Communications in Mathematics and Applications
Volume 2 (2011), Numbers 2-3, pp. 131–144

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Continuous Wavelet Transform on the One-sheeted Hyperboloid

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Abstract. In this work a constructive theory for the CWT on the one-sheeted hyperboloid, $H^{1,1}$, has been developed. The construction is based on the notion of convolution on this manifold. We defined the affine transformations on the $H^{1,1}$. These are of two types: motions and dilations. We defined the CWT on $H^{1,1}$ through a $H^{1,1}$-convolution and derived the corresponding admissibility condition.

1. **Introduction**

Several applications exist in which data living on a non-Euclidean manifold. In this situations the *continuous wavelet transform* (CWT) is a suitable tool for data analysis. In 1998 Antoine and Vandergheynst built wavelets on the 2-sphere, S^2, that are coherent states associated on the Lorentz group $SO_0(3,1)$. Let us expose here some details for this construction. This discussion is borrowed and adopted from [2]. The first step for constructing a CWT on S^2 is to identify the appropriate transformations. These are of two types: (i) motions, which are given by elements of the rotation group $SO(3)$, and (ii) dilations which acts on a point $\omega = (\theta, \varphi)$ by:

$$D_a(\theta,\varphi) = (\theta_a,\varphi), \quad \text{with} \quad \tan\frac{\theta_a}{2} = a\tan\frac{\theta}{2}.$$

Motions and dilations embed into the Lorentz group $SO_0(3,1)$, by the Iwasawa decomposition $SO_0(3,1) = SO(3)\cdot A\cdot N$, where $A \simeq SO_0(1,1) \simeq \mathbb{R} \simeq \mathbb{R}_*^+$ and $N \simeq \mathbf{C}$. Thus the parameter space of the CWT is $X = SO_0(3,1)/N$, and a natural section is $\sigma(\rho,a) = \rho\cdot a\cdot 1$, with $\rho \in SO(3)$ and $a \in A$. The next step is to find an appropriate *unitary irreducible representation* (UIR) of $SO_0(3,1)$ in $L^2(S^2,d\mu)$. A possible choice is:

$$[U(g)f](\omega) = \lambda(g,\omega)^{\frac{1}{2}} f(g^{-1}\omega), \quad g \in SO_0(3,1), f \in L^2(S^2,d\mu),$$

2010 *Mathematics Subject Classification.* 42C15, 42C40, 65T60.
Key words and phrases. Continuous wavelet transform; Fourier-Helgason transform; one-sheeted hyperboloid.

where $\lambda(g,\omega)$ is the Radon-Nikodym derivative. The representation U is square integrable on X, that is, there exist nonzero admissible vector $\eta \in L^2(S^2, d\mu)$ such that

$$\int_0^\infty \frac{da}{a^3} \int_{SO(3)} d\rho |\langle U(\sigma(\rho,a))\eta|\phi\rangle|^2 < \infty, \quad \text{for all } \phi \in L^2(S^2, d\mu),$$

where $d\rho$ is the left Haar Measure on $SO(3)$.

Proposition 1 ([2]). *An admissible wavelet is a function $\eta \in L^2(S^2, d\mu)$ for which there exists a positive constant $c < \infty$ such that*

$$\frac{8\pi^2}{2\ell+1} \sum_{|m|\leq \ell} \int_0^\infty \frac{da}{a^3} |\langle Y_\ell^m|\eta_a\rangle|^2 \leq c, \quad \textit{for all } \ell \in \mathbb{N}.$$

where Y_ℓ^m denotes the usual spherical harmonic and $\eta_a = U(\sigma(e,a))\eta$.

Given the admissible wavelet η, the spherical CWT of a function $f \in L^2(S^2, d\mu)$ with respect to η is defined by

$$\begin{aligned} W_f(\rho,a) &= \langle U(\sigma(\rho,a))\eta|f\rangle \\ &= \int_{S^2} d\mu(\omega)\overline{\eta_a(\rho^{-1}\omega)} f(\omega), \end{aligned}$$

where $\rho \in SO(3)$, $a > 0$.

In 2004 Bogdanova, Vandergheynst and Gazeau built the CWT on the upper sheet of two-sheeted hyperboloid, H_+^2 [4]. We review very briefly this construction: In polar coordinated, H_+^2 may be parametrized as $x = (x_0, x_1, x_2) = x(\chi, \varphi)$, where $x_0 = \cosh\chi$, $x_1 = \sinh\chi\cos\varphi$, $x_2 = \sinh\chi\sin\varphi$ and $\chi \geq 0$, $0 \leq \varphi < 2\pi$. Affine transformations on H_+^2 are of two types:

(i) *Motions*: hyperbolic motions, $g \in SO_0(1,2)$, act on $x(\chi,\varphi) \in H_+^2$ in the following way. A motion $g \in SO_0(1,2)$ can be factorized as $g = k_1 h k_2$, where $k_1, k_2 \in SO(2)$, $h \in SO_0(1,1)$, and the respective action of k_i's, $i = 1,2$ and h are the following:

$$\begin{aligned} &k_i(\varphi_0)\cdot x(\chi,\varphi) = x(\chi, \varphi+\varphi_0), \quad i = 1,2, \\ &h(\chi_0)\cdot x(\chi,\varphi) = x(\chi+\chi_0, \varphi). \end{aligned}$$

(ii) *Dilations through conic projection and flattening*: one projects H_+^2 onto its tangent half null-cone $C^+ = \{x = (x_0, x_1, x_2) \in \mathbb{R}^3 : x_0^2 - x_1^2 - x_2^2 = 0, x_0 > 0\}$, and then vertically onto the plane $x_0 = 0$.

$$x \in H_+^2 \to \Phi(x) \in C^+ \to \Pi_0\Phi(x) \in \mathbb{R}^2 \simeq \mathbf{C}.$$

Consider the following family of projections indexed by a positive parameter p:

$$\Pi_p\Phi(x) = \frac{1}{p}\sinh p\chi\, e^{i\varphi}, \quad x = x(\chi,\varphi).$$

The dilation operator acts on a point $x(\chi,\varphi)\in H_+^2$ by:

$$D_a(\chi,\varphi)=(\chi_a,\varphi), \quad \text{with} \quad \sinh p\chi_a = a\sinh p\chi.$$

The action of dilation on H_+^2, is shown in Figure 1.

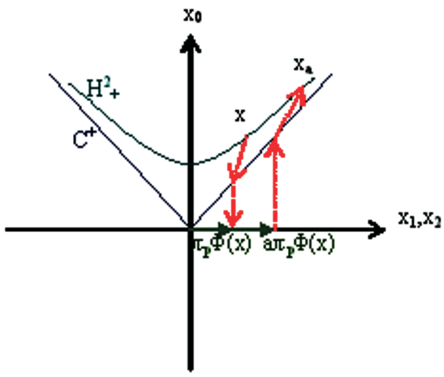

Figure 1. Cross-section of dilation on H_+^2

Theorem 2 ([4])**.** *If $\alpha(a)da$ is a homogeneous measure of the form $a^{-\beta}da$, then a square-integrable function ψ on H_+^2 with compact support is a wavelet if $\beta > \frac{2}{p}+1$ and the following zero-mean condition has to be satisfied:*

$$\int_{H_+^2}\psi(\chi,\varphi)\left[\frac{\sinh 2p\chi}{\sinh\chi}\right]^{\frac{1}{2}}d\mu(\chi,\varphi)=0,$$

where $d\mu(\chi,\varphi)=\sinh\chi\, d\chi\, d\varphi$ is the measure on H_+^2.

Given an admissible hyperbolic wavelet ψ, the hyperbolic CWT of $f \in L^2(H_+^2, d\mu)$ is

$$W_f(g,a)=\langle\psi_{a,g}|f\rangle=\int_{H_+^2}\overline{\psi_a(g^{-1}x)}f(x)d\mu(x), \quad g\in SO_0(2,1),\ a>0.$$

As in the spherical case $\psi_a(x)=\lambda(a,x)^{\frac{1}{2}}\psi(D_{\frac{1}{a}}x)$, with D_a an appropriate dilation, and $\lambda(a,x)$ is the corresponding Radon-Nikodym derivative.

We will denote by $H^{1,1}$ the one-sheeted hyperboloid. This paper is organized as follows: In section 2 we sketch the geometry of the one sheeted hyperboloid $H^{1,1}$. In section 3 we define affine transformations on the $H^{1,1}$. There are two fundamental operations: dilations and motions represented by the group $SO_o(1,2)$. In section 4, we study the harmonic analysis on $H^{1,1}$ and in section 5 define the CWT on $H^{1,1}$ through a hyperbolic convolution. Theorems 5 and 6 are our main results. The first one states an admissibility condition for the existence of $H^{1,1}$-wavelets and the second shows that the admissibility condition simplifies to a zero-mean condition. Finally we give an example of $H^{1,1}$-wavelet.

2. Geometry of the one-sheeted hyperboloid

A hyperboloid is a quadratic surface which may be one or two-sheeted. The one-sheeted hyperboloid is a surface of revolution obtained by rotating a hyperbola about the perpendicular bisector to the line between the foci, while the two-sheeted hyperboloid is a surface of revolution obtained by rotating a hyperbola about the line joining the foci. The one-sheeted hyperboloid oriented along the x_0-axis has Cartesian equation $x_0^2 - x_1^2 - x_2^2 = -1$ and parametric equations

$$x_0 = \sinh\chi, x_1 = \cosh\chi\cos\varphi, x_2 = \cosh\chi\sin\varphi, \quad \chi \in \mathbb{R}, 0 \le \varphi < 2\pi,$$

where $\chi \in \mathbb{R}$ is the arc length from the equator to the given point on the hyperboloid over meridians for fixed φ, while φ is the arc length over the orbit for fixed χ, as shown in Figure 2.

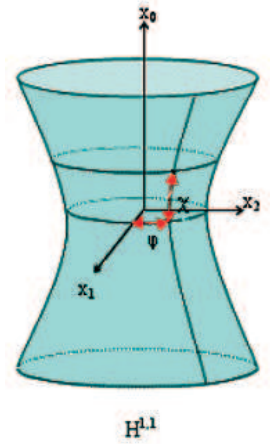

Figure 2. Geometry of the one-sheeted hyperboloid

The measure element on the hyperboloid is $d\mu = \cosh\chi\, d\chi\, d\varphi$. The measure μ is σ-finite, because $H^{1,1}$ is a countable union of sets $O_n = \{x(\chi,\varphi) \in H^{1,1} : \chi \in [-n,n], 0 \le \varphi < 2\pi\}$ with

$$\mu(O_n) = \int_{O_n} d\mu(x) = \int_0^{2\pi}\int_{-n}^{n} \cosh\chi\, d\chi\, d\varphi = 4\pi\sinh n < \infty. \tag{2.1}$$

Consider $\mathbb{R}^3$ with Lorentzian metric

$$(x_0,x_1,x_2)\cdot(y_0,y_1,y_2) = x_0y_0 - x_1y_1 - x_2y_2, (x^2 = x\cdot x),$$

we introduce the cone $C = \{\xi \in \mathbb{R}^3 : \xi^2 = 0\}$ and half cone $C^+ = \{\xi \in \mathbb{R}^3 : \xi^2 = 0, \xi_0 > 0\}$ and use the conic projection, to endow one-sheeted hyperboloid $H^{1,1} = \{x \in \mathbb{R}^3 : x^2 = -1\}$ with a local Euclidean structure. The cone C has Euclidean nature. All points of $H^{1,1}$ will be mapped onto C using a specific conic projection.

3. **Affine transformations on the one-sheeted hyperboloid**

Affine transformations on $H^{1,1}$ are of two types:

(1) Motions are given by elements of $SO_0(1,2)$. A motion $g \in SO_0(1,2)$ can be factorized as $g = k_1 h k_2$, where $k_1, k_2 \in SO(2)$, $h \in SO_0(1,1)$, and the respective action of k_i's for $i = 1,2$ and h are as follows:

$$k_i(\varphi_0) \cdot x(\chi,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_0 & -\sin\varphi_0 \\ 0 & \sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \begin{pmatrix} \sinh\chi \\ \cosh\chi\cos\varphi \\ \cosh\chi\sin\varphi \end{pmatrix} = x(\chi, \varphi + \varphi_0),$$

and

$$h(\chi_0) \cdot x(\chi, 0) = \begin{pmatrix} \cosh\chi_0 & \sinh\chi_0 & 0 \\ \sinh\chi_0 & \cosh\chi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sinh\chi \\ \cosh\chi \\ 0 \end{pmatrix} = x(\chi + \chi_0, 0).$$

Also we define:

$$\begin{aligned} h(\chi_o) \cdot x(\chi,\varphi) &= k(\varphi) h(\chi_0) k(-\varphi) \cdot x(\chi,\varphi) \\ &= x(\chi + \chi_o, \varphi). \end{aligned}$$

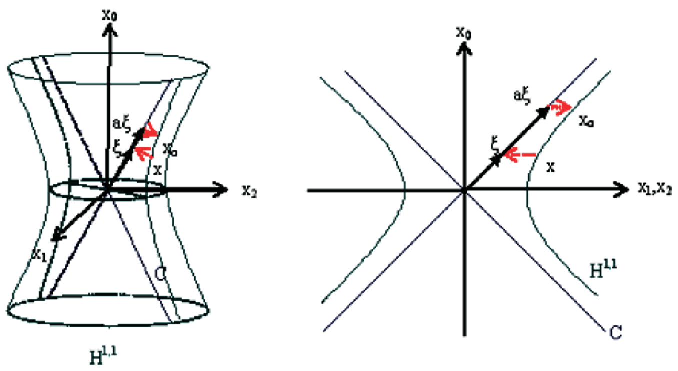

Figure 3. Action of dilation on $H^{1,1}$

The action of a motion on point $x \in H^{1,1}$ is the displacement by a hyperbolic angle $\chi \in \mathbb{R}$ and rotation by an angle φ.

(2) Dilations by a scale factor $a \in \mathbb{R}_*^+$. We use a radial dilation on $H^{1,1}$, which is obtained in three steps:

(i) given a point $x = x(\chi,\varphi) \in H^{1,1}$, project it to the point

$$\xi = (\sinh\chi, \sinh\chi\cos\varphi, \sinh\chi\sin\varphi) \in C,$$

(ii) dilate ξ to $a\xi$ on cone C,

(iii) project back $x_a = x(\chi_a, \varphi)$ with $\sinh\chi_a = a\sinh\chi$.

The action of the dilation on one-sheeted hyperboloid, $H^{1,1}$, is depicted in Figure 3.

4. **Harmonic analysis on** $H^{1,1}$

4.1. *Fourier-Helgason transform*

For all compactly supported smooth functions f on $H^{1,1}$, $f \in \mathscr{C}_c^\infty(H^{1,1})$ introduce the following pair of transforms

$$\begin{aligned}\hat{f}_\pm(\nu,\xi) &= \int_{H^{1,1}} (x\cdot\xi)_\pm^{-\frac{1}{2}-i\nu} f(x)d\mu(x)\\ &= \langle f(x), \varepsilon^\pm_{\nu,\xi}(x)\rangle, \end{aligned} \tag{4.1}$$

where $\nu \in \mathbb{R}^+$, ξ varies on the cone C^+ and the kernels $\varepsilon^\pm_{\nu,\xi}(x) = (x\cdot\xi)_\pm^{-\frac{1}{2}+i\nu}$ are hyperbolic plane waves (eigenfunctions of the Laplacian),

$$\begin{aligned}\varepsilon^\pm_{\nu,\xi}(x) &= (x\cdot\xi)_\pm^{-\frac{1}{2}+i\nu}\\ &= H(x\cdot\xi)|x\cdot\xi|^{-\frac{1}{2}+i\nu} \mp ie^{\pm\pi\nu}H(-(x\cdot\xi))|x\cdot\xi|^{-\frac{1}{2}+i\nu}, \end{aligned} \tag{4.2}$$

in which $H(t)$ is the Heaviside function. The formula (4.1) is called the *Fourier-Helgason transform*. Let us express the plane waves in polar coordinates. For points $x = x(\chi,\varphi) = (\sinh\chi, \cosh\chi\cos\varphi, \cosh\chi\sin\varphi) \in H^{1,1}$ and $\xi(\gamma) = \xi_0(1,\cos\gamma,\sin\gamma) \in C^+$ we have

$$\varepsilon^\pm_{\nu,\xi}(x) = (\xi_0[\sinh\chi - \cos(\varphi-\gamma)\cosh\chi])^{-\frac{1}{2}+i\nu}.$$

Applying any rotation $\rho \in SO(2) \subseteq SO_0(1,2)$, on these waves, we have

$$\varepsilon^\pm_{\nu,\xi}(\rho^{-1}x) = \varepsilon^\pm_{\nu,\rho\xi}(x).$$

The inverse of Fourier-Helgason transform, $\mathscr{FH}_\pm^{-1}$, is defined as

$$\mathscr{FH}_\pm^{-1}[g] = \int_{j\Xi} g(\nu,\xi)(x\cdot\xi)^{-\frac{1}{2}+i\nu}d\eta(\nu,\xi), \;\; \forall\, g \in \mathscr{C}^\infty(\mathscr{L}),$$

where $\mathscr{C}^\infty(\mathscr{L})$ is the space of smooth sections of line-bundle $\mathscr{L}$ over $\Xi = \mathbb{R}^+ \times S^1$ and

$$\begin{aligned} j:\Xi &\longrightarrow \mathbb{R}^+\times C^+\\ (\nu,\widetilde{\xi}) &\longmapsto (\nu,(1,\widetilde{\xi})). \end{aligned}$$

Since $\mathscr{C}_c^\infty(H^{1,1})$ is dense in $L^p(H^{1,1})$, $1 \le p < \infty$, the Fourier-Helgason transform extends to $L^p(H^{1,1})$. By (2.1) and Tonelli's theorem for $f \in L^2(H^{1,1})$ we have

$$\begin{aligned}\int_{j\Xi} |\hat{f}_\pm(\nu,\xi)|^2 d\eta(\nu,\xi) &= \int_{j\Xi} \hat{f}_\pm(\nu,\xi)\overline{\hat{f}_\pm(\nu,\xi)}d\eta(\nu,\xi)\\ &= \int_{j\Xi} \overline{\hat{f}_\pm(\nu,\xi)}d\eta(\nu,\xi)\int_{H^{1,1}} f(x)(x\cdot\xi)_\pm^{-\frac{1}{2}-i\nu}d\mu(x)\end{aligned}$$

$$
\begin{aligned}
&= \int_{H^{1,1}} f(x)d\mu(x)\overline{\int_{j\Xi} \hat{f}_{\pm}(\nu,\xi)(x\cdot\xi)_{\pm}^{-\frac{1}{2}+i\nu}d\eta(\nu,\xi)} \\
&= \int_{H^{1,1}} f(x)\overline{f(x)}d\mu(x) \\
&= \int_{H^{1,1}} |f(x)|^2 d\mu(x) \\
&= \|f\|_2^2 \,. \qquad (4.3)
\end{aligned}
$$

This shows that $\hat{f}_{\pm} \in L^2(\mathscr{L})$ and that $\|f\|_2 = \|\hat{f}_{\pm}\|_2$ holds.

Proposition 3. *If f is a rotation invariant function, i.e., $f(\rho^{-1}x) = f(x)$, for all $\rho \in SO(2)$, then its Fourier-Helgason transform $\hat{f}_{\pm}(\nu,\xi)$ is a function of ν alone, i.e., $\hat{f}_{\pm}(\nu)$.*

Proof. Let f be a rotation invariant function. Applying the Fourier-Helgason transform on f we write

$$
\begin{aligned}
\hat{f}_{\pm}(\nu,\xi) &= \int_{H^{1,1}} \overline{\varepsilon^{\pm}_{\nu,\xi}(x)}f(x)d\mu(x) \\
&= \int_{H^{1,1}} \overline{\varepsilon^{\pm}_{\nu,\xi}(x)}f(\rho^{-1}x)d\mu(x) \\
&= \int_{H^{1,1}} \overline{\varepsilon^{\pm}_{\nu,\xi}(\rho x')}f(x')d\mu(x') \\
&= \int_{H^{1,1}} \overline{\varepsilon^{\pm}_{\nu,\rho^{-1}\xi}(x')}f(x')d\mu(x') \\
&= \hat{f}_{\pm}(\nu,\rho^{-1}\xi),
\end{aligned}
$$

and so $\hat{f}_{\pm}(\nu,\xi)$ does not depend on ξ. □

4.2. *Convolutions on $H^{1,1}$*

Since $H^{1,1}$ is a homogeneous space of $SO_0(1,2)$, $H^{1,1} = \frac{SO_0(1,2)}{SO_0(1,1)}$, one can easily define a convolution. First, given $f \in L^2(H^{1,1})$ and $s \in L^1(H^{1,1})$, their $H^{1,1}$-convolution is defined by:

$$
(f * s)(g) = \int_{H^{1,1}} f(g^{-1}x)s(x)d\mu(x), \quad \text{for all } g \in SO_0(1,2).
$$

Next, one may take the restriction to $H^{1,1}$, using a section

$$
\begin{aligned}
[\cdot] : H^{1,1} &\longrightarrow SO_0(1,2) \\
x(\chi,\varphi) &\longmapsto g = k(\varphi)h(\chi) = [x],\ k(\varphi) \in SO(2), h(\chi) \in SO_0(1,1)
\end{aligned}
$$

and define

$$(f * s)(y) = \int_{H^{1,1}} f([y]^{-1}x)s(x)d\mu(x), \quad y \in H^{1,1}.$$

Let $f \in L^2(H^{1,1})$ and $s \in L^1(H^{1,1})$ then $s * f \in L^2(H^{1,1})$ and $\|s * f\|_2 \le \|s\|_1 \|f\|_2$ by the Young's inequality.

We will mostly deal with $H^{1,1}$-convolution kernels that are symmetric and rotation invariant functions on $H^{1,1}$. The Fourier-Helgason transform of such an element has a simpler form as shown by the following theorem.

Theorem 4. *Given $f \in L^2(H^{1,1})$ and let $s \in L^1(H^{1,1})$ be rotation invariant and symmetric i.e. $s(-\chi, \varphi) = s(\chi, \varphi)$, then $s * f \in L^2(H^{1,1})$ and*

$$\widehat{(s * f)}_\pm(\nu, \xi) = \hat{f}_\pm(\nu, \xi)\hat{s}_\pm(\nu).$$

Proof. We use polar coordinates for $x = x(\chi, \varphi)$ and $y = y(\chi', \varphi')$ in $H^{1,1}$. Since s is $SO(2)$-invariant, we have $s([y]^{-1}x) = s(h(-\chi') \cdot x(\chi, \varphi)) = s(\chi - \chi', \varphi) = s(\chi - \chi', 0)$, and similarly $s([x]^{-1}y) = s(\chi' - \chi, 0)$. Since s is symmetric, we have $s([y]^{-1}x) = s([x]^{-1}y)$. Thus, the convolution with a symmetric rotation invariant function is given by

$$\begin{aligned}
(s * f)(y) &= \int_{H^{1,1}} s([y]^{-1}x)f(x)d\mu(x) \\
&= \int_{H^{1,1}} s([x]^{-1}y)f(x)d\mu(x)
\end{aligned}$$

Applying the Fourier-Helgason transform on $s * f$ we get

$$\begin{aligned}
\widehat{(s * f)}_\pm(\nu, \xi) &= \int_{H^{1,1}} (y \cdot \xi)_\pm^{-\frac{1}{2}-i\nu}(s * f)(y)d\mu(y) \\
&= \int_{H^{1,1}} d\mu(y) \int_{H^{1,1}} d\mu(x)s([y]^{-1}x)f(x)(y \cdot \xi)_\pm^{-\frac{1}{2}-i\nu} \\
&= \int_{H^{1,1}} d\mu(x)f(x) \int_{H^{1,1}} d\mu(y)s([y]^{-1}x)(y \cdot \xi)_\pm^{-\frac{1}{2}-i\nu} \\
&= \int_{H^{1,1}} d\mu(x)f(x) \int_{H^{1,1}} d\mu(y)s([x]^{-1}y)(y \cdot \xi)_\pm^{-\frac{1}{2}-i\nu} \\
&= \int_{H^{1,1}} d\mu(x)f(x) \int_{H^{1,1}} d\mu(y)s(y)([x]y \cdot \xi)_\pm^{-\frac{1}{2}-i\nu} \\
&= \int_{H^{1,1}} d\mu(x)f(x) \int_{H^{1,1}} d\mu(y)s(y)(y \cdot [x]^{-1}\xi)_\pm^{-\frac{1}{2}-i\nu}
\end{aligned}$$

we can write $(y \cdot [x]^{-1}\xi) = ([x]^{-1}\xi)_1 \Big(y \cdot \frac{[x]^{-1}\xi}{([x]^{-1}\xi)_1} \Big)$, and using $([x]^{-1}\xi)_1 = -(x \cdot \xi)$, we finally obtain

$$\begin{aligned}
\widehat{(s * f)}_{\pm}(\nu, \xi) &= \int_{H^{1,1}} d\mu(x) f(x) (x \cdot \xi)_{\pm}^{-\frac{1}{2}-i\nu} \\
&\quad \times \int_{H^{1,1}} d\mu(y) s(y) \left(y \cdot \frac{-[x]^{-1}\xi}{([x]^{-1}\xi)_1} \right)^{-\frac{1}{2}-i\nu} \\
&= \hat{f}_{\pm}(\nu, \xi) \hat{s}_{\pm}(\nu)
\end{aligned}$$

where we used the rotation invariance of s. □

5. Continuous wavelet transform on the one-sheeted hyperboloid

Now, we are in a situation to define the CWT on the one-sheeted hyperboloid.We say $\psi \in L^1(H^{1,1})$ is admissible if the family $\{\psi_{a,[x]} : a > 0, x \in H^{1,1}\}$ is a continuous frame. Given an admissible $H^{1,1}$-wavelet ψ, the $H^{1,1}$-CWT of $f \in L^2(H^{1,1})$ is

$$W_f(a, g) := \langle \psi_{a,g} | f \rangle = \int_{H^{1,1}} \overline{\psi_a(g^{-1}x)} f(x) d\mu(x), g \in SO_0(1,2), \quad a > 0$$

where $\psi_a(x) = \lambda(a, x)^{\frac{1}{2}} \psi(D_{\frac{1}{a}} x)$ such that $\lambda(a, x) = \frac{d\mu(x_{\frac{1}{a}})}{d\mu(x)}$ is the Radon-Nikodym derivative. The function $\lambda : \mathbb{R}_*^+ \times H^{1,1} \to \mathbb{R}^+$ is a 1-cocycle, that is, it satisfies the equation $\lambda(a_1 a_2, x) = \lambda(a_1, x)\lambda(a_2, a_1^{-1}x)$ and $\lambda(1, x) = 1$. Note that $\lambda(a, x) = \frac{d \sinh \chi_{\frac{1}{a}}}{d \sinh \chi} = \frac{1}{a}$ does not depend on χ. If the wavelet ψ is symmetric and rotation invariant, the hyperbolic CWT is a convolution:

$$W_f(g, a) \equiv W_f(x, a) = (\overline{\psi}_a * f)(x), \quad \text{where} \;\; g = [x], x \in H^{1,1}.$$

It remains to state an appropriate wavelet admissibility condition.

Theorem 5. *Let $\psi \in L^1(H^{1,1})$ be a rotation invariant function, $a \mapsto \alpha(a)$ a positive function on $\mathbb{R}_*^+$, for which there exists constants m and M such that*

$$0 < m \leq \mathscr{A}_\psi(\nu) := \int_0^\infty |(\widehat{\psi_a})_{\pm}(\nu)|^2 \alpha(a) da \leq M < \infty. \tag{5.1}$$

Then the linear operator A_ψ on $L^2(H^{1,1})$ defined by

$$A_\psi f(x') = \int_0^\infty \int_{H^{1,1}} W_f(x, a) \psi_{a,[x]}(x') d\mu(x) \alpha(a) da,$$

is bounded with bounded inverse. In other words, the family $\{\psi_{a,[x]} : \; a > 0, x \in H^{1,1}\}$ is a continuous frame.

Proof. Consider the following integral

$$\Delta = \int_0^\infty \int_{H^{1,1}} |W_f(x, a)|^2 d\mu(x) \alpha(a) da.$$

By the equality $(\widehat{\psi}_{a,[x]})_{\pm}(\nu)=(x\cdot\xi)_{\pm}^{-\frac{1}{2}-i\nu}(\widehat{\psi_a})_{\pm}(\nu)$ and Tonelli's theorem, we have

$$\begin{aligned}
\Delta &= \int_0^\infty \alpha(a)da\int_{H^{1,1}} d\mu(x)W_f(x,a)\overline{W_f(x,a)}\\
&= \int_0^\infty \alpha(a)da\int_{H^{1,1}} d\mu(x)\int_{H^{1,1}}\overline{\psi_{a,[x]}(y)}f(y)d\mu(y)\int_{H^{1,1}}\psi_{a,[x]}(y)\overline{f(y)}d\mu(y)\\
&= \int_0^\infty \alpha(a)da\int_{H^{1,1}} d\mu(x)\int_{H^{1,1}}|f(y)|^2d\mu(y)\\
&\quad\times\int_{j\Xi}\overline{(x\cdot\xi)^{-\frac{1}{2}-i\nu}(\widehat{\psi_a})_{\pm}(\nu)(y.\xi)^{-\frac{1}{2}+i\nu}}d\eta(\nu,\xi)\int_{H^{1,1}}\psi_{a,[x]}(y)d\mu(y)\\
&= \int_0^\infty \alpha(a)da\int_{H^{1,1}} d\mu(x)\int_{H^{1,1}}|f(y)|^2d\mu(y)\\
&\quad\times\int_{j\Xi}(x\cdot\xi)^{-\frac{1}{2}+i\nu}\overline{(\widehat{\psi_a})_{\pm}(\nu)}d\eta(\nu,\xi)\int_{H^{1,1}}\psi_{a,[x]}(y)(y.\xi)^{-\frac{1}{2}-i\nu}d\mu(y)\\
&= \|f\|_2^2\int_0^\infty|(\widehat{\psi_a})_{\pm}(\nu)|^2\alpha(a)da\int_{j\Xi}\int_{H^{1,1}}(x\cdot\xi)^{-1}d\mu(x)d\eta(\nu,\xi)
\end{aligned}$$

If 1_K is the characteristic function on the set

$$\{x=(\chi,\varphi)\in H^{1,1}:-K\le\chi\le K,0\le\varphi<2\pi\},$$

then $1_K\in L(H^{1,1})$ and $1_K\nearrow 1$. Since

$$\begin{aligned}
&\int_{j\Xi}\int_{H^{1,1}}1_K(x\cdot\xi)^{-1}d\mu(x)d\eta(\nu,\xi)\\
&\quad=\int_{j\Xi}\int_{H^{1,1}}1_K(x\cdot\xi)^{-\frac{1}{2}-i\nu}(x\cdot\xi)^{-\frac{1}{2}+i\nu}d\mu(x)d\eta(\nu,\xi)\\
&\quad=1_K,
\end{aligned}$$

then $\int_{j\Xi}\int_{H^{1,1}}(x\cdot\xi)^{-1}d\mu(x)d\eta(\nu,\xi)=1$ by the monotone convergence theorem. Hence

$$m\|f\|_2^2\le\int_0^\infty\int_{H^{1,1}}|W_f(x,a)|^2d\mu(x)\alpha(a)da\le M\|f\|_2^2. \qquad\square$$

From Theorem 5, one obtains a reconstruction formula, which holds in $L^2(H^{1,1})$:

$$f(x')=\int_0^\infty\int_{H^{1,1}}W_f(x,a)A_\psi^{-1}\psi_{a,[x]}(x')d\mu(x)\alpha(a)da.$$

Some choices for the function α, leads to simplified admissibility condition (5.1) as we will now discuss.

Theorem 6. *Let $\alpha(a)da$ be a homogeneous measure of the form $a^{-\beta}da$, $\beta > 0$, then a positive rotation invariant function $\psi \in L^1(H^{1,1}, d\mu)$ is admissible either if $\beta < 2$, or $\beta \geq 2$ and the following zero-mean condition is satisfied:*

$$\int_{H^{1,1}} \psi(\chi, \varphi) d\mu(\chi, \varphi) = 0. \tag{5.2}$$

Proof. Let us assume $\psi(\chi)$ belongs to $\mathscr{C}_c(H^{1,1})$, i.e., it is continuous and compactly supported

$$\psi(\chi) = 0 \quad \text{if} \quad |\chi| > \widetilde{\chi}, \quad \widetilde{\chi} = \text{const.}$$

we wish to prove that

$$\int_0^\infty |\langle \varepsilon^\pm_{\nu,\xi} | \psi_a \rangle|^2 \alpha(a) da < \infty. \tag{5.3}$$

First, we compute the Fourier-Helgason coefficients of the dilated function ψ

$$\begin{aligned} \langle \varepsilon^\pm_{\nu,\xi} | \psi_a \rangle &= \int_X \psi_a(\chi, \varphi) \overline{\varepsilon^\pm_{\nu,\xi}(\chi, \varphi)} d\mu(\chi, \varphi) \\ &= \int_0^{2\pi} \int_{-\widetilde{\chi}_a}^{\widetilde{\chi}_a} \lambda(a, \chi)^{\frac{1}{2}} \psi(\chi_{\frac{1}{a}}, \varphi) \overline{\varepsilon^\pm_{\nu,\xi}(\chi, \varphi)} d\mu(\chi, \varphi) \end{aligned}$$

By performing the change of variable $\chi' = \chi_{\frac{1}{a}}$, we get $\chi = \chi'_a$ and $d\mu(\chi, \varphi) = \lambda(a^{-1}, \chi') d\mu(\chi', \varphi)$. The Fourier-Helgason coefficients become

$$\langle \varepsilon^\pm_{\nu,\xi} | \psi_a \rangle = \int_0^{2\pi} \int_{-\widetilde{\chi}}^{\widetilde{\chi}} \lambda(a, \chi'_a)^{\frac{1}{2}} \psi(\chi', \varphi) \overline{\varepsilon^\pm_{\nu,\xi}(\chi'_a, \varphi)} \lambda(a^{-1}, \chi') d\mu(\chi', \varphi)$$

since $\lambda(a, \chi'_a)^{\frac{1}{2}} \lambda(a^{-1}, \chi') = \lambda(a^{-1}, \chi')^{\frac{1}{2}} = \sqrt{a}$, we get

$$\langle \varepsilon^\pm_{\nu,\xi} | \psi_a \rangle = \sqrt{a} \int_0^{2\pi} \int_{-\widetilde{\chi}}^{\widetilde{\chi}} \psi(\chi', \varphi) \overline{\varepsilon^\pm_{\xi,\nu}(\chi'_a, \varphi)} d\mu(\chi', \varphi) \tag{5.4}$$

we split (5.4) in three parts

$$\int_0^\infty (\cdot) \alpha(a) da = \underbrace{\int_0^\sigma (\cdot) \alpha(a) da}_{I} + \underbrace{\int_\sigma^{\frac{1}{\sigma}} (\cdot) \alpha(a) da}_{II} + \underbrace{\int_{\frac{1}{\sigma}}^\infty (\cdot) \alpha(a) da}_{III}.$$

We start by studying the first integral I. Since ψ is rotation invariant, we have

$$\begin{aligned} \langle \varepsilon^\pm_{\nu,\xi} | \psi_a \rangle &= \sqrt{a} \int_0^{2\pi} \int_{-\widetilde{\chi}}^{\widetilde{\chi}} \psi(k(\varphi)(\chi', \varphi)) \overline{\varepsilon^\pm_{\xi,\nu}(\chi'_a, \varphi)} d\mu(\chi', \varphi) \\ &= \sqrt{a} \int_0^{2\pi} \int_{-\widetilde{\chi}}^{\widetilde{\chi}} \psi(\chi', \varphi) \overline{\varepsilon^\pm_{\xi,\nu}(\chi'_a, 0)} d\mu(\chi', \varphi) \end{aligned}$$

Let σ be small enough, so that a is small, and hence $\chi'_a \to 0$. So, because $\varepsilon^{\pm}_{\xi,\nu}(0,0) = \xi_0^{-\frac{1}{2}}$ the integral I is of the form

$$\xi_0^{-1}\int_0^{\sigma} a^{-\beta+1}da\left|\int_0^{2\pi}\int_{-\widetilde{\chi}}^{\widetilde{\chi}} d\mu(\chi',\varphi)\psi(\chi',\varphi)\right|^2$$

If $\beta \geq 2$, the integral I converges only if $\int_{H^{1,1}} \psi = 0$.

The second integral II is straightforward, since the dilation operator is strongly continuous and thus the integrand is bounded on $[\sigma, \frac{1}{\sigma}]$.

For the last part III we first rewrite the Fourier-Helgason coefficients for large scales. By performing the change of variable $\chi' = \chi_{a^{-2}}$, we get $\chi = \chi'_{a^2}$ and $d\mu(\chi,\varphi) = \lambda(a^{-2},\chi')d\mu(\chi',\varphi)$, and hence

$$\langle \varepsilon^{\pm}_{\nu,\xi}|\psi_a\rangle = \int_0^{2\pi}\int_{-\widetilde{\chi}_{\frac{1}{a}}}^{\widetilde{\chi}_{\frac{1}{a}}} \lambda(a,\chi'_{a^2})^{\frac{1}{2}}\psi(\chi'_a,\varphi)\overline{\varepsilon^{\pm}_{\nu,\xi}(\chi'_{a^2},\varphi)}\lambda(a^{-2},\chi')d\mu(\chi',\varphi)$$

since $\lambda(a,\chi'_{a^2})^{\frac{1}{2}}\lambda(a^{-2},\chi') = \lambda(a,\chi')^{\frac{1}{2}}\lambda(a^2,\chi')^{\frac{1}{2}} = a^{-\frac{3}{2}}$, we get

$$\langle \varepsilon^{\pm}_{\nu,\xi}|\psi_a\rangle = a^{-\frac{3}{2}}\int_0^{2\pi}\int_{-\widetilde{\chi}_{\frac{1}{a}}}^{\widetilde{\chi}_{\frac{1}{a}}} \psi(\chi'_a,\varphi)\overline{\varepsilon^{\pm}_{\xi,\nu}(\chi'_{a^2},\varphi)}d\mu(\chi',\varphi). \tag{5.5}$$

The only large scale divergence in III will never reached because the support property of ψ and this finally ensures convergence of the last term as well. If $\psi \in L^1(H^{1,1})$, then the desired result holds by Fatou's lemma. □

We conclude this section by presenting an explicit class of admissible vectors, that is, hyperbolic wavelets. First, we need the following result.

Proposition 7. *Let $\psi \in L^2(H^{1,1}, d\mu)$. Then*

$$\int_{H^{1,1}} D_a\psi(\chi,\varphi)d\mu(\chi,\varphi) = \sqrt{a}\int_{H^{1,1}} \psi(\chi,\varphi)d\mu(\chi,\varphi).$$

Proof. By a simple computation, followed by the change of variables $\chi' = \chi_{\frac{1}{a}}$, we get

$$\begin{aligned}
I &= \int_{H^{1,1}} D_a\psi(\chi,\varphi)d\mu(\chi,\varphi)\\
&= \int_{H^{1,1}} \lambda(a,\chi)^{\frac{1}{2}}\psi(\chi_{\frac{1}{a}},\varphi)d\mu(\chi,\varphi)\\
&= \int_{H^{1,1}} \lambda(a,\chi'_a)^{\frac{1}{2}}\psi(\chi',\varphi)\lambda(a^{-1},\chi')d\mu(\chi',\varphi)\\
&= \int_{H^{1,1}} \lambda(a^{-1},\chi')^{\frac{1}{2}}\psi(\chi',\varphi)d\mu(\chi',\varphi).
\end{aligned}$$

Then since $\lambda(a^{-1},\chi')^{\frac{1}{2}} = \sqrt{a}$, we get

$$I = \sqrt{a}\int_{H^{1,1}} \psi(\chi',\varphi)d\mu(\chi',\varphi).$$

which proves the proposition. □

With this result, it is easy to build a difference wavelet. Given an integrable function ψ, we define

$$f_\psi^{\upsilon}(\chi,\varphi) = \psi(\chi,\varphi) - \frac{1}{\sqrt{\upsilon}}D_\upsilon\psi(\chi,\varphi), \upsilon > 1.$$

Then it is easily checked that f_ψ^{υ} satisfies in the admissibility condition (5.3), that is, it is a $H^{1,1}$-wavelet. A typical difference wavelet for the choice $\psi(\chi,\varphi) = e^{-\sinh^2\chi}$ is

$$f_\psi^{\upsilon}(\chi,\varphi) = e^{-\sinh^2\chi} - \frac{1}{a}e^{-\frac{1}{\upsilon^2}\sinh^2\chi}.$$

Acknowledgement

This research is supported by Mahani Mathematical Research Center, Shahid Bahonar University of Kerman, Kerman, Iran.

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Received May 10, 2011
Accepted October 12, 2011