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Research Article

# Translation Surfaces in the 3-Dimensional Pseudo-Galilean Space Satisfying: $\triangle^{\mathrm{II}} r_i = \lambda_i r_i$

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**Abstract.** In this paper, we classify translation surfaces in a 3-dimensional Pseudo-Galilean space $\mathbb{G}_3^1$ under the condition $\triangle^{\mathrm{II}} r_i = \lambda_i r_i$, where $r_i$ are the components of the position vector, $\lambda_i \in \mathbb{R}$, $(i = 1,2,3)$, and $\triangle^{\mathrm{II}}$ denotes the Laplace operator with respect to the second fundamental form.

**Keywords.** Pseudo-Galilean space; Surface of finite type; Translation surfaces; II-Harmonic; Laplacian operator with respect to the second fundamental form

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## 1. Introduction

There has been a long history of studying special classes of surfaces, as surfaces with particularly interesting properties, in Euclidean Geometry, e.g., ruled surfaces, translation surfaces, surfaces of revolution, sphere, helicoidal surfaces etc.

In this paper, we mainly study the translation surfaces in the Pseudo-Galilean space under the condition $\triangle^{\mathrm{II}} r_i = \lambda_i r_i$.

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In [4], Chen give this definition, whenever the position vector $r$ of $S$ in $\mathbb{E}^n$ can be decomposed as a finite sum of $\mathbb{E}^n$-valued non-constant eigenfunctions of $\triangle^{\mathrm{II}}$, we say that $S$ is a finite type. In other words, $S$ is said to be of $k$-type if the position vector $r$ of $S$ in $\mathbb{E}^n$ can be written in the following form:

$$r = r_0 + \sum_1^n r_i,$$

where $r_0$ is a constant vector, and $r_i$ $(i = 1,2,\dots,n)$ are non-constant $\mathbb{E}^n$-valued functions on $S$.

In [2], Bekkar and Zoubir classified the surfaces of revolution with non-zero Gaussian curvature $K_G$ in the three-dimensional Lorentz-Minkowski space $\mathbb{E}_1^3$, under the condition

$$\triangle r_i = \lambda_i r_i, \quad \lambda^i \in \mathbb{R}.$$

In [7] Kaimakamis and Papantoniou investigated the surfaces of revolution without parabolic points in $\mathbb{E}_1^3$ satisfying

$$\triangle^{\mathrm{II}} r = Ar, \quad A \in Mat(3,\mathbb{R}),$$

where $Mat(3,\mathbb{R})$ is the set of $3\times 3$ real matrices.

In [12], Yoon obtained some classification of translation surfaces in Galilean 3-space which satisfy the condition

$$\triangle r_i = \lambda_i r_i, \quad \lambda_i \in \mathbb{R}, (i = 1,2,3).$$

In [1], Baba-Hamed and Bekkar studied the helicoidal surfaces without parabolic points in $\mathbb{E}_1^3$, which satisfy the condition:

$$\triangle^{\mathrm{II}} r_i = \lambda_i r_i, \quad (i = 1,2,3).$$

In [8], Senoussi and Bekkar classified the helicoidal surfaces in $\mathbb{E}_1^3$ under the condition:

$$\triangle^{\mathrm{II}} r = Ar, \quad A \in Mat(3,\mathbb{R}).$$

## 2. Basic Notions and Properties

The Pseudo-Galilean space $\mathbb{G}_3^1$ is a Cayley-Klein space with absolute figure consisting of the ordered triple $\{w, f, I\}$ where $w$ is the ideal (absolute) plane in the real 3-dimensional projective space, $f$ is a line in $w$ and $I$ is the fixed hyperbolic involution of points of $f$.

The Pseudo-Galilean scalar product is defined by:

$$\langle X,Y\rangle = \begin{cases} uu', & \text{if } u \neq 0 \text{ r } u' \neq 0, \\ vv' - ww', & \text{if } u = 0 \text{ and } u' = 0. \end{cases}$$

where $X = (u,v,w)$ and $Y = (u',v',w')$.

Now, we can write the Pseudo-Galilean norm of the vector $X = (u,v,w)$ as follows:

$$||X|| = \begin{cases} u, & \text{if } u \neq 0, \\ \sqrt{|v^2 - w^2|}, & \text{if } u = 0. \end{cases}$$

A vector $X = (u,v,w)$ of $\mathbb{G}_3^1$ is called non-isotropic if $u \neq 0$. $X$ called unit non-isotropic vector if $X = (1,u,w)$. For isotropic vectors $u = 0$ holds. There are 4 types of isotropic vectors:

spacelike ($v^2 - w^2 > 0$), timelike ($v^2 - w^2 < 0$) and 2 types of lightlike ($v = w$), ($v = -w$) vectors. A non-lightlike isotropic vector is a unit vector if $v^2 - w^2 = \pm 1$.

We introduce a pseudo-Galilean cross-product of $X$ and $Y$ on $\mathbb{G}_3^1$ in the following manner:

$$X \times Y = \begin{vmatrix} 0 & -e_2 & e_3 \\ u & v & w \\ u' & v' & w' \end{vmatrix},$$

where $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ are the standard basis vectors in $\mathbb{G}_3^1$.

Let $S$ be a regular surface in $\mathbb{G}_3^1$ parametrized by

$$X(u_1,u_2) = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2)).$$

We denote

$$x_{,i} = \frac{\partial x}{\partial u_i}, \quad y_{,i} = \frac{\partial y}{\partial u_i}, \quad z_{,i} = \frac{\partial z}{\partial u_i}, \quad i = 1,2,$$

then a surface $S$ is admissible if and only if $x_{,i} \neq 0$, for some $i = 1,2$.

Let $S$ be an admissible surface in Pseudo-Galilean space $\mathbb{G}_3^1$. The unit normal vector field of $S$ is

$$N(u_1,u_2) = \frac{1}{W}(0, x_{,1}z_{,2} - x_{,2}z_{,1}, x_{,1}y_{,2} - x_{,2}y_{,1}),$$

where

$$W = \sqrt{|(x_{,1}z_{,2} - x_{,2}z_{,1})^2 - (x_{,1}y_{,2} - x_{,2}y_{,1})^2|}.$$

We notice that the normal vector field satisfies $\langle N,N \rangle = \varepsilon = \pm 1$, therefore we have two basic types of admissible surfaces, spacelike surfaces ($\varepsilon = -1$) and timelike surfaces ($\varepsilon = +1$).

The first fundamental form of a surface $S$ in $\mathbb{G}_3^1$ is defined by

$$ds^2 = (x_{,1}du_1 + x_{,2}du_2)^2 + \delta(\widetilde{X}_{,1}du_1 + \widetilde{X}_{,2}du_2),$$

where

$$\delta = \begin{cases} 0, & \text{if direction } du_1 : du_2 \text{ is non-isotropic,} \\ 1, & \text{if direction } du_1 : du_2 \text{ is isotropic.} \end{cases}$$

and $\widetilde{X}_{,i} = \frac{\partial \widetilde{X}}{\partial u_i}$, $i = 1,2$, by $\sim$ above a vector, the projection of a vector on the pseudo-Euclidean $yz$ plane is denoted.

The coefficients of the first fundamental form are introduced as

$$g_i = x_{,i}, \; h_{ij} = \langle \widetilde{X}_{,i}, \widetilde{X}_{,j} \rangle, \quad i,j = 1,2.$$

We can write the first fundamental form as

$$ds^2 = (g_1du_1 + g_2du_2)^2 + \delta(h_{11}du_1^2 + 2h_{12}du_1du_2 + h_{22}du_2^2).$$

Now, the function $W^2$ can be written as

$$W^2 = -\varepsilon(g_1X_{,2} - g_2X_{,1})^2 = -\varepsilon(g_1^2h_{22} - 2g_1g_2h_{12} + g_2^2h_{11}) > 0.$$

The coefficients $L_{ij}$ of the second fundamental form are introduced as

$$L_{ij} = \left\langle -\varepsilon\frac{W^2}{g_1^2}(g_1\widetilde{X}_{,i,j} - g_{i,j}\widetilde{X}_{,1}), N \right\rangle, \;\text{ if } g_1 \neq 0,$$

or

$$L_{ij} = \left\langle -\varepsilon \frac{W^2}{g_2^2}(g_2 \widetilde{X}_{i,j} - g_{,i,j}\widetilde{X}_{,2}), N \right\rangle, \text{ if } g_2 \neq 0,$$

where $i,j = 1,2$.

The Gaussian curvature $K$ and the mean curvature $H$ of $S$ are defined by

$$K = -\varepsilon \frac{L_{11}L_{22} - L_{12}^2}{W^2},$$

$$H = -\frac{\varepsilon}{2W^2}(g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}).$$

More about surfaces in $\mathbb{G}_3^1$ can be found in [10], [5] and [9].

It is common knowledge that in terms of local coordinates $\{u_1, u_2\}$ of $S$. The Laplacian operator with respect to the second fundamental form on $S$, $\Delta^{\mathrm{II}}$ is defined by

$$\Delta^{\mathrm{II}} = \frac{-1}{\sqrt{\mathscr{L}}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i}\left(\sqrt{\mathscr{L}} L^{ij} \frac{\partial}{\partial u_j}\right), \tag{1}$$

where $L_{ij}$ $(i,j = 1,2)$ are the components of the second fundamental form II on $S$, and $(L^{ij})$ (resp. $\mathscr{L}$) the inverse matrix (resp. the determinant of the matrix $(L_{ij})$).

**Definition 2.1.** A surface in the 3-dimensional Pseudo-Galilean space is said to be II-Harmonic if it satisfies the condition $\Delta^{\mathrm{II}} r_i = 0$, where $r_i$ are the components of the position vector field $r$.

## 3. Translation Surfaces in Pseudo-Galilean Space $\mathbb{G}_3^1$

The following is devoted to the classification of a non II-Harmonic translation surfaces with non-degenerate second fundamental form in Pseudo-Galilean space $\mathbb{G}_3^1$ satisfying:

$$\Delta^{\mathrm{II}} r_i = \lambda_i r_i, \quad \lambda_i \in \mathbb{R}, \ (i = 1,2,3), \tag{2}$$

where $\Delta^{\mathrm{II}}$ is the Laplacian operator with respect to the second fundamental form on $S$ in $\mathbb{G}_3^1$.

Let $S$ a translation surface in $\mathbb{G}_3^1$ that is obtained by translating two planar generating curves, that is a non-isotropic curve $\alpha(u) = (u, 0, \varphi(u))$ in plane $y = 0$ and an isotropic curve $\beta(v) = (0, v, \psi(v))$ in plane $x = 0$. Then, it is parametrized by:

$$S : r(u,v) = (u, v, \varphi(u) + \psi(v)), \tag{3}$$

where $\varphi(u)$ and $\psi(v)$ are smooth functions.

By the local coordinate system $\{u, v\}$, a simple calculation implies that

$$g_1 = 1,\ g_2 = 0,\ h_{11} = -\varphi'^2,\ h_{12} = -\varphi'\psi',\ h_{22} = 1 - \psi'^2;$$

$$L_{11} = \frac{-\varepsilon\varphi''}{W},\ L_{12} = 0,\ L_{22} = \frac{-\varepsilon\psi''}{W},$$

where we put $W^2 = -\varepsilon(1 - \psi'^2)$.

The Gaussian curvature $K$ is given by

$$K = \frac{-\varepsilon\varphi''\psi''}{(1 - \psi'^2)^2}, \quad \psi'(v) \neq \varepsilon,\ \forall\ v \in \Omega \subset \mathbb{R}.$$

Suppose that the surface has non zero Gaussian curvature, so

$$\varphi''\psi'' \neq 0, \quad \forall\, u, v \in \Omega \subset \mathbb{R}.$$

The Laplacian operator $\Delta^{\mathrm{II}}$ of $S$ can be expressed as:

$$\Delta^{\mathrm{II}} = \varepsilon W \left( \frac{1}{\varphi''} \frac{\partial^2}{\partial u^2} - \frac{\varphi'''}{2\varphi''^2} \frac{\partial}{\partial u} + \frac{1}{\psi''} \frac{\partial^2}{\partial v^2} - \frac{\psi'''}{2\psi''^2} \frac{\partial}{\partial v} \right). \tag{4}$$

Using (3) and (4), we get:

$$\begin{cases} \Delta^{\mathrm{II}} u = -\dfrac{-\varepsilon W \varphi'''}{2\varphi''^2}, \\ \Delta^{\mathrm{II}} v = -\dfrac{-\varepsilon W \psi'''}{2\psi''^2}, \\ \Delta^{\mathrm{II}} (\varphi + \psi) = 2\varepsilon W - \dfrac{\varepsilon W \varphi'''}{2\varphi''^2} \varphi' - \dfrac{\varepsilon W \psi'''}{2\psi''^2} \psi'. \end{cases} \tag{5}$$

As the surface $S$ has no parabolic points we must have $\varphi''\psi'' \neq 0$.

Combining (2) and (5) we get the following equations:

$$\frac{-\varepsilon W \varphi'''}{2\varphi''^2} = \lambda_1 u, \tag{6}$$

$$\frac{-\varepsilon W \psi'''}{2\psi''^2} = \lambda_2 v, \tag{7}$$

$$2\varepsilon W - \frac{\varepsilon W \varphi'''}{2\varphi''^2} \varphi' - \frac{\varepsilon W \psi'''}{2\psi''^2} \psi' = \lambda_3 (\varphi + \psi). \tag{8}$$

Substituting (6) and (7) into (8) implies that

$$\lambda_1 u \varphi' - \lambda_3 \varphi = \lambda_3 \psi - \lambda_2 v \psi' - 2\varepsilon \sqrt{-\varepsilon(1 - \psi'^2)}. \tag{9}$$

We will presently discuss eight cases according to constants $\lambda_i$ $(i = 1, 2, 3)$.

| | |
|---|---|
| **Case 1:** $\lambda_1 = \lambda_2 = \lambda_3 = 0$ | **Case 5:** $\lambda_1 \neq 0,\ \lambda_2 = \lambda_3 = 0$ |
| **Case 2:** $\lambda_1 = \lambda_3 = 0,\ \lambda_2 \neq 0$ | **Case 6:** $\lambda_1 \lambda_2 \neq 0,\ \lambda_3 = 0$ |
| **Case 3:** $\lambda_1 = \lambda_2 = 0,\ \lambda_3 \neq 0$ | **Case 7:** $\lambda_1 \lambda_3 \neq 0,\ \lambda_2 = 0$ |
| **Case 4:** $\lambda_1 = 0,\ \lambda_2 \lambda_3 \neq 0$ | **Case 8:** $\lambda_1 \lambda_2 \lambda_3 \neq 0$ |

From the above, we will examine two parts one of them spacelike surfaces and other timelike surfaces.

**Part 1:** In this part, we suppose that $S$ is spacelike surface i.e. $(\varepsilon = -1)$.

We have

$$\lambda_1 u \varphi' - \lambda_3 \varphi = \lambda_3 \psi - \lambda_2 v \psi' + 2\sqrt{1 - \psi'^2}. \tag{10}$$

In the Cases 1, 3, 4 and 5, we can easily prove that there are no non II-Harmonic translation surfaces with non-degenerate second fundamental form satisfying (2).

We will study the Cases 2, 6, 7 and 8.

**Case 2.** Let $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$, we obtain

$$-\lambda_2 v \psi' + 2\sqrt{1 - \psi'^2} = 0, \tag{11}$$

we get

$$\psi'^2 = \frac{4}{\lambda_2^2 v^2 + 4}, \tag{12}$$

this differential equation admits the solution

$$\psi(v) = \pm\frac{2}{\lambda_2}\ln\left|\sqrt{\lambda_2^2 v^2 + 4} + \lambda_2 v\right| + c, \quad c \in \mathbb{R}.$$

Here, the function $\varphi(u)$ is independent of the function $\psi(v)$.

In this case,the translation surfaces $S$ are given by

$$r(u,v) = \left(u, v, \varphi(u) \pm \frac{2}{\lambda_2}\ln\left|\sqrt{\lambda_2^2 v^2 + 4} + \lambda_2 v\right| + c\right), \quad c \in \mathbb{R}.$$

where $\varphi'' \neq 0$.

**Case 6.** Let $\lambda_1\lambda_2 \neq 0$, $\lambda_3 = 0$, we obtain

$$\lambda_1 u\varphi' = -\lambda_2 v\psi' + 2\sqrt{1-\psi'^2}$$

where $u$ and $v$ are independent variables. So, we can write

$$\lambda_1 u\varphi' = -\lambda_2 v\psi' + 2\sqrt{1-\psi'^2} = k, \tag{13}$$

where $k \in \mathbb{R}$.

For the function $\varphi(u)$, we get

$$\varphi(u) = \frac{k}{\lambda_1}\ln|u| + c_1, \quad c_1 \in \mathbb{R}, \tag{14}$$

and for the function $\psi(v)$, we get

$$\psi'(v) = -\frac{\lambda_2 k v}{\lambda_2^2 v^2 + 4} \pm 2\frac{\sqrt{\lambda_2^2 v^2 - k^2 + 4}}{\lambda_2^2 v^2 + 4}. \tag{15}$$

The function $\psi(v)$ is given by

$$\psi(v) = -\frac{k}{2\lambda_2}\ln(\lambda_2^2 v^2 + 4) \mp \frac{k}{\lambda_2}\ln\left|\frac{2\sqrt{\lambda_2^2 v^2 - k^2 + 4} - k\lambda_2 v}{2\sqrt{\lambda_2^2 v^2 - k^2 + 4} + k\lambda_2 v}\right| \pm \frac{2}{\lambda_2}\ln\left|\sqrt{\lambda_2^2 v^2 - k^2 + 4} + \lambda_2 v\right| + c_2. \tag{16}$$

In particular, if $k = 0$, then we have

$$\varphi(u) = c_1, \quad \psi(v) = \pm\frac{2}{\lambda_2}\ln\left|\sqrt{\lambda_2^2 v^2 + 4} + \lambda_2 v\right| + c_2 \tag{17}$$

so, $k \neq 0$.

In this case, the translation surfaces $S$ are given by

$$r(u,v) = \left(u, v, \frac{k}{\lambda_1}\ln|u| - \frac{k}{2\lambda_2}\ln(\lambda_2^2 v^2 + 4) \mp \frac{k}{2\lambda_2}\ln\left|\frac{2\sqrt{\lambda_2^2 v^2 - k^2 + 4} - k\lambda_2 v}{2\sqrt{\lambda_2^2 v^2 - k^2 + 4} + k\lambda_2 v}\right| \right.$$
$$\left. \pm \frac{2}{\lambda_2}\ln\left|\sqrt{\lambda_2^2 v^2 - k^2 + 4} + \lambda_2 v\right| + c\right),$$

where $c \in \mathbb{R}$ and $k \neq 0$.

**Case 7.** Let $\lambda_1\lambda_3 \neq 0$, $\lambda_2 = 0$, we obtain

$$\lambda_1 u\varphi' - \lambda_3\varphi = k = \lambda_3\psi + 2\sqrt{1-\psi'^2}. \tag{18}$$

We have

$$\lambda_1 u\varphi' - \lambda_3\varphi = k. \tag{19}$$

A calculation gives

$$\varphi(u) = \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} - \frac{k}{\lambda_3}, \quad c_1 \in \mathbb{R}^*, \tag{20}$$

where $\lambda_3 \neq \lambda_1$.

The equation (18) yields

$$\frac{\psi''}{\sqrt{1-\psi'^2}} = \frac{\lambda_3}{2}, \tag{21}$$

the function $\psi(v)$ is given by

$$\psi(v) = \frac{-2}{\lambda_3}\sin\left(\frac{\lambda_3}{2}v + c_2\right) + \frac{k}{\lambda_3}, \quad c_2 \in \mathbb{R}. \tag{22}$$

In this case, the translation surfaces $S$ are given by

$$r(u,v) = \left(u, v, \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} - \frac{2}{\lambda_3}\sin\left(\frac{\lambda_3}{2}v + c_2\right)\right), \quad c_1, c_2 \in \mathbb{R},$$

where $\lambda_3 \neq \lambda_1$.

**Case 8.** Let $\lambda_1\lambda_2\lambda_3 \neq 0$, we obtain

$$\lambda_1 u\varphi' - \lambda_3\varphi = \lambda_3\psi - \lambda_2 v\psi' + 2\sqrt{1-\psi'^2} = k. \tag{23}$$

A calculation implies that

$$\varphi(u) = \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} - \frac{k}{\lambda_3}, \quad c_1 \in \mathbb{R}^*. \tag{24}$$

For the function $\psi(v)$, we have

$$\lambda_3\psi = \lambda_2 v\psi' - 2\sqrt{1-\psi'^2} + k. \tag{25}$$

Differentiating this equation with respect to $v$ we obtain

$$(\lambda_3 - \lambda_2)\psi' - \lambda_2 v\psi'' = \frac{2\psi'\psi''}{\sqrt{1-\psi'^2}}, \tag{26}$$

- If $\lambda_2 = \lambda_3 = \mu$, thus (26) becomes

$$\mu v = \frac{-2\psi'}{\sqrt{1-\psi'^2}}, \tag{27}$$

which give $\psi(v) = -\frac{1}{\mu}\sqrt{\mu^2 v^2 + 4} + \frac{k}{\mu}$.

- If $\lambda_2 \neq \lambda_3$, putting $\psi' = \frac{d\psi}{dv} = t$, we get

$$(\lambda_3 - \lambda_2)t\frac{dv}{dt} - \lambda_2 v = \frac{2t}{\sqrt{1-t^2}}. \tag{28}$$

The general solution of this equation is given by

$$v(t) = \frac{2t^{\frac{\lambda_2}{\lambda_3-\lambda_2}}}{\lambda_3-\lambda_2}\int \frac{t^{\frac{\lambda_2}{\lambda_2-\lambda_3}}}{\sqrt{1-t^2}}dt. \tag{29}$$

So, $\psi(t)$ is given by

$$\psi(t) = \frac{\lambda_2}{\lambda_3}v(t)t - \frac{2}{\lambda_3}\sqrt{1-t^2} + \frac{k}{\lambda_3}. \tag{30}$$

Combining the equation (29) and (30) we have

$$\psi(t) = \frac{2\lambda_2}{\lambda_3(\lambda_3-\lambda_2)}t^{\frac{\lambda_3}{\lambda_3-\lambda_2}}\int \frac{t^{\frac{\lambda_2}{\lambda_2-\lambda_3}}}{\sqrt{1-t^2}}dt - \frac{2}{\lambda_3}\sqrt{1-t^2} + \frac{k}{\lambda_3}.$$

**Remark 3.1.**

$$\int \frac{t^{\frac{\lambda_2}{\lambda_2-\lambda_3}}}{\sqrt{1-t^2}}dt = \frac{(\lambda_2-\lambda_3)t^{\frac{\lambda_2}{\lambda_2-\lambda_3}+1}\ {}_2F_1\left(1/2, \frac{2\lambda_2-\lambda_3}{2\lambda_2-2\lambda_3}; \frac{4\lambda_2-3\lambda_3}{2\lambda_2-2\lambda_3}; t^2\right)}{2\lambda_2-\lambda_3} + c, \quad c \in \mathbb{R},$$

where ${}_2F_1\left(1/2, \frac{2\lambda_2-\lambda_3}{2\lambda_2-2\lambda_3}; \frac{4\lambda_2-3\lambda_3}{2\lambda_2-2\lambda_3}; t^2\right)$ is the hypergeometric function (see [6]).

Thus, the translation surfaces $S$ in this case are given by

$$r(u,v) = \left(u, v, \frac{\lambda_1}{\mu}cu^{\frac{\mu}{\lambda_1}} - \frac{1}{\mu}\sqrt{\mu^2v^2+4}\right), \quad c \in \mathbb{R} \text{ and } \mu \in \mathbb{R}^*,$$

where $\mu = \lambda_2 = \lambda_3$ and $\lambda_1 \neq \mu$, $\mu\lambda_1 \neq 0$, or

$$r(u,v) = \left(u, v(t), \frac{\lambda_1}{\lambda_3}cu^{\frac{\lambda_3}{\lambda_1}} + \frac{2\lambda_2}{\lambda_3(\lambda_3-\lambda_2)}t^{\frac{\lambda_3}{\lambda_3-\lambda_2}}\int \frac{t^{\frac{\lambda_2}{\lambda_2-\lambda_3}}}{\sqrt{1-t^2}}dt - \frac{2}{\lambda_3}\sqrt{1-t^2}\right),$$

where $c \in \mathbb{R}$ and $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$.

**Part 2:** In this part, we suppose that $S$ is timelike surface i.e. ($\varepsilon = 1$).

We have

$$\lambda_1 u\varphi' - \lambda_3\varphi = \lambda_3\psi - \lambda_2 v\psi' - 2\sqrt{\psi'^2 - 1}. \tag{31}$$

By using the same methods as in Part 1, we obtain:

In Cases 1, 3, 4, 5 and 6, we can easily prove that there are no non II-Harmonic translation surfaces with non-degenerate second fundamental form satisfying (2).

We will study only the Cases 2, 7 and 8.

**Case 2.** Let $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$, we obtain

$$\lambda_2 v\psi' + 2\sqrt{\psi'^2 - 1} = 0, \tag{32}$$

we get

$$\psi'^2 = \frac{4}{4-\lambda_2^2 v^2}, \tag{33}$$

this differential equation admits the solution

$$\psi(v) = -\frac{2}{\lambda_2}\arcsin\left(\frac{\lambda_2 v}{2}\right) + c, \quad c \in \mathbb{R}.$$

So, the translation surfaces $S$ in this case are given by

$$r(u,v) = \left(u, v, \varphi(u) - \frac{2}{\lambda_2}\arcsin\left(\frac{\lambda_2 v}{2}\right) + c\right), \quad c \in \mathbb{R},$$

where $\varphi'' \neq 0$.

**Case 7.** Let $\lambda_1\lambda_3 \neq 0$, $\lambda_2 = 0$, we obtain

$$\lambda_1 u\varphi' - \lambda_3\varphi = k = \lambda_3\psi - 2\sqrt{\psi'^2 - 1}, \tag{34}$$

we get

$$\varphi(u) = \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} - \frac{k}{\lambda_3}, \quad c_1 \in \mathbb{R}^*, \tag{35}$$

where $\lambda_3 \neq \lambda_1$, and for the function $\psi(v)$, we get

$$\frac{\psi''}{\sqrt{\psi'^2 - 1}} = \frac{\lambda_3}{2}, \tag{36}$$

this equation give

$$\psi(v) = \frac{2}{\lambda_3}\sinh\left(\frac{\lambda_3}{2}v + c\right) + \frac{k}{\lambda_3}, \quad c \in \mathbb{R}. \tag{37}$$

Thus, the translation surfaces $S$ in this case are given by

$$r(u,v) = \left(u, v, \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} + \frac{2}{\lambda_3}\sinh\left(\frac{\lambda_3}{2}v + c_2\right)\right), \quad c_1, c_2 \in \mathbb{R},$$

where $\lambda_3 \neq \lambda_1$.

**Case 8.** Let $\lambda_1\lambda_2\lambda_3 \neq 0$, we obtain

$$\lambda_1 u\varphi' - \lambda_3\varphi = k = \lambda_3\psi - \lambda_2 v\psi' - 2\sqrt{\psi'^2 - 1}. \tag{38}$$

For the function $\varphi(u)$, we have the solution

$$\varphi(u) = \frac{\lambda_1}{\lambda_3} c u^{\frac{\lambda_3}{\lambda_1}} - \frac{k}{\lambda_3}, \quad c \in \mathbb{R}^*. \tag{39}$$

where $\lambda_1 \neq \lambda_3$.

By using the similar method of (Case 8 of Part 1), we obtain

- If $\lambda_2 = \lambda_3 = \mu$, we get

$$\psi(v) = -\frac{1}{\mu}\sqrt{\mu^2 v^2 - 4} + \frac{k}{\mu}.$$

- If $\lambda_2 \neq \lambda_3$, we get

$$\psi(t) = \frac{2\lambda_2}{\lambda_3}\frac{t^{\frac{\lambda_3}{\lambda_3 - \lambda_2}}}{\lambda_3 - \lambda_2}\int \frac{t^{\frac{\lambda_2}{\lambda_2 - \lambda_3}}}{\sqrt{t^2 - 1}}dt + \frac{2}{\lambda_3}\sqrt{t^2 - 1} + \frac{k}{\lambda_3}. \tag{40}$$

Thus, the translation surfaces $S$ in this case are given by

$$r(u,v) = \left(u, v, \frac{\lambda_1}{\mu} c u^{\frac{\mu}{\lambda_1}} - \frac{1}{\mu}\sqrt{\mu^2 v^2 + 4}\right), \quad c \in \mathbb{R} \text{ and } \mu \in \mathbb{R}^*,$$

where $\mu = \lambda_2 = \lambda_3$, $\lambda_1 \neq \mu$, or

$$r(u,v) = \left(u, v(t), \frac{\lambda_1}{\lambda_3} c u^{\frac{\lambda_3}{\lambda_1}} + \frac{2\lambda_2}{\lambda_3(\lambda_3 - \lambda_2)} t^{\frac{\lambda_3}{\lambda_3 - \lambda_2}} \int \frac{t^{\frac{\lambda_2}{\lambda_2 - \lambda_3}}}{\sqrt{t^2 - 1}}dt - \frac{2}{\lambda_3}\sqrt{t^2 - 1}\right), \quad c \in \mathbb{R},$$

where $c \in \mathbb{R}$, $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$.

In light of the above results, we will introduce the following theorem:

**Theorem 3.1.** *Let $S$ be a non* II*-Harmonic translation surface with non-degenerate second fundamental form given by* (3) *in $\mathbb{G}_3^1$. If the surface $S$ satisfies the condition $\Delta^{\mathrm{II}} r_i = \lambda_i r$, where $\lambda_i \in \mathbb{R}$, $(i = 1,2,3)$, then it is congruent to open part of the surfaces.*

1. $r(u,v) = \left(u, v, \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} - \frac{2}{\lambda_3} \sin\left(\frac{\lambda_3}{2} v + c_2\right)\right)$, $c_1, c_2 \in \mathbb{R}$, *where* $\lambda_3 \neq \lambda_1$, $\lambda_1\lambda_3 \neq 0$*;*
2. $r(u,v) = \left(u, v, \varphi(u) - \frac{2}{\lambda_2} \arcsin\left(\frac{\lambda_2 v}{2}\right) + c\right)$, $c \in \mathbb{R}$, *where* $\lambda_2 \neq 0$, $\varphi'' \neq 0$*;*
3. $r(u,v) = \left(u, v, \frac{\lambda_1}{\lambda_3} c_1 u^{\frac{\lambda_3}{\lambda_1}} + \frac{2}{\lambda_3} \sinh\left(\frac{\lambda_3}{2} v + c_2\right)\right)$, $c_1, c_2 \in \mathbb{R}$, *where* $\lambda_3 \neq \lambda_1$ *and* $\lambda_1\lambda_3 \neq 0$*;*
4. $r(u,v) = \left(u, v, \frac{\lambda_1}{\mu} c u^{\frac{\mu}{\lambda_1}} - \frac{1}{\mu}\sqrt{\mu^2 v^2 + 4}\right)$, $c \in \mathbb{R}$ *and* $\mu \in \mathbb{R}^*$, *where* $\mu = \lambda_2 = \lambda_3$, $\lambda_1 \neq \mu$ *and* $\mu\lambda_1 \neq 0$*;*
5. $r(u,v) = \left(u, v, \frac{\lambda_1}{\mu} c u^{\frac{\mu}{\lambda_1}} - \frac{1}{\mu}\sqrt{\mu^2 v^2 - 4}\right)$, $c \in \mathbb{R}$ *and* $\mu \in \mathbb{R}^*$ *where* $\mu = \lambda_2 = \lambda_3 \neq \lambda_1$ *and* $\lambda_1\mu \neq 0$*;*
6. $r(u,v) = \left(u, v, \varphi(u) \pm \frac{2}{\lambda_2} \ln\left|\sqrt{\lambda_2^2 v^2 + 4} + \lambda_2 v\right| + c\right)$, $c \in \mathbb{R}$ *where* $\varphi'' \neq 0$, $\lambda_2 \neq 0$*;*
7. $$r(u,v) = \left(u, v, \frac{k}{\lambda_1} \ln|u| - \frac{k}{2\lambda_2} \ln(\lambda_2^2 v^2 + 4) \mp \frac{k}{\lambda_2} \ln\left|\frac{2\sqrt{\lambda_2^2 v^2 - k^2 + 4} - k\lambda_2 v}{2\sqrt{\lambda_2^2 v^2 - k^2 + 4} + k\lambda_2 v}\right|\right.$$
$$\left. \pm \frac{2}{\lambda_2} \ln\left|\sqrt{\lambda_2^2 v^2 - k^2 + 4} + \lambda_2 v\right| + c\right),\ c \in \mathbb{R},$$
8. $$r(u,v) = \left(u, v(t), \frac{\lambda_1}{\lambda_3} c u^{\frac{\lambda_3}{\lambda_1}} + \frac{2\lambda_2}{\lambda_3(\lambda_3 - \lambda_2)} t^{\frac{\lambda_3}{\lambda_3 - \lambda_2}} \int \frac{t^{\frac{\lambda_2}{\lambda_2 - \lambda_3}}}{\sqrt{1 - t^2}} dt - \frac{2}{\lambda_3}\sqrt{1 - t^2}\right),\ c \in \mathbb{R},$$
*where* $v(t) = \frac{2t^{\frac{\lambda_2}{\lambda_3 - \lambda_2}}}{\lambda_3 - \lambda_2} \int \frac{t^{\frac{\lambda_2}{\lambda_2 - \lambda_3}}}{\sqrt{1 - t^2}} dt$ *and* $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$, $\lambda_1\lambda_2\lambda_3 \neq 0$,
9. $$r(u,v) = \left(u, v(t), \frac{\lambda_1}{\lambda_3} c u^{\frac{\lambda_3}{\lambda_1}} + \frac{2\lambda_2}{\lambda_3(\lambda_3 - \lambda_2)} t^{\frac{\lambda_3}{\lambda_3 - \lambda_2}} \int \frac{t^{\frac{\lambda_2}{\lambda_2 - \lambda_3}}}{\sqrt{t^2 - 1}} dt - \frac{2}{\lambda_3}\sqrt{t^2 - 1}\right),\ c \in \mathbb{R},$$
*where* $v(t) = \frac{2t^{\frac{\lambda_2}{\lambda_3 - \lambda_2}}}{\lambda_3 - \lambda_2} \int \frac{t^{\frac{\lambda_2}{\lambda_2 - \lambda_3}}}{\sqrt{t^2 - 1}} dt$ *and* $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$, $\lambda_1\lambda_2\lambda_3 \neq 0$.

## 4. Conclusion

In a three-dimensional Pseudo-Galilean space $\mathbb{G}_3^1$, the translation surfaces $S$ which are non II-Harmonic, with non-degenerate second fundamental form and given by the parametrisation (3). If these surfaces satisfy the condition (2), they are congruent to open part of one of the surfaces given in Theorem 3.1.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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