Research Article

Construction of a Family of C^1 Convex Integro Cubic Splines

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Abstract. We construct a family of monotone and convex C^1 integro cubic splines under a strictly convex position of the dataset. Then, we find an optimal spline by considering its approximation properties. Finally, we give some examples to illustrate the convex-preserving properties of these splines.

Keywords. Shape-preserving; Approximation; Integro spline

MSC. 41A15; 65D10

1. Introduction

Let Δ_k be the non-uniform partition on $[a,b]$, $a = x_0 < x_1 < \cdots < x_k = b$, and $h_{i+1} = x_{i+1} - x_i$, $i = 0, \cdots, k-1$, are step sizes. Let $S(x)$ be a cubic spline that approximates a function $u(x)$. We assume that the function values $u_i = u(x_i)$ are not given, but the integral values $h_{i+1}I_{i+1}$ on the subintervals $[x_i, x_{i+1}]$ of $u(x)$ are known. The problem of the construction of an integro cubic spline (see [2]) is to find $S(x)$ such that

$$\int_{x_i}^{x_{i+1}} S(x)dx = \int_{x_i}^{x_{i+1}} u(x)dx = h_{i+1}I_{i+1}, \quad i = 0, \cdots, k-1. \tag{1}$$

We use a notation $m_i = S'(x_i)$ and piecewise polynomial representation

$$S(x) = (1-t)^2(1+2t)S(x_i) + t^2(3-2t)S(x_{i+1}) + h_{i+1}t(1-t)\{(1-t)m_i - tm_{i+1}\},$$

$$x \in [x_i, x_{i+1}], \quad t = \frac{x - x_i}{h_{i+1}}, \quad i = 0, \cdots, k-1, \quad t \in [0,1]. \tag{2}$$

The sufficient conditions for convexity of cubic histosplines derived in [8,9] can be written in the form:

$$2m_{i-1} + m_i \le \frac{3}{h_i}(S(x_i) - S(x_{i-1})) \le m_{i-1} + 2m_i, \quad i = 1, \cdots, k,$$

$$\frac{1}{2}(S(x_{i-1}) + S(x_i)) + \frac{h_i}{12}(m_{i-1} - m_i) = I_i, \quad i = 1, \cdots, k. \tag{3}$$

Obviously, the system (3) has an infinite number of solutions. In [9–11], the staircase algorithm with three terms of recurrence relations was used to find the solutions of (3). Moreover, shape-preserving approximations of histosplines have been studied in [3, 4, 15] and references therein. There are two traditional approaches to constructing shape-preserving histosplines: additional knots of spline and splines of a higher order with less smoothness [4]. It is well known that the interpolating cubic spline of the class C^2 does not preserve the monotonicity and convexity of the input data. Recently, the shape-preserving properties of the C^2 local integro cubic spline have been investigated only on a uniform partition in [14].

Usually, the monotonicity and convexity preserving property of the spline $S(x)$ are discussed based on the properties of the data $u_i = u(x_i)$. Now, we discuss the properties of monotonicity and convexity of the spline $S(x)$ based on data I_i. We construct a family of monotone and convex C^1 integro cubic splines under a strictly convex position of the data set. In this paper, we will give a simple constructive algorithm for C^1 integro cubic splines (or histosplines) that preserve monotonicity and convexity. The remainder of this paper is organized as follows. In Section 2, a simple method for constructing the family of C^1 integro cubic splines (depending on the parameter α) is given. We discuss sufficient conditions of monotonicity and convexity of the presented integro cubic splines. We also consider an error analysis of the integro cubic splines in Section 3. Some numerical examples are given in Section 4 to illustrate the convexity preserving property.

2. Construction of Convex Integro Cubic Splines

Using the ideas in [12, 13] instead of inequalities in (3), we consider the following relations:

$$\frac{3}{h_i}(S(x_i) - S(x_{i-1})) = \alpha(m_{i-1} + 2m_i) + (1-\alpha)(2m_{i-1} + m_i)$$

$$= (2-\alpha)m_{i-1} + (1+\alpha)m_i, \quad \alpha \in [0,1]. \tag{4}$$

The right-hand side of (4) is a linear combination of $m_{i-1} + 2m_i$ and $2m_{i-1} + m_i$, and is a linear function with respect to α. Hence, from (4), it follows when $m_{i-1} \le m_i$ that

$$2m_{i-1} + m_i \le \frac{3}{h_i}(S(x_i) - S(x_{i-1})) \le m_{i-1} + 2m_i. \tag{5}$$

That is, instead of (3), it is possible to consider

$$\frac{3}{h_i}(S(x_i) - S(x_{i-1})) = (2 - \alpha)m_{i-1} + (1 + \alpha)m_i,$$

$$\frac{3}{h_i}(S(x_i) + S(x_{i-1})) = \frac{6}{h_i}I_i - \frac{1}{2}(m_{i-1} - m_i). \tag{6}$$

By adding and subtracting these two equations, we get

$$S(x_i) = I_i + \frac{h_i}{12}\{(3 - 2\alpha)m_{i-1} + (3 + 2\alpha)m_i\}, \quad i = 1, \cdots, k, \tag{7}$$

$$S(x_{i-1}) = I_i + \frac{h_i}{12}\{(2\alpha - 5)m_{i-1} - (2\alpha + 1)m_i\}, \quad i = 1, \cdots, k. \tag{8}$$

From (7) and (8), it follows that

$$\lambda_i(3 - 2\alpha)m_{i-1} + \{\lambda_i(3 + 2\alpha) + \mu_i(5 - 2\alpha)\}m_i + \mu_i(2\alpha + 1)m_{i+1} = 6\delta I_i, \quad i = 1, \cdots, k - 1, \tag{9}$$

where

$$\lambda_i = \frac{h_i}{h_i + h_{i+1}}, \quad \mu_i = 1 - \lambda_i, \quad \delta I_i = \frac{I_{i+1} - I_i}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}. \tag{10}$$

Using the eq. (7), (8), and (9), we get a closed system of equations

$$(5 - 2\alpha)m_0 + (2\alpha + 1)m_1 = \frac{12}{h_1}(I_1 - S(x_0)),$$

$$a_i m_{i-1} + c_i m_i + b_i m_{i+1} = 6\delta I_i, \quad i = 1, \cdots, k - 1, \tag{11}$$

$$(3 - 2\alpha)m_{k-1} + (3 + 2\alpha)m_k = \frac{12}{h_k}(S(x_k) - I_k),$$

where

$$a_i = \lambda_i(3 - 2\alpha) > 0, \quad b_i = \mu_i(1 + 2\alpha) > 0, \quad c_i = a_i + b_i + 4(\alpha\lambda_i + (1 - \alpha)\mu_i) > a_i + b_i > 0. \tag{12}$$

Since,

$$5 - 2\alpha - 2\alpha - 1 = 4(1 - \alpha) \geq 0,$$

$$\lambda_i(3 + 2\alpha) + \mu_i(5 - 2\alpha) - \lambda_i(3 - 2\alpha) - \mu_i(2\alpha + 1) = 4\lambda_i\alpha + 4\mu_i(1 - \alpha) > 0, \quad i = 1, \cdots, k - 1,$$

$$3 + 2\alpha - 3 + 2\alpha = 4\alpha \geq 0,$$

the matrix of the system (11) has diagonal dominance. Hence, the system (11) has a unique solution $(m_0, m_1, \cdots, m_k)$ for each $\alpha \in [0, 1]$, and it can be easily solved by using the tridiagonal *LU* decomposition algorithm. Here, $S(x_0)$ and $S(x_k)$ are assumed to be given for now. The values of $S(x_i)$ are determined by means of (7) or (8), and the spline S is given by (2). Then, $S(x)$ will be C^1 cubic integro splines depending on the parameter $\alpha \in [0, 1]$. Thus, the family of $S(x, \alpha)$ depending on the parameter α is determined completely. As usual, the given data I_i is called monotonically increasing if

$$\delta I_i = \frac{I_{i+1} - I_i}{\hbar_i} \geq 0, \quad i = 1, \cdots, k - 1, \tag{13}$$

and convex if

$$\frac{I_{i+1} - I_i}{\hbar_i} - \frac{I_i - I_{i-1}}{\hbar_{i-1}} \geq 0, \quad i = 2, \cdots, k - 1, \tag{14a}$$

or

$$\delta I_i \geq \delta I_{i-1}. \tag{14b}$$

Using the Taylor expansion of $S(x)$ in (2), we obtain

$$S(x_0) = I_1 + \frac{h_1}{12} \left\{ \frac{\mu_1(1+2\alpha)(2\alpha-5)(\delta I_1 - \delta I_2)}{\lambda_1(3-2\alpha)} - 6\delta I_1 \right\},$$

$$S(x_k) = I_k + \frac{h_k}{12} \left\{ \frac{\lambda_{k-1}(9-4\alpha^2)(\delta I_{k-1} - \delta I_{k-2})}{\mu_{k-1}(1+2\alpha)} + 6\delta I_{k-1} \right\}.$$

To study the shape-preserving properties of (2), one must use the derivatives of (2), which are

$$S'(x) = 6t(1-t)\frac{S(x_{i+1}) - S(x_i)}{h_{i+1}} + (1-t)(1-3t)m_i + t(3t-2)m_{i+1}, \tag{15}$$

and

$$S''(x) = 6(1-2t)\frac{S(x_{i+1}) - S(x_i)}{h_{i+1}^2} + \frac{1}{h_{i+1}}\{(6t-4)m_i + (6t-2)m_{i+1}\}. \tag{16}$$

Using (6) in (15) and (16), we obtain

$$S'(x) = (1-t)(1+(1-2\alpha)t)m_i + t(2\alpha + t(1-2\alpha))m_{i+1}, \tag{17}$$

and

$$S''(x) = 2\frac{\alpha + t(1-2\alpha)}{h_{i+1}}(m_{i+1} - m_i). \tag{18}$$

It is easy to show that

$$1 + (1-2\alpha)t \geq 0, \quad 2\alpha + t(1-2\alpha) \geq 0, \quad \text{for } \alpha \in [0,1].$$

Hence, from (17), it follows that

$$S'(x) \geq 0, \quad x \in [x_i, x_{i+1}] \text{ if } m_i \geq 0 \text{ and } m_{i+1} \geq 0. \tag{19}$$

Since $\alpha + t(1-2\alpha) \geq 0$ then from (18), it follows that

$$S''(x) \geq 0, \quad x \in [x_i, x_{i+1}] \text{ if } m_{i+1} - m_i \geq 0. \tag{20}$$

Thus, from (19), we conclude that $S(x, \alpha)$ will monotonically increase if the solution to (11) is nonnegative. In order to study the solution to (11), we use the following theorem given in [5].

Theorem 1. *For the system $Ax = f$, suppose that*

$$a_{ij} \geq 0, \quad a_{ii} > 0, \quad f_i > 0, \quad i,j = 1,\cdots,k, \quad i \neq j.$$

If for all i, $i = 1,\cdots,k$,

$$f_i > \sum_{j=1, \, j \neq i}^{k} a_{ij}\frac{f_j}{a_{jj}},$$

then A is invertible, and $x_i = (A^{-1}f)_i > 0$ for all i.

We show that the assumptions given in Theorem 1 are fulfilled for our system (11) under conditions

$$\frac{2a_1(I_1 - S(x_0))}{h_1(5-2\alpha)} + \frac{b_1\delta I_2}{c_2} < \delta I_1 < \frac{2c_1(I_1 - S(x_0))}{h_1(2\alpha+1)}, \quad I_1 - S(x_0) > 0, \tag{21a}$$

$$\frac{a_j \delta I_{j-1}}{c_{j-1}} + \frac{b_j \delta I_{j+1}}{c_{j+1}} < \delta I_j, \quad j = 2, \cdots, k-2, \tag{21b}$$

$$\frac{2b_{k-1}(S(x_k) - I_k)}{h_k(3+2\alpha)} + \frac{a_{k-1}\delta I_{k-2}}{c_{k-2}} < \delta I_{k-1} < \frac{2c_{k-1}(S(x_k) - I_k)}{h_k(3-2\alpha)}, \quad S(x_k) - I_k > 0. \tag{21c}$$

Let us summarize the obtained above results as:

Theorem 2. *Let the integro cubic splines $S(x, \alpha) \in C^1[a,b]$ be defined by (2), (7), and (11), and the data I_i monotonically increase. If the inequalities (21) are valid then $m_i > 0$ for all $i = 0, \cdots, k$ and thereby $S'(x) > 0$ on $[x_0, x_k]$, that is, S is monotonically increasing on $[a,b]$.*

Now, we proceed to study the convexity property of $S(x, \alpha)$. To this end, we pass from (11) to the following system

$$(2\alpha + 1)(m_1 - m_0) = \frac{12}{h_1}(I_1 - S(x_0)) - 6m_0, \tag{22a}$$

$$a_i(m_{i-1} - m_{i-2}) + c_i(m_i - m_{i-1}) + b_i(m_{i+1} - m_i)$$
$$= 6(\delta I_i - \delta I_{i-1}) + (a_{i-1} - a_i)m_{i-2} + (c_{i-1} - c_i)m_{i-1} + (b_{i-1} - b_i)m_i, \quad i = 2, \cdots, k-1, \tag{22b}$$

$$(3 + 2\alpha)(m_k - m_{k-1}) = \frac{12}{h_k}(S(x_k) - I_k) - 6m_{k-1}. \tag{22c}$$

If the following equalities hold:

$$a_{i-1} - a_i = 0, \quad c_{i-1} - c_i = 0, \quad b_{i-1} - b_i = 0, \tag{23}$$

then the equation (22b) for $i = 2, \cdots, k-1$ leads to

$$a_i(m_{i-1} - m_{i-2}) + c_i(m_i - m_{i-1}) + b_i(m_{i+1} - m_i) = 6(\delta I_i - \delta I_{i-1}), \quad i = 2, \cdots, k-1. \tag{24}$$

As above, it is easy to verify that the assumptions of Theorem 1 are fulfilled for the system (22a), (22c), and (24) under conditions

$$\frac{a_2(\frac{2(I_1 - S(x_0))}{h_1} - m_0)}{2\alpha + 1} + \frac{b_2(\delta I_3 - \delta I_2)}{c_3} < \delta I_2 - \delta I_1, \tag{25a}$$

$$\frac{a_j(\delta I_{j-1} - \delta I_{j-2})}{c_{j-1}} + \frac{b_j(\delta I_{j+1} - \delta I_j)}{c_{j+1}} < \delta I_j - \delta I_{j-1}, \quad j = 3, \cdots, k-2, \tag{25b}$$

$$\frac{b_{k-1}(\frac{2}{h_k}(S(x_k) - I_k) - m_{k-1})}{3 + 2\alpha} + \frac{a_{k-1}(\delta I_{k-2} - \delta I_{k-3})}{c_{k-2}} < \delta I_{k-1} - \delta I_{k-2}, \tag{25c}$$

where $\frac{2}{h_1}(I_1 - S(x_0)) - m_0 > 0$ and $\frac{2}{h_k}(S(x_k) - I_k) - m_{k-1} > 0$. Thus, we have:

Theorem 3. *Let the integro cubic splines $S(x, \alpha) \in C^1[a,b]$ be defined by (2), (7), and (11), and the data I_i are convex, and m_0 and m_{k-1} are given. If (23) and (25) are valid then $m_i - m_{i-1} > 0$ for all $i = 1, \cdots, k$ and thereby $S''(x) > 0$ on $[x_0, x_k]$, that is, $S(x)$ is convex on $[a,b]$.*

Note that the equalities (23) hold true if the step sizes of grid satisfy

$$h_i = \sqrt{h_{i-1}h_{i+1}}, \quad i = 1, \cdots, k-1. \tag{26}$$

Of course, the conditions (26) are fulfilled on a uniform partition. Now, we are interested in the dependence of m_i on parameter α. To this end, differentiating the system (11) with respect to

α, we obtain

$$(5-2\alpha)m_0'(\alpha)+(2\alpha+1)m_1'(\alpha)=2(m_0-m_1),$$

$$a_i m_{i-1}'(\alpha)+c_i m_i'(\alpha)+b_i m_{i+1}'(\alpha)=2\lambda_i(m_{i-1}-m_i)+2\mu_i(m_i-m_{i+1}), \quad i=1,\cdots,k-1, \qquad (27)$$

$$(3-2\alpha)m_{k-1}'(\alpha)+(3+2\alpha)m_k'(\alpha)=2(m_{k-1}-m_k).$$

From (27), it is clear that the right-hand side of the system (27) is negative if (23) and (25) are fulfilled.

Theorem 4. *Assume that* (23) *and* (25) *are fulfilled. Then, $m_i(\alpha)$ is a decreasing function with respect to α for all $i=1,\cdots,k-1$, i.e.,*

$$m_i(0) \geq m_i(\alpha) \geq m_i(1), \quad i=1,\cdots,k-1. \qquad (28)$$

Proof. By Theorems 1 and 3, the solution to system (27) is negative, that is, $m_i'(\alpha)<0$ for all $i=1,\cdots,k-1$. Hence, (28) is valid. $\qquad\square$

Thus, we obtain feasible intervals $[m_i(1),m_i(0)]$ of $m_i(\alpha)$ that ensure the monotonicity and convexity of splines $S(x,\alpha)$. From (7) and (28), we also derive the interval of $S(x_i,\alpha)$:

$$S(x_i,\alpha) \in \left[I_i + \frac{h_i}{2}m_{i-1}(1), I_i + \frac{h_i}{2}m_i(0) \right], \quad i=1(1)k-1. \qquad (29)$$

Theorem 5. *Let the assumptions of Theorem 3 be fulfilled and $m_0>0$. Then, $S(x_i)$ are in strictly convex positions as the data I_i, that is,*

$$\delta S(x_i) > \delta S(x_{i-1}) \geq 0, \quad i=1,\cdots,k-1. \qquad (30)$$

Proof. By (4) and Theorem 3, we have

$$\delta S(x_i) = \frac{S(x_{i+1})-S(x_i)}{h_{i+1}} = \frac{1}{3}\{(2-\alpha)m_i+(1+\alpha)m_{i+1}\} \geq m_i \geq 0.$$

By Theorem 3, we have

$$m_{i+1}-m_i>0, \quad i=0,\cdots,k-1,$$

$$m_i-m_{i-1}>0, \quad i=1,\cdots,k.$$

From this, we obtain

$$(1+\alpha)(m_{i+1}-m_i)+(2-\alpha)(m_i-m_{i-1})>0,$$

which leads to

$$(1+\alpha)m_{i+1}+(2-\alpha)m_i > (1+\alpha)m_i+(2-\alpha)m_{i-1}, \quad i=1,\cdots,k-1. \qquad (31)$$

Using (6) in (31), we get (30). $\qquad\square$

The well-known convex interval interpolation problem was solved by three-term staircase algorithm in [7]. From (14) and (29), one can easily see that we solved the convex interval interpolation problem $S(x_i) \in [l_i,v_i]$, $i=0,\cdots,k$, for a particular case with $l_i = I_i + \frac{h_i}{2}m_{i-1}(1)$, $v_i = I_i + \frac{h_i}{2}m_i(0)$.

3. Error Analysis

Now, we consider the approximation properties of convex integro cubic splines $S(x, \alpha)$. Using the Taylor expansion of $u \in C^3[a, b]$, one can easily obtain

$$\delta I_i = u'_i + \frac{h_{i+1} - h_i}{3} u''_i + O(\bar{h}^2), \tag{32}$$

$$I_i = u_i - \frac{h_i}{2} u'_i + \frac{h_i^2}{6} u''_i + O(\bar{h}^3), \tag{33}$$

where $\bar{h} = \max_{1 \le i \le k} h_i$.

From (32), it is clear that

$$\delta I_i = u'_i + O(\bar{h}^2), \tag{34}$$

under condition

$$h_{i+1} - h_i = O(\bar{h}^2). \tag{35}$$

Theorem 6. *Let $S(x, \alpha)$ be C^1 integro cubic splines defined by (2), (7), (11), and $S(x_i) = u_i + O(\bar{h}^3)$, $i = 0, k$. Then, for $u \in C^3$, we have estimations*

$$S^{(r)}(x_i) - u_i^{(r)} = O(\bar{h}^{\sigma+1-r}), \quad r = 0, 1, \ i = 0, \cdots, k. \tag{36}$$

under (35). Here $\sigma = 1$ when $\alpha \ne \frac{1}{2}$ and $\sigma = 2$ when $\alpha = \frac{1}{2}$.

Proof. First, let us estimate $q_i = m_i - u'_i$, $i = 0, \cdots, k$. To this end, we pass from (11) to the system

$$(5 - 2\alpha)q_0 + (2\alpha + 1)q_1 = d_0,$$

$$a_i q_{i-1} + c_i q_i + b_i q_{i+1} = d_i, \quad i = 1, \cdots, k-1, \tag{37}$$

$$(3 - 2\alpha)q_{k-1} + (3 + 2\alpha)q_k = d_k,$$

where

$$d_0 = \frac{12}{h_1}(I_1 - S(x_0)) - \{(5 - 2\alpha)u'_0 + (2\alpha + 1)u'_1\},$$

$$d_i = 6\delta I_i - \{a_i u'_{i-1} + c_i u'_i + b_i u'_{i+1}\}), \tag{38}$$

$$d_k = \frac{12}{h_k}(S(x_k) - I_k) - \{(3 - 2\alpha)u'_{k-1} + (3 + 2\alpha)u'_k\}.$$

Using (33), (34), (35), and the Taylor expansion of function $u \in C^3[a, b]$ in (38), one can easily obtain

$$d_i = O(\bar{h}^\sigma). \tag{39}$$

Then, from (37) and (39), it follows (36) for $r = 1$. From (7), we get

$$S(x_i) - u_i = I_i - u_i + \frac{h_i}{12}\{(3 - 2\alpha)(m_{i-1} - u'_{i-1}) + (3 + 2\alpha)(m_i - u'_i)\}$$

$$+ \frac{h_i}{12}\{(3 - 2\alpha)u'_{i-1} + (3 + 2\alpha)u'_i\}, \quad i = 1, \cdots, k-1. \tag{40}$$

As above, using (33), (36) for $r = 1$ and the Taylor expansion of function $u \in C^3[a, b]$ in (40), we obtain

$$S(x_i) - u_i = O(\bar{h}^{\sigma+1}), \quad i = 1, \cdots, k-1,$$

i.e., the estimate (36) is proved for $r = 0$. This completes the proof of Theorem 6. $\qquad \square$

Using (34) and (36) for $r = 1$, it is easy to show that

$$S''_{i+0} - S''_{i-0} = O(\bar{h}^{\sigma-1}), \quad i = 1, \cdots, k-1. \tag{41}$$

From the estimations (36) and (41), it is clear that the best or optimal C^1 integro cubic spline (abbr. OCICS) is derived when $\alpha = \frac{1}{2}$ in the sense of approximation properties. This selection shows that using an optimal choice of parameter one can rise the order of approximation.

4. Numerical Experiments

In this section, we apply the proposed method to some numerical examples.

Example 1. We consider the histogram $I = \{1, 2, 4\}$ on $\Delta_3 = \{0 < 4 < 6 < 7\}$ in [9]. A convex integro cubic spline (abbr. CICS) curve with $\alpha = 0.5$ is shown in Figure 1, and with $\alpha = 1$ is shown in Figure 2.
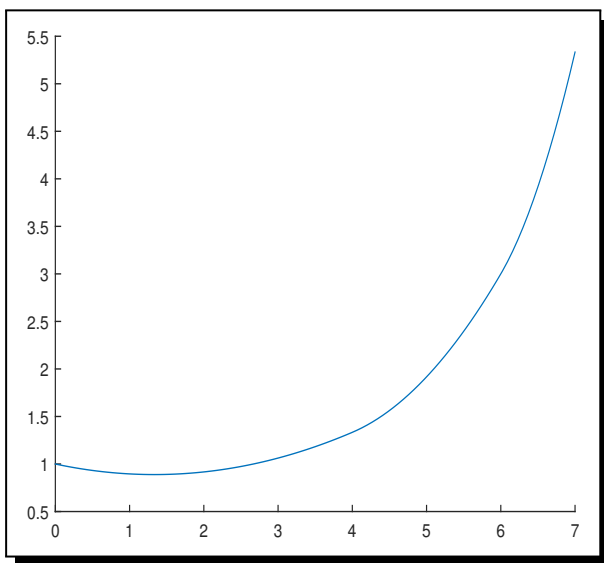

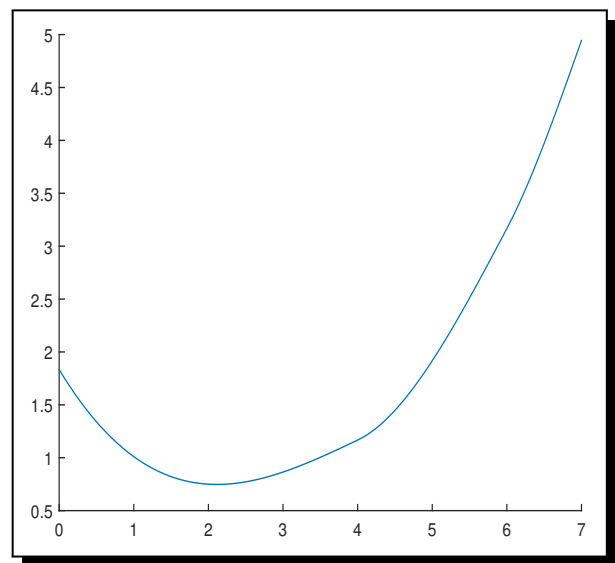**Figure 1.** Approximation by OCICS with $\alpha = 0.5$ for Example 1

Figure 2. Approximation by CICS with $\alpha = 1$ for Example 1

Example 2. Next, we take $u(x) = 2 - \sqrt{x(2-x)}$, $0 \le x \le 2$ [6]. This function is approximated by the CICS on a uniform mesh in x, for $k = 10$, in Figure 3. In Figure 4, we consider the CICS for this function on a non-uniform grid $\Delta_{10} = \{0 < 0.05 < 0.1 < 0.4 < 0.7 < 1 < 1.3 < 1.6 < 1.9 < 1.95 < 2\}$. Near the end knots, the fitting result of the spline curve in Figure 4 is better than that of the spline curve in Figure 3. From this example, we can see that the constructed CICS possesses

convexity-preserving property and convergence. The purpose of this example is to observe the effects of the changes in the step size.
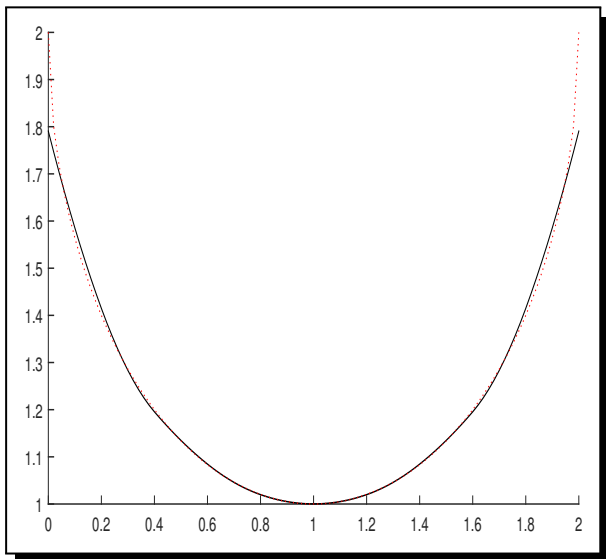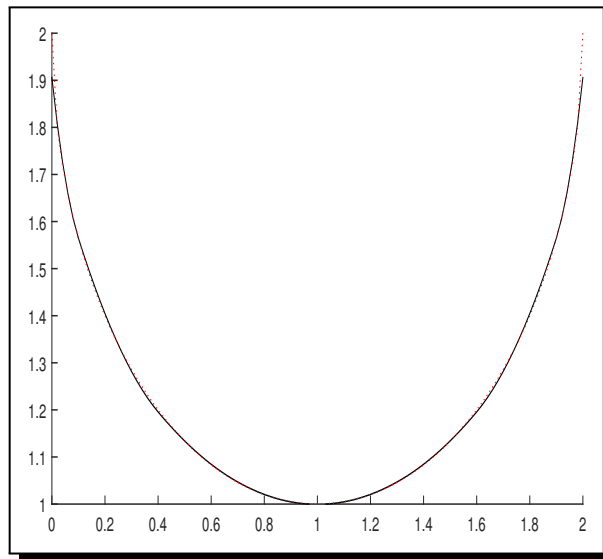

Figure 3. Approximation by OCICS for $u(x)$ on Δ_{10}

Figure 4. Approximation by OCICS for Akima's data

Example 3. Then, we consider the histogram $I = \{2.86, 1, 0.5, 1, 2, 2.86\}$ on $\Delta_6 = \{0 < 1 < 2 < 4 < 6 < 7 < 8\}$ which is in convex positions. Figure 5 shows that the fitting result is the same as that presented in [9].

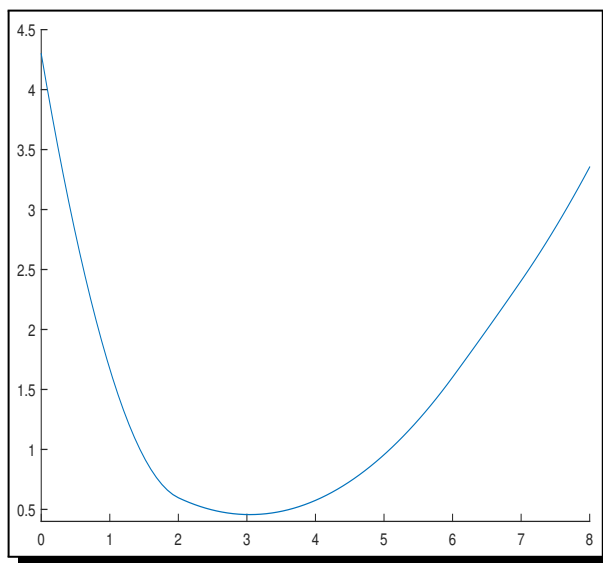Fortunately, for the data of the examples above, the conditions (25) are fulfilled.

Figure 5. Approximation by OCICS for Example 3

Example 4. The data for the last example were taken from [1] (see Table 1), where the conditions (25) are not fulfilled. As for Akima's data, the solution of the system (11) does not satisfy the condition $m_i \le m_{i+1}$, $i = 0, \cdots, k-1$, so we can not construct the shape-preserving integro spline with the proposed method. Now, we can simply choose m_i by

$$m_i = \delta I_i, \quad i = 1, \cdots, k-1,$$

because of Theorem 6. The remainder m_0 and m_k are obtained from (11) setting $i = 1, k-1$, respectively. The values of $S(x_i)$ are completely determined by (7) and (8). Figure 6 shows that the last integro cubic spline with $\alpha = \frac{1}{2}$ has a better convexity and monotonicity property.

Table 1. Akima's data [1]

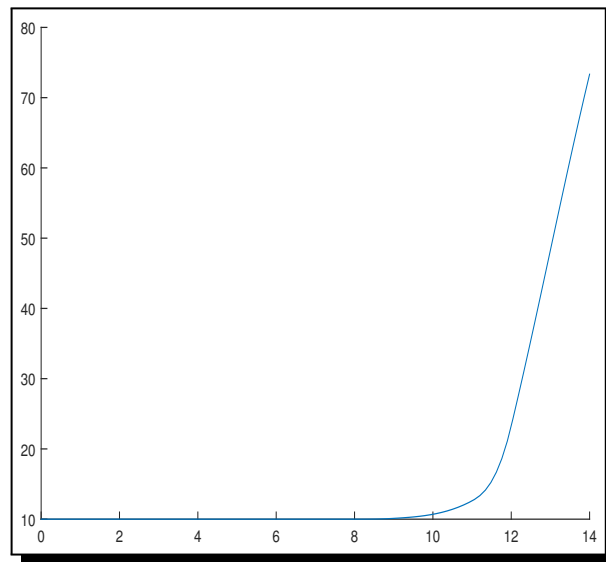| x_i	0	2	3	5	6	8	9	11	12	14
I_i	10	10	10	10	10	10	10.5	15	50	

Figure 6. Approximation by OCICS for Example 3

5. Conclusion

In this paper, we derive a family of C^1 convex integro cubic splines based on sufficient conditions for convexity. We give some sufficient convexity and monotonicity conditions for constructed integro splines. The proposed family of splines has good approximation properties. The best convex integro spline is obtained when the α parameter is equal to $\frac{1}{2}$. The shape-preserving properties of splines are demonstrated by numerical examples.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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