim_{n\to\infty} d(x_n,z_n) = 0$*;*
3. $\lim_{n\to\infty} d(x_n,T_1x_n) = \lim_{n\to\infty} d(x_n,T_2x_n)$.

Proof. First, to show (1), we prove that the sequence $\{x_n\}$ is spherical bounded. Note that $z_n = R_{\lambda_n}x_n$ for each $n \geq 1$. Let $q \in \Omega$. Then, by (2.3) of Lemma 2.6, we have

$$\begin{aligned}\min\{\cos d(z_n,x_n),\cos d(q,z_n)\} &\geq \cos d(z_n,x_n) = \cos d(q,z_n)\\ &\geq \cos d(q,x_n)\end{aligned} \tag{3.1}$$

which implies that

$$\max\{d(z_n,x_n),d(q,x_n)\} \le d(q,x_n). \tag{3.2}$$

Since T_1 and T_2 are two nonexpansive mappings and X is admissible, it from (2.2), we get

$$\begin{aligned}\cos d(q,y_n) &= \cos d(q,(1-\beta_n)x_n \oplus \beta_n T_1 z_n)\\ &\ge (1-\beta_n)\cos d(q,x_n)+\beta_n \cos d(q,T_1 z_n)\\ &\ge (1-\beta_n)\cos d(q,x_n)+\beta_n \cos d(q,z_n)\\ &\ge (1-\beta_n)\cos d(q,x_n)+\beta_n \cos d(q,x_n)\\ &= \cos d(q,x_n)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\cos d(q,x_{n+1}) &= \cos d(q,(1-\alpha_n)x_n \oplus \alpha_n T_2 y_n)\\ &\ge (1-\alpha_n)\cos d(q,x_n)+\alpha_n \cos d(q,T_2 y_n)\\ &\ge (1-\alpha_n)\cos d(q,x_n)+\alpha_n \cos d(q,y_n)\\ &\ge (1-\alpha_n)\cos d(q,x_n)+\alpha_n \cos d(q,x_n)\\ &= \cos d(q,x_n),\end{aligned} \tag{3.4}$$

which implies that

$$d(q,x_{n+1}) \le d(q,x_n) \le d(q,x_1) < \frac{\pi}{2}. \tag{3.5}$$

So, the sequence $\{x_n\}$ and $\{z_n\}$ are spherically bounded. Thus, assertion (1) follows. Next, we show that

$$\sup_{n\ge 1} d(x_n,z_n) < \frac{\pi}{2}$$

and $\lim_{n\to\infty} d(q,x_n) < \frac{\pi}{2}$ exists for all $q \in \Omega$. This means that we obtain

$$\lim_{n\to\infty} d(q,x_n) = r \ge 0. \tag{3.6}$$

Thus, this claim that $\lim_{n\to\infty} d(x_n,q)$ exists, for all $q \in \Omega$. Now, we claim that $\lim_{n\to\infty} d(x_n,z_n) = 0$. By (3.4), it follows that

$$\begin{aligned}\cos d(q,x_{n+1}) &= \cos d(q,(1-\alpha_n)x_n \oplus \alpha_n T_2 y_n)\\ &\ge (1-\alpha_n)\cos d(q,x_n)+\alpha_n \cos d(q,T_2 y_n)\\ &\ge (1-\alpha_n)\cos d(q,x_n)+\alpha_n \cos d(q,y_n),\end{aligned}$$

i.e.,

$$\begin{aligned}\cos d(q,x_{n+1}) &\ge \cos d(q,x_n)-\alpha_n \cos d(q,x_n)+\alpha_n \cos d(q,y_n),\\ \alpha_n \cos d(q,x_n) &\ge \cos d(q,x_n)-\cos d(q,x_{n+1})+\alpha_n \cos d(q,y_n),\\ \cos d(q,x_n) &\ge \frac{1}{\alpha_n}[d(q,x_n)-\cos d(q,x_{n+1})]+\cos d(q,y_n).\end{aligned}$$

Since $\alpha_n \geq a_1 > 0$ for each $n \geq 1$, we obtain

$$\cos d(q,x_n) \geq \frac{1}{a_1}[d(q,x_n) - \cos d(q,x_{n+1})] + \cos d(q,y_n). \tag{3.7}$$

So, by (3.6) and (3.7), we obtain

$$r = \liminf_{n\to\infty} \cos d(q,x_n) \geq \liminf_{n\to\infty} \cos d(q,y_n). \tag{3.8}$$

On the other hand, by (3.3), we observe that

$$\limsup_{n\to\infty} \cos d(q,y_n) \geq \limsup_{n\to\infty} \cos d(q,x_n) = r. \tag{3.9}$$

Thus, by (3.8) and (3.9), we obtain

$$\lim_{n\to\infty} \cos d(q,y_n) = r. \tag{3.10}$$

From (3.1) and (3.2), we obtain

$$\begin{aligned}
\cos d(q,y_n) &= (1-\beta_n)\cos d(q,x_n) + \beta_n \cos d(q,T_1 z_n) \\
&\geq (1-\beta_n)\cos d(q,x_n) + \beta_n \cos d(q,z_n) \\
&\geq (1-\beta_n)\cos d(q,x_n) + \beta_n \frac{\cos d(q,x_n)}{\cos d(z_n,x_n)} \\
&= \cos d(q,x_n) + \beta_n \cos d(q,x_n)\left[\frac{1}{\cos d(z_n,x_n)} - 1\right],
\end{aligned}$$

that is,

$$\frac{\cos d(q,y_n)}{\cos d(q,x_n)} - 1 \geq \beta_n \left[\frac{1}{\cos d(z_n,x_n)} - 1\right].$$

Since $\beta_n \geq a_1 > 0$ for each $n \geq 1$, by (3.6) and (3.10), it follows that

$$1 \leq \frac{1}{\cos d(z_n,x_n)},$$

that is,

$$\lim_{n\to\infty} d(z_n,x_n) = 0.$$

So, we get

$$\lim_{n\to\infty} d(R_{\lambda_n} x_n) = 0.$$

Since $\lambda_n \geq \lambda > 0$ for each $n \geq 1$, we have

$$\lim_{n\to\infty} d(R_\lambda x_n, x_n) = 0.$$

Thus, this claim that $\lim_{n\to\infty} d(x_n,z_n) = 0$. Finally, we prove that $\lim_{n\to\infty} d(x_n,T_2 x_n) = \lim_{n\to\infty} d(x_n,T_1 x_n) = 0$. By the inequality (2.2), we have

$$\begin{aligned}
d^2(q,y_n) &= d^2(q,(1-\beta_n)x_n \oplus \beta_n T_1 z_n) \\
&\leq (1-\beta_n)d^2(q,x_n) + \beta_n d^2(q,T_1 z_n) - \frac{R}{2}(1-\beta_n)\beta_n d^2(x_n,T_1 z_n) \\
&\leq (1-\beta_n)d^2(q,x_n) + \beta_n d^2(q,z_n) - \frac{R}{2}a_1 a_2 d^2(x_n,T_1 z_n) \\
&\leq (1-\beta_n)d^2(q,x_n) + \beta_n d^2(q,x_n) - \frac{R}{2}a_1 a_2 d^2(x_n,T_1 z_n)
\end{aligned}$$

$$= d^2(q,x_n) - \frac{R}{2}a_1a_2d^2(x_n,T_1z_n),$$

which is equivalent to

$$d^2(q,y_n) \le d^2(q,x_n) - \frac{R}{2}a_1a_2d^2(x_n,T_1z_n),$$

$$\frac{R}{2}a_1a_2d^2(x_n,T_1z_n) \le d^2(q,x_n) - d^2(q,y_n),$$

$$d^2(x_n,T_1z_n) \le \frac{2}{Ra_1a_2}[d^2(q,x_n) - d^2(q,y_n)].$$

This yields

$$\lim_{n\to\infty} d(x_n,T_1z_n) = 0.$$

So, by the triangle inequality, we have

$$\begin{aligned} d(x_n,Tx_n) &\le d(x_n,T_1z_n) + d(T_1z_n,T_1x_n) \\ &\le d(x_n,T_1z_n) + d(z_n,x_n) \\ &\to 0, \quad \text{as } n\to\infty \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_n,T_1x_n) = 0.$$

Now, we have

$$\begin{aligned} d^2(q,x_{n+1}) &= d^2(q,(1-\alpha_n)x_n \oplus \alpha_nT_2y_n) \\ &\le (1-\alpha_n)d^2(q,x_n) + \alpha_nd^2(q,T_2y_n) - \frac{R}{2}(1-\alpha_n)\alpha_nd^2(x_n,T_2y_n) \\ &\le (1-\alpha_n)d^2(q,x_n) + \alpha_nd^2(q,T_2y_n) - \frac{R}{2}a_1a_2d^2(x_n,T_2y_n) \\ &\le (1-\alpha_n)d^2(q,x_n) + \alpha_nd^2(q,y_n) - \frac{R}{2}a_1a_2d^2(x_n,T_2y_n) \\ &= d^2(q,x_n) - \frac{R}{2}a_1a_2d^2(x_n,T_2y_n), \end{aligned}$$

which implies that

$$d^2(q,x_{n+1}) \le d^2(q,x_n) - \frac{R}{2}a_1a_2d^2(x_n,T_2y_n),$$

$$\frac{R}{2}a_1a_2d^2(x_n,T_2y_n) \le d^2(q,x_n) - d^2(q,x_{n+1}),$$

$$d^2(x_n,T_2y_n) \le \frac{2}{Ra_1a_2}[d^2(q,x_n) - d^2(q,x_{n+1})].$$

This gives

$$\lim_{n\to\infty} d(x_n,T_2y_n) = 0.$$

It follows that

$$\begin{aligned} d(y_n,x_n) &\le d((1-\beta_n)x_n \oplus \beta_nT_1z_n,x_n) \\ &\le \beta d(T_1z_n,x_n) \end{aligned}$$

$$\to 0, \quad \text{as } n \to \infty.$$

Using the triangle inequality, we have

$$\begin{aligned} d(x_n, T_2x_n) &\le d(x_n, T_2y_n) + d(T_2y_n, Sx_n) \\ &\le d(x_n, T_2y_n) + d(y_n, x_n) \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, the assertion 3. as follows. This completes the proof.

Next, assume that the conclusion of Lemma 3.1 hold. We prove some Δ-convergence results as follows.

Theorem 3.2. *Let (X,d) be an admissible complete CAT(1) spaces and $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. Then the sequence $\{x_n\}$ generated by* (1.8) *Δ-converges to an element of Ω, where $\Omega = F(T_1) \cap F(T_1) \cap \arg\min_{x\in X} g(x)$.*

Proof. Let $\rho \in \Omega$. Then we have $g(\rho) \le g(z_n)$ for each $n \ge 1$. By Lemma 2.6, we have

$$\lambda_n(g(z_n) - g(\rho)) \le \frac{\pi}{2}\left(\frac{1}{\cos^2 d(z_n,x_n)} + 1\right)(\cos d(z_n,x_n)\cos d(\rho,z_n) - \cos d(\rho,x_n)), \tag{3.11}$$

which gives

$$\begin{aligned} 0 &\le \lambda_n(g(z_n) - g(\rho)) \\ &\le \frac{\pi}{2}\left(\frac{1}{\cos^2 d(z_n,x_n)} + 1\right)(\cos d(z_n,x_n)\cos d(\rho,z_n) - \cos d(\rho,x_n)), \end{aligned} \tag{3.12}$$

Since $\lambda_n > \lambda > 0$ for each $n \ge 1$, by Lemma 3.1, we show that

$$d(z_{n,n}) \to 0, \quad \lim_{n\to\infty} d(\rho,x_n) \text{ and } \lim_{n\to\infty} d(\rho,z_n) \text{ exist.} \tag{3.13}$$

By (3.11) and (3.12), we obtain

$$\lim_{n\to\infty} g(z_n) = \inf g(X). \tag{3.14}$$

Next, it remains to show that $W_\Delta(\{x_n\}) \subset \Omega$. Let $z \in W_\Delta(\{x_n\})$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to the point z. Since $\lim_{n\to\infty} d(z_n,x_n)$, we can see, by the definition of the Δ-convergence, the subsequence $\{z_{n_i}\}$ of $\{z_n\}$ also Δ-converges to the point z. By using Lemma 2.7 and (3.14), we get

$$g(z) \le \liminf_{i\to\infty} g(z_{n_i}) \le \lim_{n\to\infty} g(z_n) = \inf g(X).$$

Hence, $z \in \arg\min_{x\in X} g(x)$ and so $W_\Delta(\{x_n\}) \subset \arg\min_{x\in X} g(x)$. Moreover, since

$$\lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_1x_n) = 0,$$

and $\{x_n\}$ Δ-converges to z, it follows from Corollary 2.9 that $z \in F(T_1)$. Thus we conclude that $W_\Delta(\{x_n\}) \subset \Omega$, we can see that $d(z,x_n)$ is convergent for all $z \in W_\Delta(\{x_n\})$. Using Lemma 2.8, $\{x_n\}$ is Δ-convergent to element in Ω. This complete the proof. □

Theorem 3.3. *Let (X,d) be an admissible complete $CAT(1)$ spaces and $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. Then the following are equivalent*

(A) *The sequence $\{x_n\}$ generated by (1.8) strongly converges to an element of Ω.*

(B) $\liminf_{n\to\infty} d(x_n,\Omega) = 0$*, when* $d(x,\Omega) = \inf\{d(x,x^*) : q \in \Omega\}$.

Proof. First, we prove that (A)⇒(B). It is obvious.

Second, we prove that (B)⇒(A). Suppose that $\liminf_{n\to\infty} = 0$. Since, for all $q \in \Omega$

$$d(x_{n+1},q) \le d(x_n,q),$$

we get

$$d(x_{n+1},\Omega) \le d(x_n,\Omega).$$

Therefore, $\lim_{n\to\infty} d(x_n,\Omega) = 0$. Then, by the techniques in proof of Khan and Abbas [19], we obtain $\{x_n\}$ is Cauchy sequence in X. This implies that $\{x_n\}$ converges to point $c \in X$ and therefore $d(c,\Omega) = 0$. Since Ω is closed, $c \in \Omega$. This completes the proof. □

The mappings T_1, T_2, T_3 are called to satisfy the condition Q if there exists a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(k) \ge 0$ for all $k \in (0,\infty)$ such that

$$d(x,T_1x) \ge h(d(x,H))$$

or

$$d(x,T_2x) \ge h(d(x,H))$$

or

$$d(x,T_3x) \ge h(d(x,H)),$$

for all $x \in X$, where $H = H(T_1) \cap H(T_2) \cap H(T_3)$.

By applying the condition Q, we the result as follows.

Theorem 3.4. *Let (X,d) be an admissible complete $CAT(1)$ spaces and $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. If the mappings R_λ, T_1 and T_2 satisfy the condition Q, then the sequence $\{x_n\}$ generated by (1.8) strongly converges to an element of Ω.*

Proof. By Lemma 3.1, we prove $\lim_{n\to\infty} d(x_n,q)$ exists for all $q \in \Omega$. Also, it follows that $\lim_{n\to\infty} d(x_n,\Omega)$ exists. Later, by applying the condition Q, we have

$$\lim_{n\to\infty} h(d(x_n,\Omega)) \le \lim_{n\to\infty} d(x_n,R_\lambda x_n) = 0,$$

or

$$\lim_{n\to\infty} h(d(x_n,\Omega)) \le \lim_{n\to\infty} d(x_n,T_1x_n) = 0,$$

or

$$\lim_{n\to\infty} h(d(x_n,\Omega)) \le \lim_{n\to\infty} d(x_n,T_2x_n) = 0.$$

Hence, we get

$$\lim_{n\to\infty} h(d(x_n,\Omega)) = 0$$

which by the property of h yields $\lim_{n\to\infty} d(x_n,\Omega) = 0$. Likewise, from the remained proof can be followed by the proof of Theorem 3.3 and thus the desirous result follows. This complete the proof. □

Lemma 3.5. *Let (X,d) be an admissible complete $CAT(1)$ spaces and $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. If the mappings R_λ or T_1 or T_2 is demi-compact, then the sequence $\{x_n\}$ generated by (1.8) strongly converges to an element of Ω.*

Proof. By Lemma 3.1, we have

$$\lim_{n\to\infty} d(x_n,R_\lambda x_n) = \lim_{n\to\infty} d(x_n,T_1x_n) = \lim_{n\to\infty} d(x_n,T_2x_n) = 0 \tag{3.15}$$

as $n\to\infty$. Without loss of generality, we suppose that T_1, T_2 or R_λ is demi-compact. Therefore, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $\rho^* \in X$. Hence, from (3.15) and the nonexpansiveness of mappings T, S, R_λ, it followed that

$$d(\rho^*,R_\lambda\rho^*) = d(\rho^*,T_1\rho^*) = d(\rho^*,T_2\rho^*) = 0,$$

which mean that $\rho^* \in \Omega$. Later, we can prove the strong convergence of $\{x_n\}$ to an element of Ω. This complete the proof. □

4. Some Applications

In this section, we show some applications to some convex optimization problems and the common fixed point in CAT(κ) with the bounded positive real number κ.

Throughout this section, we give the following assumptions:

(A_1) X is a complete CAT(κ) space such that $d(v,v') < \frac{D_\kappa}{2}$;

(A_2) κ is a positive real number and $D_x = \frac{\pi}{\sqrt{\kappa}}$;

(A_3) $g : X \to (-\infty,\infty]$ be a proper lower semicontinuous convex function;

(A_4) $\widehat{R}_\lambda$ is the resolvent mapping on X defined by

$$\widehat{R}_\lambda(x) = \operatorname*{argmin}_{y\in X}\left[g(y) + \frac{1}{\lambda}\tan(\sqrt{\kappa}d(y,x))\sin(\sqrt{\kappa}d(y,x))\right]$$

for all $\lambda > 0$ and $x \in X$.

Since $(X,\sqrt{\kappa}d)$ be the admissible complete CAT(1) space, the mapping $\widehat{R}_\lambda$ is well-defined [20]. By Theorem 3.2, 3.3, 3.4 and Lemma 3.5 and assume that assumptions A_1, A_2, A_3 and A_4 hold, we obtain Corollary 4.1, 4.2, 4.3 and 4.4, respectively.

Corollary 4.1. *Suppose that the assumptions A_1-A_4 hold. Let T_1 and T_2 be two nonexpansive self mappings on the set C such that $\Omega \neq \emptyset$. Assume that the sequence $\{\alpha_n\},\{\beta_n\} \subseteq [a_1,a_2]$ for some $a_1,a_2 \in (0,1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ.*

For any $x_1 \in X$, define the sequence $\{x_n\} \in C$ by

$$\begin{cases} z_n = \arg\min_{y \in X} \left[g(y) + \frac{1}{\lambda_n} \tan(\sqrt{\kappa} d(y, x_n)) \sin(\sqrt{\kappa} d(y, x_n)) \right], \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_2 y_n \end{cases} \tag{4.1}$$

for each $n \geq 1$. Then sequence $\{_n\}$ Δ-converges to an element of Ω.

Corollary 4.2. *Suppose that the assumptions A_1-A_4 hold. Let T_1 and T_2 be two nonexpansive self mappings on the set C such that $\Omega \neq \emptyset$. Assume that the sequence $\{\alpha_n\}, \{\beta_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0,1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. Then the following are equivalent:*

(1) *The sequence $\{x_n\}$ generated by (4.1) converges strongly to an element of Ω.*

(2) $\liminf_{n\to\infty} d(x_n, \Omega) = 0$ *where* $d(x, \Omega) = \inf\{d(x,^*) : q \in \Omega\}$.

Corollary 4.3. *Suppose that the assumptions A_1-A_4 hold. Let T_1 and T_2 be two nonexpansive self mappings on the set C such that $\Omega \neq \emptyset$. Assume that the sequence $\{\alpha_n\}, \{\beta_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0,1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. If the mappings R_λ, T_1, T_1 satisfy the Condition (Q) then the sequence $\{x_n\}$ generated by (4.1) converges strongly to an element of Ω.*

Corollary 4.4. *Suppose that the assumptions A_1-A_4 hold. Let T_1 and T_2 be two nonexpansive self mappings on the set C such that $\Omega \neq \emptyset$. Assume that the sequence $\{\alpha_n\}, \{\beta_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0,1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ.*

5. Conclusion

The main objectives of this paper is to introduced a new modified proximal point algorithm for solving minimization problems and common fixed point problem in CAT(1) spaces. We prove some convergence theorems under some mild conditions. Further, an application on convex minimization and common fixed point problem over CAT(κ) spaces with the bounded positive real number κ are presented. Our results extent and improve the corresponding recent results announced by many authors from the literature. Further attention is needed for the study of applications of the established result in the real world problems.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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