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Research Article

Approximate Solution of Time-Fractional Helmholtz and Coupled Helmholtz Equations

Mine Aylin Bayrak* and Semiha Tetik

Department of Mathematics, Kocaeli University, Kocaeli, Turkey
***Corresponding author:** aylin@kocaeli.edu.tr

Abstract. The approximate analytical solution of the time-fractional Helmholtz and coupled Helmholtz equations have been acquired successfully via *Residual Power Series Method* (RPSM). The approximate solutions obtained by RPSM are compared with the exact solutions through different graphics and tables. The fractional derivatives are described in the Caputo sense. The numerical results demonstrate that the new method is quite accurate and readily implemented.

Keywords. Residual power series method; Time-fractional Helmholtz equation; Caputo derivative; Mittag-Leffler function

MSC. 35K35; 26A33; 35F25

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1. Introduction

In recent years, fractional calculus has found countless applications in different branches of engineering and science such as *Fractional Differential Equations* (FDE), fluid flow, electrical network, mathematical physics, biology, image and signal processing, visco-elasticity and control [3, 4, 8, 15, 22–24, 27].

There are some common methods that are used to obtain approximate or analytical solutions of fractional partial differential equations in literature. *Adomian Decomposition Method* (ADM), *Laplace Analysis Method* (LAM), *Homotopy Analysis Method* (HAM), *Homotopy Perturbation Method* (HPM), *Differential Transformation Method* (DTM) and *Perturbation-Iteration Algorithm* (PIA) are among them [1, 2, 6, 11, 13, 14, 16–21, 25, 26, 28, 29].

The Helmholtz equation in two dimensional case was suggested as follows:

$$D_x^{\alpha}u+D_y^{\alpha}u+w^{\alpha}u=f(x,y), \tag{1.1}$$

with the initial conditions

$$u(0,y)=\mu_1(y),\ \ u_x(0,y)=\mu_2(y), \tag{1.2}$$

where $f(x,y)$ is the known source function and $1<\alpha\le 2$. While coupled Helmholtz equations in two dimensional case was introduced as follows:

$$D_x^{2\alpha}u+D_{2y}^{\alpha}v+w_1^{2\alpha}u=f_1(x,y), \tag{1.3}$$

$$D_x^{2\alpha}v+D_{2y}^{\alpha}u+w_2^{2\alpha}v=f_2(x,y), \tag{1.4}$$

with the initial conditions

$$u(0,y)=\mu_1(y),\ \ D_x^{\alpha}u(0,y)=\mu_2(y), \tag{1.5}$$

$$v(0,y)=\nu_1(y),\ \ D_x^{\alpha}v(0,y)=\nu_2(y), \tag{1.6}$$

where $f_1(x,y)$, $f_2(x,y)$ are the known source functions and $0<\alpha\le 1$. In this article, a new technique, namely, *Residual Power Series Method* (RPSM), is used to obtain approximate solution of time-fractional Helmholtz equation. In this method, the coefficients of the power series are calculated by means of the concept of residual error with the help of one or more variable algebraic equation chains, and finally, in practice, a so-called truncated series solution is obtained.

The main advantage of this method over other methods is that it can be applied directly to the problem without linearization, perturbation or discretization and without any transformation by selecting appropriate initial conditions.

2. Preliminaries

The fundamentals for fractional calculus theory are given [23].

Definition 1. The Riemann-Liouville fractional integral of order α ($\alpha\ge 0$) is given as

$$J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\quad \alpha>0,\ x>0, \tag{2.1}$$

$$J^0f(x)=f(x). \tag{2.2}$$

Definition 2. The Caputo fractional derivative with order α is given as

$$D^{\alpha}f(x)=J^{m-\alpha}D^mf(x)=\int_0^x (x-t)^{m-\alpha-1}\frac{d^m}{dt^m}f(t)dt,\quad m-1<\alpha<m,\ x>0, \tag{2.3}$$

where D^m is the classic differential operator with order m.

Definition 3. The Caputo's time-fractional derivative of order α of $u(x,t)$ is defined as

$$D_t^{\alpha}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\xi)^{m-\alpha-1}\frac{\partial^m u(x,\xi)}{\partial t^m}d\xi, & m-1<\alpha<m,\\[2ex] \dfrac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\in N.\end{cases} \tag{2.4}$$

Definition 4. A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \dots, \quad 0 \le m-1 < \alpha \le m,\ t \ge t_0 \tag{2.5}$$

is called fractional power series about $t = t_0$ [1].

The power series expansions about $t = t_0$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{m-1} f_k(x)(t-t_0)^{k\alpha+l}, \quad 0 \le m-1 < \alpha \le m,\ t \ge t_0 \tag{2.6}$$

are called multiple fractional power series, where $f_k(x)$ is called the coefficients of the series.

Definition 5. The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by [19]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in C. \tag{2.7}$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ generalizes the exponential function e^z in that $E_{1,1}(z) = e^z$. It is an entire function in z with order $\frac{1}{\alpha}$ and type one.

3. Basic Idea of RPSM

To give the approximate solution of nonlinear fractional order differential equations by means of the RPSM [1,5,7,9,10], we consider a general nonlinear fractional differential equation:

$$D^{\alpha} u = N(u) + R(u), \tag{3.1}$$

where $N(u)$ is nonlinear term and $R(u)$ is a linear term. Subject to the initial condition

$$u(x,0) = f(x). \tag{3.2}$$

The RPSM proposes the solution for (3.1) as a fractional power series about the initial point $y = 0$

$$u(x,y) = \sum_{n=0}^{\infty} f_n(x) \frac{y^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \le 1,\ -\infty < x < \infty,\ 0 \le y < R. \tag{3.3}$$

Next, we let $u_k(x,y)$ denote the kth truncated series of $u(x,y)$ as follows:

$$u_k(x,y) = \sum_{n=0}^{k} f_n(x) \frac{y^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{3.4}$$

The 0th RPS approximate solution of $u(x,y)$ is used in the following form

$$u_0(x,y) = u(x,0) = f(x). \tag{3.5}$$

Equation (3.4) can be written as

$$u_k(x,y) = f(x) + \sum_{n=1}^{k} f_n(x) \frac{y^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 1,2,3,\dots. \tag{3.6}$$

We define the residual function for (3.1)

$$Res_u(x,y) = D_y^{\alpha} u - N(u) - R(u). \tag{3.7}$$

Therefore, the k^{th} residual function $Res_{u,k}$ is

$$Res_{u_k}(x,y) = D_y^{\alpha} u_k - N(u_k) - R(u_k). \tag{3.8}$$

As in [1, 23], $Res_{u,k} = 0$ and $\lim_{k\to\infty} Res_k(x,y) = Res(x,y)$. Therefore, $D_y^{n\alpha} Res(x,y) = 0$ since the fractional derivative of a constant in the Caputo sense is zero and the fractional derivatives $D_y^{n\alpha}$ of $Res(x,y)$ and $Res_k(x,y)$ are matching at $y = 0$ for each $n = 0,1,2,\dots$; that is, $D_y^{n\alpha} Res(x,0) = D_y^{n\alpha} Res(x_k,0) = 0$, $n = 0,1,2,\dots$.

To determine $f_1(x), f_2(x), f_3(x),\dots$ we consider $k = 1,2,3,\dots$ in (3.6) and substitute it into (3.8), applying the fractional derivative $D_y^{(k-1)\alpha}$ in both sides $k = 1,2,3,\dots$ and finally we solve in the following form:

$$D_y^{(k-1)\alpha} Res_{u_k}(x,0) = 0, \quad k = 1,2,3,\dots. \tag{3.9}$$

In the case of unknown exact solution, for the accuracy and comparison purposes of RPSM, absolute and relative errors are respectively given as follows:

$$\Delta_k = |u_{k+1}(x,y) - u_k(x,y)| \tag{3.10}$$

and

$$\delta_k = \frac{|u_{k+1}(x,y) - u_k(x,y)|}{|u_{k+1}(x,y)|}. \tag{3.11}$$

4. Numerical Examples

To illustrate the basic idea of RPSM, we consider the following time-fractional Helmholtz and coupled Helmholtz equations [12].

Example 1. Consider the time-fractional homogeneous Helmholtz equation

$$D_x^\alpha u + D_y^\alpha u - u = 0, \quad 1 < \alpha \le 2 \tag{4.1}$$

subject to the boundary condition

$$u(0,y) = y, u_x(0,y) = y + \cosh(y). \tag{4.2}$$

We define the residual function for the kth residual function as follows:

$$Res_{u_k}(x,y) = D_x^\alpha u_k + D_y^\alpha u_k - u_k. \tag{4.3}$$

Applying to the 0th RPS approximate solution of $u(x,y)$, we have in the following form

$$u_0(x,y) = u(x,0) = u(0,y) + xu_x(0,y) = y(1+x) + x\cosh(y) = f_0(y). \tag{4.4}$$

To determine $f_1(x)$, $u_1(x,y)$ and $Res_{u_1}(x,y)$ are constructed as follows:

$$u_1(x,y) = f_0(y) + f_1(y)\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{4.5}$$

and

$$Res_{u_1}(x,y) = f_1(y) + D_y^\alpha f_0(y) + D_y^\alpha f_1(y)\frac{x^\alpha}{\Gamma(1+\alpha)} - f_0(y) - f_1(y)\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{4.6}$$

From (3.6) and the properties of Caputo derivative, (4.6) leads to the following

$$f_1(y) = y. \tag{4.7}$$

To determine $f_2(x)$, $u_2(x,y)$ is constructed as follows

$$u_2(x,y) = f_0(y) + f_1(y)\frac{x^\alpha}{\Gamma(1+\alpha)} + f_2(y)\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}. \tag{4.8}$$

From (3.9) and the properties of Caputo derivative, we obtain

$$f_2(y) = y. \tag{4.9}$$

Using (3.4), we have

$$u(x,y) = y(1+x) + x\cosh(y) + y\frac{x^{\alpha}}{\Gamma(1+\alpha)} + y\frac{x^{\alpha+1}}{\Gamma(2+\alpha)} + y\frac{x^{\alpha+2}}{\Gamma(3+\alpha)} + \dots. \tag{4.10}$$

It can be easily observe that using (3.3) leads immediately to the solution of (3.10) given by

$$u(x,y) = x\cosh(y) + yE_{\alpha,1}(x^{\alpha}). \tag{4.11}$$

It is clear from Figure 1-3, for different orders of fractional derivatives that the approximate solutions of gives better results for small values x and t. As it can be seen from Table 1-3 the absolute error of the approximate solution gets closer to the exact solution as fractional order α increases to 2 for 3rd RPS approximate solution.

Table 1. Absolute and relative errors of Example 1 with exact solution for $\alpha = 2$

k	x	y	Exact	Δ_k	δ_k
0	0.4	0.4	1.02916	3.20e-02	3.02e-02
		0.8	1.72843	6.40e-02	3.57e-02
	0.8	0.4	1.75491	1.28e-01	6.80e-02
		0.8	2.85005	2.56e-01	8.24e-02
1	0.4	0.4	1.02916	4.27e-03	4.00e-03
		0.8	1.72843	8.53e-03	4.74e-03
	0.8	0.4	1.75491	3.41e-02	1.78e-02
		0.8	2.85005	6.83e-02	2.15e-02
2	0.4	0.4	1.02916	4.27e-04	4.00e-04
		0.8	1.72843	8.53e-04	4.74e-04
	0.8	0.4	1.75491	6.83e-03	3.55e-03
		0.8	2.85005	1.37e-02	4.28e-03

Table 2. Absolute and relative errors of Example 1 with exact solution for $\alpha = 1.8$

k	x	y	Exact	Δ_k	δ_k
0	0.4	0.4	1.09192	4.59e-02	4.03e-02
		0.8	1.85395	9.17e-02	4.71e-02
	0.8	0.4	1.84061	1.60e-01	7.98e-02
		0.8	3.02145	3.19e-01	9.56e-02
1	0.4	0.4	1.09192	6.55e-03	5.72e-03
		0.8	1.85395	1.31e-02	6.69e-03
	0.8	0.4	1.84061	4.56e-02	2.23e-02
		0.8	3.02145	9.12e-02	2.66e-02
2	0.4	0.4	1.09192	6.90e-04	6.02e-04
		0.8	1.85395	1.38e-03	7.04e-04
	0.8	0.4	1.84061	9.60e-03	4.67e-03
		0.8	3.02145	1.92e-02	5.57e-03

Table 3. Absolute and relative errors of Example 1 with exact solution for $\alpha = 1.6$

k	x	y	Exact	Δ_k	δ_k
0	0.4	0.4	1.16649	6.46e-02	5.25e-02
		0.8	2.00310	1.29e-01	6.06e-02
	0.8	0.4	1.92817	1.96e-01	9.22e-02
		0.8	3.19658	3.92e-01	1.09e-01
1	0.4	0.4	1.16649	9.94e-03	8.01e-03
		0.8	2.00310	1.99e-02	9.23e-03
	0.8	0.4	1.92817	6.02e-02	2.76e-02
		0.8	3.19658	1.20e-01	3.25e-02
2	0.4	0.4	1.16649	1.10e-03	8.89e-04
		0.8	2.00310	2.21e-03	1.02e-03
	0.8	0.4	1.92817	1.34e-02	6.09e-03
		0.8	3.19658	2.68e-02	7.17e-02

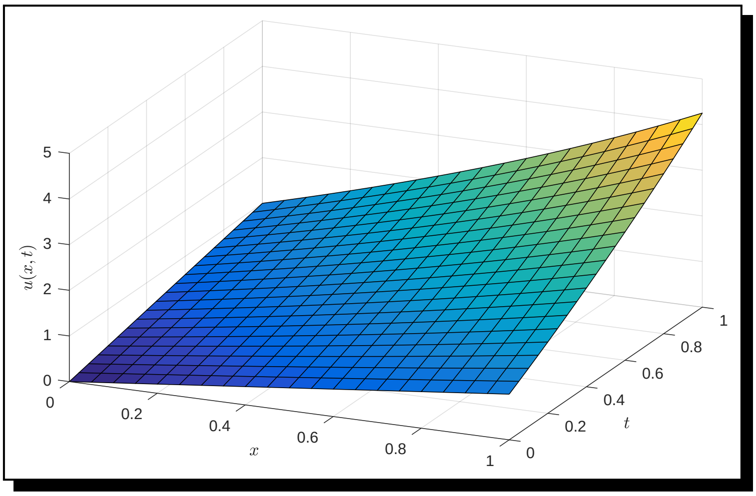

Figure 1. The approximate solution when $\alpha = 2$ of Example 1

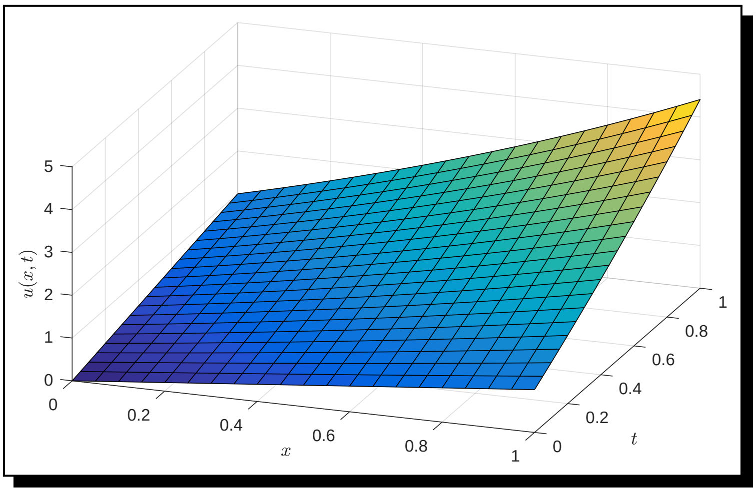

Figure 2. The approximate solution when $\alpha = 1.8$ of Example 1

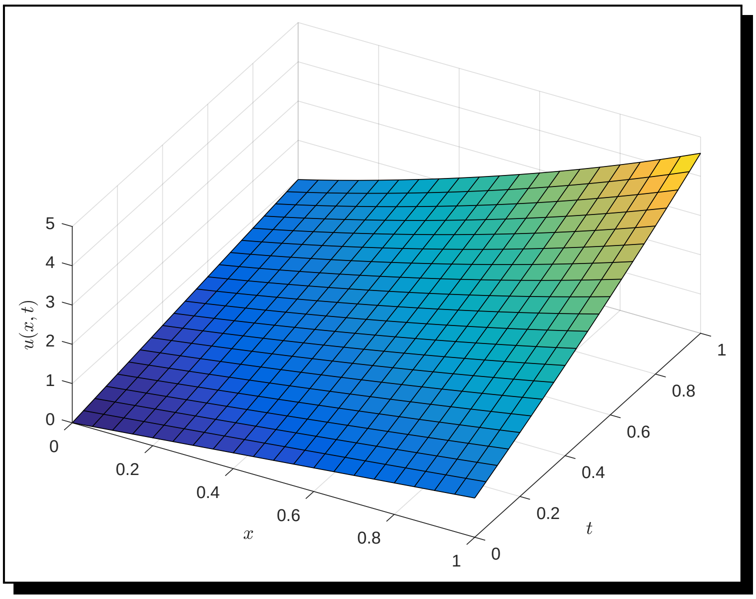

Figure 3. The approximate solution when $\alpha = 1.6$ of Example 1

Example 2. Consider the time-fractional homogeneous coupled Helmholtz equation

$$D_x^{2\alpha}u + D_y^{2\alpha}v - u = 0, \tag{4.12}$$

$$D_x^{2\alpha}v + D_y^{2\alpha}u - v = 0, \tag{4.13}$$

subject to the boundary condition

$$u(0,y) = 0, \ \ D_x^{\alpha}u(0,y) = E_{\alpha,1}(y^{\alpha}), \tag{4.14}$$

$$v(0,y) = 0, \ \ D_x^{\alpha}v(0,y) = -E_{\alpha,1}(y^{\alpha}), \tag{4.15}$$

where $0 < \alpha \le 1$. Applying to the 0^{th} RPS approximate solution of $u(x,y)$ we have

$$u_0(x,y) = u(x,0) = u(0,y) + \frac{x^{\alpha}}{\Gamma(1+\alpha)}u_x(0,y) = E_{\alpha,1}(y^{\alpha})\frac{x^{\alpha}}{\Gamma(1+\alpha)} = f_0(y), \tag{4.16}$$

$$v_0(x,y) = v(x,0) = v(0,y) + \frac{x^{\alpha}}{\Gamma(1+\alpha)}v_x(0,y) = -E_{\alpha,1}(y^{\alpha})\frac{x^{\alpha}}{\Gamma(1+\alpha)} = g_0(y). \tag{4.17}$$

To determine $f_1(x)$ and $g_1(x)$, we consider

$$u_1(x,y) = f_0(y) + f_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{4.18}$$

$$v_1(x,y) = g_0(y) + g_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{4.19}$$

and

$$Res_{u_1}(x,y) = f_1(y) + D_y^{2\alpha}g_0(y) + D_y^{2\alpha}g_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} - f_0(y) - f_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{4.20}$$

$$Res_{v_1}(x,y) = g_1(y) + D_y^{2\alpha}f_0(y) + D_y^{2\alpha}f_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} - g_0(y) - g_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{4.21}$$

From (3.7) and the properties of Caputo derivative we obtain

$$f_1(y) = 0, \tag{4.22}$$

$$g_1(y) = 0. \tag{4.23}$$

To determine $f_2(x)$ and $g_2(x)$, we consider

$$u_1(x,y) = f_0(y) + f_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + f_2(y)\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{4.24}$$

$$v_1(x,y) = g_0(y) + g_1(y)\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + g_2(y)\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}. \tag{4.25}$$

From (3.9) and the properties of Caputo derivative we get

$$f_2(x) = 2E_{\alpha,1}(y^\alpha), \tag{4.26}$$

$$g_2(x) = -2E_{\alpha,1}(y^\alpha). \tag{4.27}$$

Using (2.5) leads immediately to the solution of (3.10) given by

$$u(x,y) = E_{\alpha,1}(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + 2\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \dots\right) = E_{\alpha,1}(y^\alpha)\frac{\sinh_\alpha(\sqrt{2}x^\alpha)}{\sqrt{2}}, \tag{4.28}$$

$$v(x,y) = -E_{\alpha,1}(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + 2\frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \dots\right) = -E_{\alpha,1}(y^\alpha)\frac{\sinh_\alpha(\sqrt{2}x^\alpha)}{\sqrt{2}}. \tag{4.29}$$

It is clear from Figure 4-6 for different orders of fractional derivatives that the approximate solutions of gives better results for small values x and t. As it can be seen from Table 4-6 the absolute error of the approximate solution gets closer to the exact solution as fractional order α increases to 1 for 3rd RPS approximate solution.

Table 4. Absolute and relative errors of Example 2 with exact solution for $\alpha = 1$

k	x	y	Exact u	Exact v	Δ_k	δ_k
0	0.4	0.4	0.62903	−0.62903	3.18e-02	5.06e-02
		0.8	0.93714	−0.93714	4.74e-02	5.06e-02
	0.8	0.4	1.46478	−1.46478	2.55e-01	1.76e-01
		0.8	2.18224	−2.18224	3.79e-01	1.76e-01
1	0.4	0.4	0.62903	−0.62903	5.09e-04	8.10e-04
		0.8	0.93714	−0.93714	7.59e-04	8.10e-04
	0.8	0.4	1.46478	−1.46478	1.63e-02	1.12e-02
		0.8	2.18224	−2.18224	2.43e-02	1.12e-02
2	0.4	0.4	0.62903	−0.62903	3.88e-06	6.17e-06
		0.8	0.93714	−0.93714	5.78e-06	6.17e-06
	0.8	0.4	1.46478	−1.46478	4.97e-04	3.40e-04
		0.8	2.18224	−2.18224	7.40e-04	3.41e-04

Table 5. Absolute and relative errors of Example 2 with exact solution for $\alpha = 0.8$

k	x	y	Exact u	Exact v	Δ_k	δ_k
0	0.4	0.4	0.89216	−0.89216	1.28e-01	1.26e-01
		0.8	1.37193	−1.37193	1.97e-01	1.26e-01
	0.8	0.4	1.80012	−1.80012	6.76e-01	3.04e-01
		0.8	2.76817	−2.76817	1.04e-01	3.04e-01
1	0.4	0.4	0.89216	−0.89216	7.34e-03	7.20e-03
		0.8	1.37193	−1.37193	1.13e-02	7.21e-03
	0.8	0.4	1.80012	−1.80012	1.18e-01	5.13e-02
		0.8	2.76817	−2.76817	1.81e-01	5.18e-02
2	0.4	0.4	0.89216	−0.89216	2.36e-04	2.31e-04
		0.8	1.37193	−1.37193	3.63e-04	2.32e-04
	0.8	0.4	1.80012	−1.80012	1.15e-02	4.94e-03
		0.8	2.76817	−2.76817	1.76e-02	5.04e-03

Table 6. Absolute and relative errors of Example 2 with exact solution for $\alpha = 0.6$

k	x	y	Exact u	Exact v	Δ_k	δ_k
0	0.4	0.4	0.89216	−0.89216	4.81e-01	2.62e-01
		0.8	1.37193	−1.37193	7.49e-01	2.62e-01
	0.8	0.4	1.80012	−1.80012	1.68e+00	4.49e-01
		0.8	2.76817	−2.76817	2.61e+00	4.49e-01
1	0.4	0.4	0.89216	−0.89216	8.96e-02	4.65e-02
		0.8	1.37193	−1.37193	1.39e-01	6.65e-02
	0.8	0.4	1.80012	−1.80012	7.17e-01	1.61e-01
		0.8	2.76817	−2.76817	1.12e+00	1.61e-01
2	0.4	0.4	0.89216	−0.89216	1.10e-02	5.67e-03
		0.8	1.37193	−1.37193	1.71e-02	5.67e-03
	0.8	0.4	1.80012	−1.80012	2.02e-01	4.34e-02
		0.8	2.76817	−2.76817	3.14e-01	4.34e-02

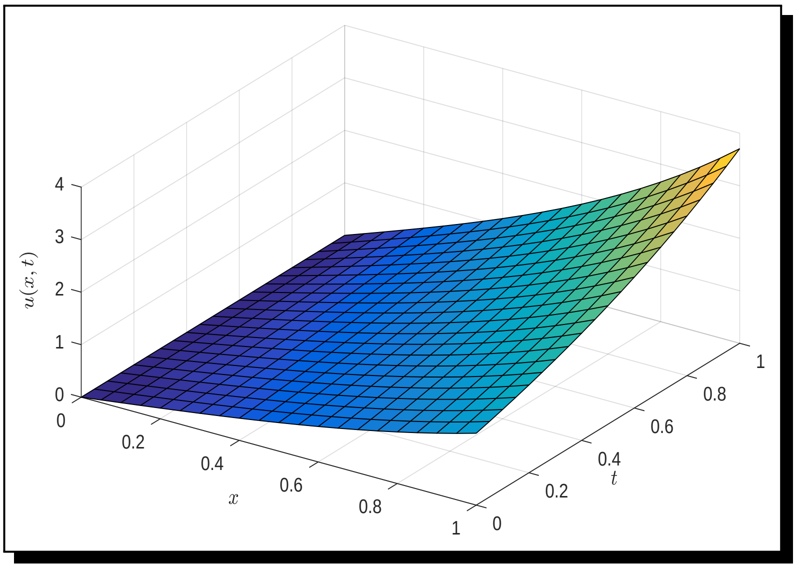

Figure 4. The approximate solution when $\alpha = 1$ of Example 2

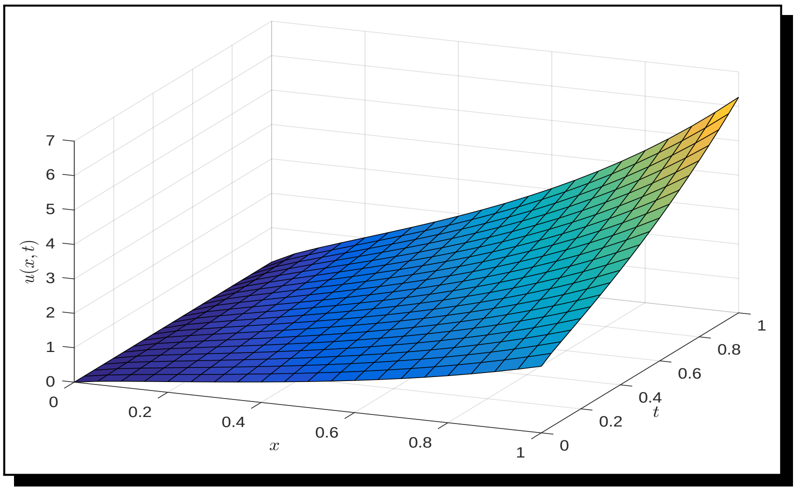

Figure 5. The approximate solution when $\alpha = 0.8$ of Example 2

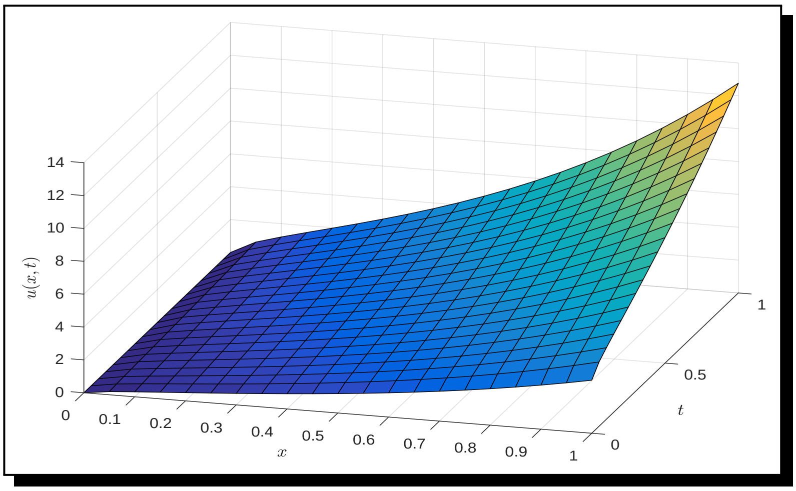

Figure 6. The approximate solution when $\alpha = 0.6$ of Example 2

5. Conclusion

The approximate analytical solution of the time-fractional Helmholtz and coupled Helmholtz equations are constructed by RPSM. The results show that the obtained approximation is one of the best, since it can be applied directly to the problem without linearization, perturbation or discretization. In the future research, we apply this method or modification of this method to various problem in science.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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