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Research Article

Strong and Δ-Convergence Results for Generalized Nonexpansive Mapping in Hyperbolic Space

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Abstract. In this paper, we propose a new iteration process which is faster than Picard Normal S-iteration process, Normal S-iteration process, Mann iteration process and Picard iteration process in hyperbolic space for generalized nonexpansive mapping. We also present strong and $\Delta-$convergence results for proposed iteration process. An illustrative example with different set of parameters is also given in this paper.

Keywords. Hyperbolic space; Generalized nonexpansive mappings; Picard normal S-iteration process; Normal S-iteration process

MSC. 47H10; 47H09

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1. Introduction

Let X be a normed space and C be a nonempty subset of X. The mapping $T : C \to C$ is said to be

(1) *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

(2) *quasi-nonexpansive*, if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$, where $F(T)$ denotes the set of all fixed point of T, i.e.,

$$F(T) = \{x : Tx = x, x \in C\}.$$

Many nonlinear problems are formulated as fixed point problems

$$x = Tx \tag{1.1}$$

where fixed point mapping T may be nonlinear. The solution of the problem is called a fixed point of the mapping T.

The fixed point iteration is given by

$$x_{n+1} = Tx_n, \quad \forall\ n \in N. \tag{1.2}$$

The iteration process (1.2) is also known as Picard iteration, Richardson iteration or method of successive substitution. For the Banach contraction mapping, the Picard iteration converges to unique fixed point of T, but this iteration fails to approximate fixed point for nonexpansive mapping. On the other hand, this iteration method enable us to identify the existence of fixed point of T.

In last fifty years, many researchers designed iterative process to approximate fixed point of nonexpansive mapping and for some wider class of nonexpansive mappings (see [1,4,10,11,13]) and compare which iteration process is faster.

In 1953, Mann [10] introduced an iteration process: For an arbitrary $x_0 \in C$, construct sequence $\{x_n\}$ as follows

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ is real sequence in the interval $(0,1)$.

In 2011, Sahu [12] introduced Normal S-iteration process: For convex subset C of normed linear space X and a mapping $T : C \to T$, $x_0 \in C$, construct a sequence $\{x_n\}$ in C as follows

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \in N, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is real sequence in $(0,1)$.

The rate of convergence of Normal S-iteration is similar to the Picard iteration process and it is faster than other fixed point iteration process (see [12, Theorem 3.5]).

Remark 1.1. If $\alpha = 0$, then Normal S-iteration process (1.4) reduces to Picard iteration process.

In 2014, Kadioglu and Yildirim [5] introduced Picard Normal S-iteration process: With C, X, T as in Normal S-Iteration, for $x_1 \in C$, construct a sequence $\{x_n\}$ in C as follows:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \in N. \end{cases} \tag{1.5}$$

The rate of convergence of Picard Normal S-iteration is faster than Normal S-iteration iteration (see [5, Theorem 5]).

Remark 1.2. (1) If $\beta_n = 0$, then Picard Normal S-iteration process (1.5) reduces to Normal S-iteration iteration process.

(2) If $\alpha_n = 0$, $\beta_n = 0$, then Picard Normal S-iteration process (1.5) reduces to Picard iteration process.

In 2016, Imdad and Dashputre [3] established strong and Δ-convergence results for Picard Normal S-iteration process (1.5) for generalized nonexpansive mappings in uniformly convex hyperbolic space. In this paper, we design a new iteration process as follows: With C, X, T as in Normal S-iteration, for $x_1 \in C$, construct a sequence $\{x_n\}$ in C as follows:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ w_n = (1-\beta_n)w_n + \beta_n T w_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \qquad n \in N. \end{cases} \tag{1.6}$$

Remark 1.3. (1) If $\gamma_n = 0$, then new iteration process (1.6) reduces to Picard Normal S-iteration iteration.

(2) If $\beta_n = 0$, $\gamma_n = 0$, then new iteration process (1.6) reduces to Normal S-iteration process.

(3) If $\alpha_n = 0$, $\beta_n = 0$ and $\gamma_n = 0$, then new iteration process (1.6) reduces to Picard iteration process.

Since the new iterative process is generalization of Picard Normal S-iteration Process, therefore we can call it by Generalized Picard Normal S-iteration process.

In this paper, we will show Generalized Picard Normal S-iteration Process is faster than Picard Normal S-iteration process, Normal S-iteration process, Mann iteration process and Picard iteration process. Also, we will established strong and Δ-convergence theorems for Generalized Picard Normal S-iteration process for generalized nonexpansive mappings in uniformly convex hyperbolic space.

2. Preliminaries

Let (X,d) be a metric space and C be a nonempty subset of X. In 2008, Suzuki [14] defined a class of single valued mapping as follows

$$\frac{1}{2}d(x,Tx) \le d(x,y) \to d(Tx,Ty) \le d(x,y) \tag{2.1}$$

The equation (2.1) is Suzuki-generalized nonexpansive mapping (or Condition (C)).

Remark 2.1. The Suzuki-generalized nonexpansive mapping (or Condition (C)) is weak than nonexpansive mapping and stronger than quasi-nonexpansive mapping. But converse may not be true.

$$\boxed{\text{Nonexpansive mapping} \Rightarrow \text{Suzuki-generalized nonexpansive mapping} \Rightarrow \text{Quasi-nonexpansive mapping}}$$

In [2], author introduced following generalization of nonexpansive mappings.

Definition 2.2. Let T be a mapping defined on a subset C of a metric space X and $\lambda \in (0,1)$. Then T is said to satisfy the condition (C_λ) if for all $x, y \in C$

$$\lambda d(x,Tx) \le d(x,y) \to d(Tx,Ty) \le d(x,y). \tag{2.2}$$

For $0 < \lambda_1 < \lambda_2 < 1$, condition $C_{\lambda_1} \to$ condition C_{λ_2}.

Definition 2.3. Let T be a mapping defined on a subset C of a metric space X and $\mu \geq 1$. Then T is said to satisfy the condition (E_μ) if for all $x,y \in C$

$$d(x,Ty) \leq \mu d(x,Tx) + d(x,y). \tag{2.3}$$

We consider the following definition of a hyperbolic space introduced by Kohlenbach [7].

Definition 2.4. A metric space (X,d) is a hyperbolic space if there exists a map $W: X^2 \times [0,1] \to X$ satisfying

(i) $d(u,W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$,

(ii) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta| d(x,y)$,

(iii) $W(x,y,\alpha) = W(y,x,(1-\alpha))$,

(iv) $d(W(x,z,\alpha),W(y,w,\alpha)) \leq \alpha d(x,y) + (1-\alpha)d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$.

Definition 2.5 ([15])**.** A metric space is said to be convex metric space when a triple (X,d,W) satisfy only (i) in Definition 2.4.

Definition 2.6 ([15])**.** A subset $K$of a hyperbolic space X is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$.

If $x,y \in X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for more general setting of convex metric space [15]: for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(x,(1-\lambda)x \oplus \lambda y) = \lambda d(x,y) \quad \text{and} \quad d(y,(1-\lambda)x \oplus \lambda y) = (1-\lambda)d(x,y).$$

Thus

$$1x \oplus 0y = x \quad \text{and} \quad 0x \oplus 1y = y.$$

$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Definition 2.7 ([8])**.** A hyperbolic space (X,d,W) is uniformly convex if for any $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that $d\left(\frac{1}{2}x \oplus \frac{1}{2}y, u\right) \leq (1-\delta)r$ whenever $d(x,u) \leq r$, $d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$.

Definition 2.8. A map $\eta: (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

In [8], Luestean proved that CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ quadratic in ε.

Now, we give the concept of Δ-convergence and some of its basic properties.

Let C be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X while diam(C) denotes the diameter of C. Consider a continuous functional $r_a(\cdot,\{x_n\}): X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad x \in X.$$

The infimum of $r_a(\cdot,\{x_n\})$ over C is said to be the asymptotic radius of $\{x_n\}$ with respect to C and it is denoted by $r_a(C,\{x_n\})$.

A point $z \in C$ is said to be asymptotic center of the sequence $\{x_n\}$ with respect to C if

$$r_a(z,\{x_n\}) = \inf\{r_a(x,\{x_n\}) : x \in C\}.$$

The set of all asymptotic center of $\{x_n\}$ with respect to C is denoted by $AC(C,\{x_n\})$. The set $AC(C,\{x_n\})$ may be empty, singleton or have infinitely many points.

If the asymptotic radius and asymptotic center are taken with respect to X, then they are denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $AC(X,\{x_n\}) = AC(\{x_n\})$, respectively. We know that for $x \in X, r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$ and every bounded sequence has a unique asymptotic center with respect to closed convex subset in uniformly convex Banach spaces.

Definition 2.9. The sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$, if x is unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Lemma 2.10 ([9]). *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Consider the following Lemma of Khan *et al.* [6] which we use in the sequel.

Lemma 2.11. *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and t_n be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \le c,$$

$$\limsup_{n\to\infty} d(y_n,x) \le c,$$

$$\limsup_{n\to\infty} d(W(x_n,y_n,t_n),x) = c,$$

for some $c \ge 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 2.12 ([3]). *Let (X,d) be complete uniformly convex hyperbolic space with monotone modulus of convexity η. Let C be a nonempty convex closed subset of a hyperbolic space X and $T : C \to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda \in (0,1)$ and conditionE_μ on C. Suppose that $\{x_n\}$ is bounded sequence in C such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, then T has a fixed point.*

Definition 2.13. Let C be a nonempty convex closed subset of a hyperbolic space $X$and $\{x_n\}$ is a sequence in X. Then $\{x_n\}$ is Fejér monotone with respect to C if for all $x \in C$ and $n \in N$,

$$d(x_{n+1},x) \le d(x_n,x).$$

We can easily prove the following proposition:

Proposition 2.14. *Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $\{x_n\}$ is Fejér monotone with respect to C. Then we have the followings:*

(1) *$\{x_n\}$ is bounded.*

(2) *The sequence $\{d(x_n,p)\}$ is decreasing and converges for all $p \in F(T)$.*

3. Strong and Δ-Convergence of Generalized Picard Normal S-iteration Process

Firstly, we define Generalized Picard Normal S-iteration process in hyperbolic space as follows: Let C be nonempty closed convex subset of a hyperbolic space X and $T : C \to C$ be a mapping which satisfies the condition C_λ for some $\lambda \in (0,1)$. If for any $x_1 \in C$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Ty_n, 0, 0), \\ y_n = W(z_n, Tz_n, \alpha_n), \\ z_n = W(w_n, Tw_n, \beta_n), \\ w_n = W(x_n, Tx_n, \gamma_n), \quad n \in N. \end{cases} \tag{3.1}$$

where α_n, β_n and γ_n are in $[\varepsilon, 1-\varepsilon]$ for all $n \in N$ and some $\varepsilon \in (0,1)$.

Lemma 3.1. *Let C be a nonempty convex closed subset of a hyperbolic space X and $T : C \to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda \in (0,1)$. If $\{x_n\}$ is a sequence defined by* (3.1)*, then $\{x_n\}$ is Fejér monotone with respect to $F(T)$.*

Proof. Since for some $\lambda \in (0,1)$ and $p \in F(T)$, T satisfies condition (C_λ), therefore for all $n \in N$,

$$\lambda d(p,Tp) = 0 \le d(p,w_n),$$

$$\lambda d(p,Tp) = 0 \le d(p,z_n),$$

$$\lambda d(p,Tp) = 0 \le d(p,y_n)$$

and

$$\lambda d(p,Tp) = 0 \le d(p,x_n).$$

Therefore, we have

$$d(Tp,Tw_n) \le d(p,w_n),$$

$$d(Tp,Tz_n) \le d(p,z_n),$$

$$d(Tp,Ty_n) \le d(p,y_n)$$

and

$$d(Tp,Tx_n) \le d(p,x_n).$$

By (3.1), we have

$$\begin{aligned} d(w_n,p) &= d(W(x_n,Tx_n,\gamma_n),p) \\ &= d((1-\gamma_n)x_n + \gamma_n Tx_n, p) \end{aligned}$$

$$\begin{aligned} &\le (1-\gamma_n)d(x_n,p)+\gamma_n d(Tx_n,p). \\ &\le d(x_n,p). \end{aligned} \tag{3.2}$$

From (3.2), we have

$$\begin{aligned} d(z_n,p) &= d(W(w_n,Tw_n,\beta_n),p) \\ &= d((1-\beta_n)w_n+\beta_n Tw_n,p) \\ &\le (1-\beta_n)d(w_n,p)+\beta_n d(Tw_n,p) \\ &\le d(w_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{3.3}$$

From (3.2), (3.3), we have

$$\begin{aligned} d(y_n,p) &= d(W(z_n,Tz_n,\alpha_n),p) \\ &= d((1-\alpha_n)z_n+\alpha_n Tz_n,p) \\ &\le (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\le d(z_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{3.4}$$

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} d(x_{n+1},p) &= d(W(Ty_n,0,0),p) \\ &= d(Ty_n,p) \\ &\le d(y_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{3.5}$$

Thus

$$d(x_{n+1},p)\le d(x_n,p) \quad \text{for all } p\in F(T).$$

Therefore, $\{x_n\}$ is Fejéer monotone with respect to $F(T)$. □

Lemma 3.2. *Let C be a nonempty convex closed subset of a hyperbolic space X with monotone modulus of convexity η and $T:C\to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda\in(0,1)$. If $\{x_n\}$ is a sequence defined by* (3.1)*, then $F(T)$ is nonempty if and only if the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,Tx_n)=0$.*

Theorem 3.3. *Let C be a nonempty convex closed subset of a hyperbolic space X with monotone modulus of convexity η and $T:C\to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda\in(0,1)$ and (E_μ) for some $\mu\ge 1$ on C with $F(T)\ne\emptyset$. If $\{x_n\}$ is a sequence defined by* (3.1)*, then sequence $\{x_n\}$ is Δ-converges to a fixed point of T.*

Proof. From Lemma 3.2, we observe that $\{x_n\}$ is a bounded sequence, therefore $\{x_n\}$ has a Δ-convergent subsequence. Now, we will prove that every Δ-convergent subsequence of $\{x_n\}$ has unique Δ-limit $F(T)$. For this, let u and v be Δ-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$, respectively. Now, by Lemma 2.10, $AC(C,\{u_n\})=\{u_n\}$ and $AC(C,\{v_n\})=\{v_n\}$. By Lemma 3.2, we have $\lim_{n\to\infty} d(u_n,Tu_n)=0$.

Now, we will prove u and v are fixed points of T and they are unique.

By Lemma 2.12, u and v are fixed points of T.

Now, we will show $u=v$. If not, then by the uniqueness of asymptotic center

$$\begin{aligned}\limsup_{n\to\infty} d(x_n,u) &= \limsup_{n\to\infty} d(u_n,u)\\ &< \limsup_{n\to\infty} d(u_n,v)\\ &= \limsup_{n\to\infty} d(x_n,v)\\ &= \limsup_{n\to\infty} d(v_n,v)\\ &< \limsup_{n\to\infty} d(v_n,u)\\ &= \limsup_{n\to\infty} d(x_n,u)\end{aligned}$$

which is a contradiction. Hence $u=v$, and sequence $\{x_n\}$ Δ-converges to a fixed point of T. □

Theorem 3.4. *Let C be a nonempty convex closed subset of a hyperbolic space X with monotone modulus of convexity η and $T:C\to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda\in(0,1)$ and E_μ for some $\mu\ge 1$ on C with $F(T)\ne\emptyset$. If $\{x_n\}$ is a sequence defined by (3.1), then sequence $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} D(x_n,F(T))=0$, where $D(x_n,F(T))=\inf_{x\in F(T)} d(x_n,x)$.*

Proof. It is easy to prove necessary part. We only prove sufficient part. First we show that the fixed point set $F(T)$ is closed, let $\{x_n\}$ be a sequence in $F(T)$ which converges to some fixed point $w\in C$. As

$$\lambda d(x_n,Tx_n)=0\le d(x_n,w).$$

By condition of C_λ, we have

$$d(x_n,Tw)=d(x_n,Tw)\le d(x_n,w).$$

On taking limit as $n\to\infty$ both the sides, we get

$$\lim_{n\to\infty} d(x_n,Tw)\le \lim_{n\to\infty} d(x_n,w)=0.$$

In view of the uniqueness of the limit, we have $w=Tw$, so that $F(T)$ is closed. Suppose

$$\liminf_{n\to\infty} D(x_n,F(T))=0.$$

From (3.5), we have

$$D(x_{n+1},F(T))\le D(x_n,F(T)).$$

From Lemma 3.1 and Proposition 2.14, we have $\lim_{n\to\infty} d(x_n,F(T))$ exists. Hence

$$\lim_{n\to\infty} D(x_n,F(T))=0.$$

Consider the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k},p_k)<\frac{1}{2^k}$ for all $k\ge 1$, where $\{p_k\}$ is in $F(T)$.

From Lemma 3.1, we have

$$d(x_{n_{k+1}},p_k) \le d(x_{n_k},p_k) < \frac{1}{2^k}$$

which implies that

$$\begin{aligned} d(p_{k+1},p_k) &\le d(p_{k+1},x_{n_{k+1}}) + d(x_{n_{k+1}},p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{p_k\}$ is a convergent sequence. Let $\lim_{k\to\infty} p_k = p$. Then we know that $\{x_n\}$ converges to p. Since

$$d(x_{n_k},p) \le d(x_{n_k},p_k) + d(p_k,p).$$

As $k \to \infty$, we have

$$\lim_{k\to\infty} d(x_{n_k},p) = 0.$$

Since $\lim_{n\to\infty} d(x_n,p)$ exists, therefore the sequence $\{x_n\}$ converges to p. □

4. Example

Consider the mapping $T:[0,1] \to [0,1]$ defined by

$$Tx = \begin{cases} \frac{1}{2} + 2x, & x \in \left[0,\frac{1}{6}\right); \\ 1 - \frac{x}{3}, & x \in \left[\frac{1}{6},1\right]. \end{cases} \tag{4.1}$$

Put $x = \frac{1}{6}$ and $y = \frac{6}{42}$, then $\|Tx - Ty\| = \frac{10}{21}$ and $\|x - y\| = \frac{1}{42}$. Thus, we have $\|Tx - Ty\| \ge \|x - y\|$. Therefore, T is not nonexpansive mapping. Now, we will check the mapping T satisfies Condition (C). For this we consider two cases.

Case I: Let $x \in [0,\frac{1}{6})$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{4}(2x+1)$. For $\frac{1}{2}\ \|x - Tx\| \le \|x - y\|$, we have $\frac{1}{4}(2x+1) \le y - x$, i.e. $y \ge \frac{3}{2} + \frac{1}{4}$. Hence $y \in \left(\frac{1}{2},1\right]$.
Now, for all $x \in \left[0,\frac{1}{6}\right)$ and $y \in \left(\frac{1}{2},1\right]$, we have

$$\|Tx - Ty\| = \left|\frac{12x + 2y - 3}{6}\right| < \frac{1}{6} \quad \text{and} \quad \|x - y\| = |x - y| > \frac{1}{4}.$$

Hence $\frac{1}{2}\|x - Tx\| \le \|x - y\| \to \|Tx - Ty\| \le \|x - y\|$. Thus T satisfies Condition (C).

Case II: Let $x \in \left[\frac{1}{6},1\right]$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\left(\frac{4x-3}{3}\right)$. For $\frac{1}{2}\|x - Tx\| \le \|x - y\|$, we have $\frac{1}{2}\left(\frac{4x-3}{3}\right) \le |x - y|$.

Again there are two possibilities:

A1: If $y > x$, then $\frac{1}{2}\left(\frac{4x-3}{3}\right) \le y - x$. Here $y \in \left[\frac{1}{6},1\right]$. Now, for $x, y \in \left[\frac{1}{6},1\right]$, we have $\|Tx - Ty\| \le \frac{1}{3}\|x - y\|$. Thus $\frac{1}{2}\|x - Tx\| \le \|x - y\| \to \|Tx - Ty\| \le \|x - y\|$. Thus T satisfies Condition (C).

A2: If $y < x$, then $\frac{1}{2}\left(\frac{4x-3}{3}\right) \le x - y$, i.e., $y \le \frac{2x+3}{6}$. Therefore, $y \in \left[\frac{5}{9},\frac{5}{6}\right]$. For all $x \in \left[\frac{1}{6},1\right]$ and $y \in \left[\frac{5}{9},\frac{5}{6}\right]$, we have $\|Tx - Ty\| \le \frac{1}{3}\|x - y\|$. Thus $\frac{1}{2}\|x - Tx\| \le \|x - y\| \to \|Tx - Ty\| \le \|x - y\|$. Thus T satisfies Condition (C).

In above example, we see that T is not nonexpansive but T satisfies Condition (C), i.e. T is Suzuki-generalized nonexpansive mapping.

5. Numerical Results

We compare the new iteration process, i.e., Generalized Picard Normal S-iteration process with Picard Normal S-iteration process, Normal S-iteration process, Mann iteration process and Picard iteration process. We set the stop parameter to $d(x_n,p) \le 10^{-14}$.

For above example, consider the following parameter:

PI: $\alpha_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{n}{(2n+7)^{\frac{1}{2}}}$, $\gamma_n = \frac{1}{(3n+5)^{\frac{1}{2}}}$.

Also, we compare the number of iterations for following set of parameters:

PII: $\alpha_n = \frac{1}{(n+11)^{1/8}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\gamma_n = \frac{1}{(n+1)^{\frac{2}{5}}}$.

PIII: $\alpha_n = \frac{n^2}{n^2+61}$, $\beta_n = \left(\frac{1}{2n+1}\right)^{\frac{2}{9}}$, $\gamma_n = \frac{n}{4n+3}$.

PIV: $\alpha_n = \left(\frac{n^2}{(n^3+1)}\right)^{\frac{1}{9}}$, $\beta_n = \frac{n}{2n+1}$, $\gamma_n = \left(\frac{1}{9n+2}\right)^{\frac{1}{5}}$.

Table 1. Comparison of iteration of different iteration process for initial value 0.1

Initial value	New iteration process	Picard Normal S-iteration	Normal S-iteration	Mann iteration	Picard iteration
1	0.74017268433791	0.73182071694926	0.79848738361592	0.60453784915223	0.7
2	0.74999496573282	0.74998442643975	0.75021195851703	0.75190762665326	0.76666666666667
3	0.74999999324209	0.74999996752023	0.74999595929642	0.75010909899660	0.74444444444444
4	0.74999999993773	0.74999999955770	0.75000014593171	0.75001182046826	0.75185185185185
5	0.74999999999867	0.74999999998653	0.74999999279701	0.75000175032601	0.74938271604938
6	0.74999999999995	0.74999999999932	0.75000000043286	0.75000031555308	0.75020576131687
7	0.75000000000000	0.74999999999995	0.74999999997010	0.75000006538110	0.74993141289438
8		0.75000000000000	0.75000000000229	0.75000001505071	0.75002286236854
9			0.74999999999981	0.75000000376586	0.74999237921049
10			0.75000000000002	0.75000000100875	0.75000254026317
11			0.75000000000000	0.75000000028610	0.74999915324561
12				0.75000000008520	0.75000028225146
13				0.75000000002647	0.74999990591618
14				0.75000000000854	0.75000003136127
15				0.75000000000285	0.74999998954624
16				0.75000000000098	0.75000000348459
17				0.75000000000034	0.74999999883847
18				0.75000000000012	0.75000000038718
19				0.75000000000005	0.74999999987094
20				0.75000000000002	0.75000000004302
21				0.75000000000001	0.74999999998566
22				0.75000000000000	0.75000000000478
23					0.74999999999841
24					0.75000000000053
25					0.74999999999982
26					0.75000000000006
27					0.74999999999998
28					0.75000000000001
29					0.75000000000000

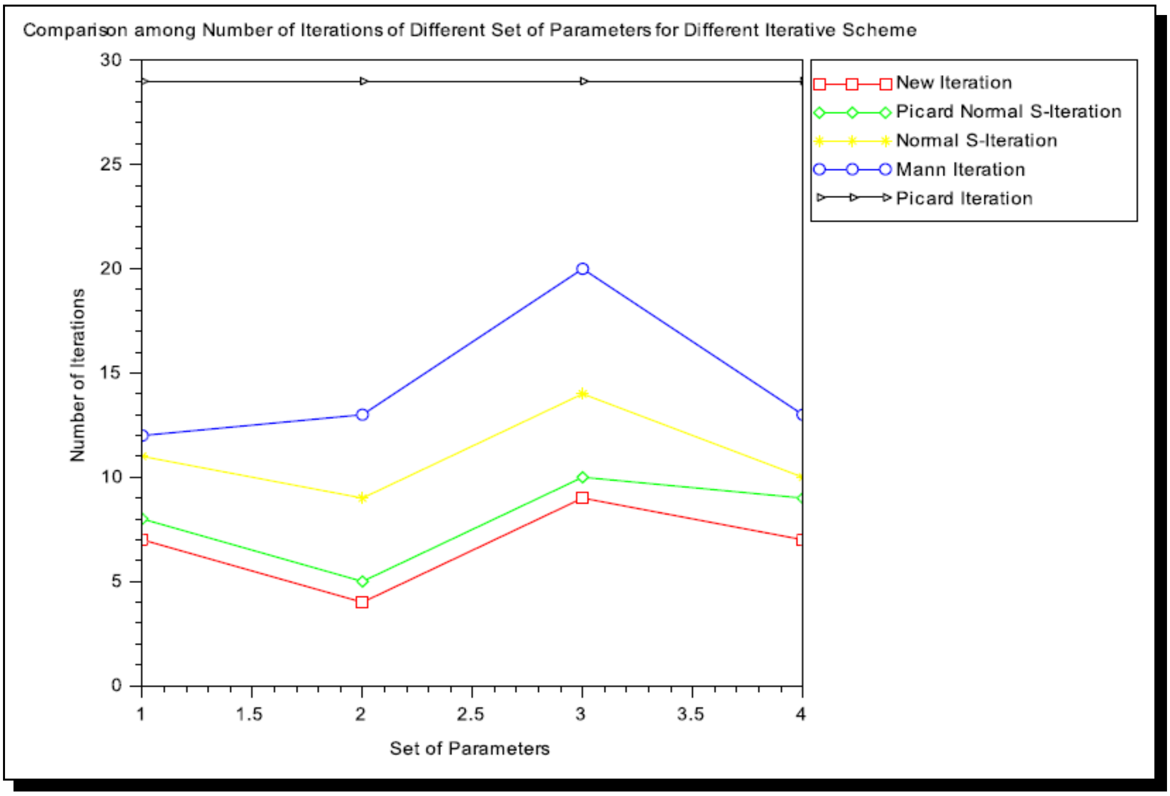

Figure 1. Comparison among number of iterations

Table 2. Comparison of number of iteration of different iteration process for different set of parameters

Initial Value	New iteration				Picard Normal S-iteration				Normal S-iteration				Mann iteration				Picard iteration			
	PI	PII	PIII	PIV	PI	PII	PIII	PIV	PI	PII	PIII	PIV	PI	PII	PIII	PIV	PI	PII	PIII	PIV
0.1	7	4	9	7	8	5	10	9	11	9	14	10	21	13	20	13	29	29	29	29
0.2	7	4	9	7	8	5	10	8	11	8	14	10	21	12	20	13	30	30	30	30
0.3	7	4	9	7	8	5	10	8	10	8	14	10	21	12	20	13	30	30	30	30
0.4	7	4	9	7	8	5	10	8	10	8	14	10	21	12	19	13	30	30	30	30
0.5	7	4	9	7	8	5	10	8	10	8	14	10	20	12	19	12	29	29	29	29
0.6	7	4	9	7	8	5	9	8	10	8	14	10	20	11	19	12	29	29	29	29
0.7	6	3	8	7	7	5	9	8	10	8	13	10	19	11	19	12	28	28	28	28
0.8	6	3	8	7	7	5	9	8	10	8	13	10	19	11	19	12	28	28	28	28
0.9	7	4	9	7	8	5	9	8	10	8	14	10	20	11	19	12	29	29	29	29
1.0	7	4	9	7	8	5	10	8	10	8	14	10	20	12	19	13	29	29	29	29

By the above results, i.e., Table 2 and Figure 1, it is clear that Generalized Picard Normal S-iteration process is faster than Picard Normal S-iteration process, Normal S-iteration process, Mann iteration process and Picard iteration process.

6. Conclusion

We introduced a new iteration process named as Generalized Picard Normal S-iteration process in Hyperbolic space. Also we proved the strong and Δ-convergence results for this iteration process. We presented an example of Suzuki-generalized nonexpansive mapping which is not nonexpansive mapping. With the help of numerical example, we showed that Generalized Picard Normal S-iteration process is faster than some famous iteration process such as Picard Normal S-iteration process, Normal S-iteration process, Mann iteration process and Picard iteration process.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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