Communications in Mathematics and Applications
Vol. 11, No. 3, pp. 357–364, 2020
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v11i3.1355

Research Article

Coincidence Point and Common Fixed Point Results in Cone Metric Spaces Under c-Distance

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Abstract. In this paper, by using the concept of c-distance in a cone metric space, we prove some coincidence point and common fixed point results with different type of contractive condition. Our results, extend, generalize and improve the corresponding results of Fadail *et al.* [8–11].

Keywords. Cone metric spaces; c-distance; Common fixed point; Coincidence point

MSC. 47H09; 47H10; 54H25

Received: January 18, 2020 **Accepted:** February 27, 2020

1. Introduction

The first important result on fixed points for contractive type mapping was the Banach's contractive principle by Banach in 1922. In 1976, Jungck [15], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. Jungck [14, 16] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. In 2007, Huang and Zhang [12] introduced the concept of cone metric space, which is a generalized version of metric spaces. Many fixed point theorems have been proved in normal or non-normal cone metric spaces by some authors (see [2], [13], [17], [1]).

In 2011, Cho *et al.* [3] introduced the concept of c-Distance in a cone metric spaces (also see [20]) and proved some fixed point theorems in ordered cone metric spaces. This was cone version of w-Distance of Kada *et al.* [18]. Then several authors have proved fixed point theorems for c-Distance in cone metric spaces (see [4–11, 19]).

In this paper, we extend and generalize the common fixed point theorem on c-Distance of Fadail *et al.* [8–11] and Dubey *et al.* [4, 6]. In our theorems, by replacing the constant numbers in the contractive condition with functions without assumption of normality for cones..

2. Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is the subset of E such that

(i) P is closed, non-empty, and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geqslant 0$; $x, y \in P$ and $\mathbb{R}$ as a set of real number $\Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that, for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leqslant K\|y\|$. The least positive number satisfying above is called the normal constant of P.

Definition 2.1 ([12])**.** Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(i) If $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 2.2. Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geqslant 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \propto \max\{|x_1 - y_1|, |x_2 - y_2|\})$, where $\propto \geqslant 0$ is a constant. Then (X, d) is cone metric space. It is easy to see that d is a cone metric and hence (X, d) becomes a cone metric space over (E, P). Also, we have P is a solid and normal cone where the normal constant $K = 1$.

Definition 2.3 ([12])**.** Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$.

(i) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$is said to be convergent and $\{x_n\}$ converges to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $(n \to \infty)$.

(ii) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent in X then (X, d) is called a complete cone metric space.

Lemma 2.4 ([17])**.** (1) *If E be a real Banach space with a cone P and $a \preceq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(2) *If* $c \in \operatorname{int}P$, $\theta \preceq a_n$ *and* $a_n \to \theta$, *then there exists a positive integer* N *such that* $a_n \ll c$ *for all* $n \geqslant N$.

Definition 2.5 ([3])**.** Let (X,d) be a cone metric space. A function $q : X \times X \to E$ is called a c-Distance on X if the following coditions hold:

(q1) $\theta \preceq q(x,y)$ for all $x,y \in X$,

(q2) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x,y,z \in X$,

(q3) for each $x \in X$ and $n \geq 1$ if $q(x,y_n) \preceq u$ for some $u = u_x \in P$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 2.6 ([3])**.** Let $E = \mathbb{R}$ and $P = \{x \in E : x \geqslant 0\}$, $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ is defined by $d(x,y) = |x-y|$, for all $x,y \in X$. Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y$ for all $x,y \in X$. Then q is a c-distance on X.

The following lemma is useful in our work.

Lemma 2.7 ([3])**.** *Let* (X,d) *be a cone metric space and* q *be a* c*-Distance on* X. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be a sequences in* X *and* $x,y,z \in X$. *Suppose that* $\{u_n\}$ *is a sequence in* P *converging to* θ. *Then the following hold:*

(1) *If* $q(x_n,y) \preceq u_n$ *and* $q(x_n,z) \preceq u_n$, *then* $y = z$.

(2) *If* $q(x_n,y_n) \preceq u_n$ *and* $q(x_n,z) \preceq u_n$, *then* $\{y_n\}$ *converges to* z.

(3) *If* $q(x_n,x_m) \preceq u_n$ *for* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

(4) *If* $q(y,x_n) \preceq u_n$ *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Remark 2.8 ([3])**.** (1) If $q(x,y) = q(y,x)$ does not necessarily for all $x,y \in X$.

(2) If $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

Definition 2.9. An element $x \in X$ is called:

(1) A coincidence point of mappings $f : X \to X$ and $g : X \to X$, if $w = gx = fx$ and w is called a point of coincidence.

(2) A common fixed pont of mappings $f : X \to X$ and $g : X \to X$ if $x = gx = fx$.

Definition 2.10. The mappings $f : X \to X$ and $g : X \to X$ are called weakly compatible if $gfx = fgx$ whenever $gx = fx$.

Now, we are ready to state and prove our main results.

3. Main Results

In our main theorems, the only assumptions are that, the mappings are weakly compatible and the cone P is solid, that is $\operatorname{int}P \neq \phi$.

Theorem 3.1. *Let (X,d) be a cone metric space over a solid cone P and q is a c-Distance on X. Let $S:X\to X$ and $T:X\to X$ be two self mappings and suppose there exists mappings $k,l,r,t:X\to[0,1)$ such that the following conditions hold:*

(a) $k(Sx)\le k(Tx)$, $l(Sx)\le l(Tx)$, $r(Sx)\le r(Tx)$ *and* $t(Sx)\le t(Tx)$ *for all* $x\in X$,

(b) $(k+l+r+2t)(x)<1$ *for all* $x\in X$,

(c) $q(Sx,Sy)\le k(Tx)q(Tx,Ty)+l(Tx)q(Tx,Sx)+r(Tx)q(Ty,Sy)+t(Tx)[q(Sx,Ty)+q(Sy,Tx)]$ *for all* $x,y\in X$.

If $S(X)\subseteq T(X)$ and $T(X)$ is a complete subspace of X, then S and T have a coincidence point x^* in X. Further, if $w=Tx^*=Sx^*$ then $q(w,w)=\theta$. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $Tx_1=Sx_0$. This can be done because $S(X)\subseteq T(X)$. Continuing this process we obtain a sequence $\{x_n\}$ in X such that $Tx_{n+1}=Sx_n$. Then, we have

$$\begin{aligned}
q(Tx_n,Tx_{n+1}) &= q(Sx_{n-1},Sx_n)\\
&\le k(Tx_{n-1})q(Tx_{n-1},Tx_n)+l(Tx_{n-1})q(Tx_{n-1},Sx_{n-1})\\
&\quad +r(Tx_{n-1})q(Tx_n,Sx_n)+t(Tx_{n-1})[q(Sx_{n-1},Tx_n)+q(Sx_n,Tx_{n-1})]\\
&= k(Sx_{n-2})q(Tx_{n-1},Tx_n)+l(Sx_{n-2})q(Tx_{n-1},Tx_n)\\
&\quad +r(Sx_{n-2})q(Tx_n,Tx_{n+1})+t(Sx_{n-2})[q(Tx_{n+1},Tx_{n-1})]\\
&\le k(Tx_{n-2})q(Tx_{n-1},Tx_n)+l(Tx_{n-2})q(Tx_{n-1},Tx_n)\\
&\quad +r(Tx_{n-2})q(Tx_n,Tx_{n+1})+t(Tx_{n-2})[q(Tx_{n-1},Tx_n)+q(Tx_n,Tx_{n+1})]\\
&\ \ \vdots\\
&\le k(Tx_0)q(Tx_{n-1},Tx_n)+l(Tx_0)q(Tx_{n-1},Tx_n)\\
&\quad +r(Tx_0)q(Tx_n,Tx_{n+1})+t(Tx_0)q[q(Tx_{n-1},Tx_n)+q(Tx_n,Tx_{n+1})]\\
&\le \frac{k(Tx_0)+l(Tx_0)+t(Tx_0)}{1-r(Tx_0)-t(Tx_0)}q(Tx_{n-1},Tx_n)\\
&= \mu q(Tx_{n-1},Tx_n)\le \mu^2 q(Tx_{n-2},Tx_{n-1})\\
&\le\\
&\ \ \vdots\\
&\le \mu^n q(Tx_0,Tx_1),
\end{aligned}$$

where $\mu=\dfrac{k(Tx_0)+l(Tx_0)+t(Tx_0)}{1-r(Tx_0)-t(Tx_0)}<1$.

Note that,

$$q(Tx_n,Tx_{n+1})=q(Sx_{n-1},Sx_n)\le \mu q(Tx_{n-1},Tx_n). \tag{3.1}$$

Let $m>n\ge 1$. Then it follows that

$$q(Tx_n,Tx_m)\le q(Tx_n,Tx_{n+1})+q(Tx_{n+1},Tx_{n+2})+\cdots+q(Tx_{m-1},Tx_m)$$

$$\begin{aligned}&\preceq (\mu^n+\mu^{n+1}+\cdots+\mu^{m-1})q(Tx_0,Tx_1)\\&\preceq \frac{\mu^n}{1-\mu}q(Tx_0,Tx_1).\end{aligned}$$

Thus, Lemma 2.7(3) shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since $T(X)$ is complete, there exists $x^*\in X$ such that $Tx_n\to Tx^*$ as $n\to\infty$. By Definition 2.5(q_3), we have

$$q(Tx_n,Tx^*)\preceq \frac{\mu^n}{1-\mu}q(Tx_0,Tx_1). \tag{3.2}$$

Now, by using (3.1), we have

$$\begin{aligned}q(Tx_n,Sx^*)=q(Sx_{n-1},Sx^*)&\preceq \mu q(Tx_{n-1},Tx^*)\\&\preceq \mu\frac{\mu^{n-1}}{1-\mu}q(Tx_0,Tx_1)\\&=\frac{\mu^n}{1-\mu}q(Tx_0,Tx_1).\end{aligned} \tag{3.3}$$

Thus, Lemma 2.7(1), (3.2) and (3.3) show that $Tx^*=Sx^*$.

Therefore, x^* is a coincidence point of S and T and w is a point of coincidence of S and T where $w=Tx^*=Sx^*$ for some x^* in X.

Suppose that $w=Tx^*=Sx^*$. Then, we have

$$\begin{aligned}q(w,w)&=q(Sx^*,Sx^*)\\&\preceq k(Tx^*)q(Tx^*,Tx^*)+l(Tx^*)q(Tx^*,Sx^*)\\&\quad+r(Tx^*)q(Tx^*,Sx^*)+t(Tx^*)[q(Sx^*,Tx^*)+q(Sx^*,Tx^*)]\\&=k(w)q(w,w)+l(w)q(w,w)+r(w)q(w,w)+t(w)[q(w,w)+q(w,w)]\\&=(k+l+r+2t)(w)q(w,w).\end{aligned}$$

Since $(k+l+r+2t)(w)<1$, Lemma 2.4(1) shows that $q(w,w)=\theta$.

Finally, suppose there is another point of coincidence u of S and T such that $u=Sy^*=Ty^*$ for some y^* in X. Then, we have

$$\begin{aligned}q(w,u)&=q(Sx^*,Sy^*)\\&\preceq k(Tx^*)q(Tx^*,Ty^*)+l(Tx^*)q(Tx^*,Sx^*)\\&\quad+r(Tx^*)q(Ty^*,Sy^*)+t(Tx^*)[q(Sx^*,Ty^*)+q(Sy^*,Tx^*)]\\&=k(w)q(w,u)+l(w)q(w,w)+r(w)q(u,u)+t(w)[q(w,u)+q(u,w)]\\&=(k+2t)(w)q(w,u).\end{aligned}$$

Since $(k+2t)(w)<1$, Lemma 2.4(1) shows that $q(w,u)=\theta$.

Also, we have $q(w,w)=\theta$. Thus, Lemma 2.7(1) shows that $w=u$. Therefore, w is the unique point of coincidence.

Now, let $w=Tx^*=Sx^*$. Since S and T are weakly compatible, we have

$$Tw=TTx^*=TSx^*=STx^*=Sw.$$

Hence Tw is a point of coincidence. The uniqueness of the point of coincidence implies that $Tw=Tx^*$. Therefore, $w=Tw=Sw$. Hence, w is the unique common fixed point of S and T. □

We have the following result (immediate consequence of Theorem 3.1).

Theorem 3.2. *Let* (X,d) *be a complete cone metric space over a solid cone* P *and* q *is a* c*-Distance on* X*. Let* $S:X\to X$ *be a self mapping and suppose there exists mappings* $k,l,r,t:X\to[0,1)$ *such that the following conditions hold:*

(a) $k(Sx)\le k(x), l(Sx)\le l(x), r(Sx)\le r(x)$ *and* $t(Sx)\le t(x)$ *for all* $x\in X$,

(b) $(k+l+r+2t)(x)<1$ *for all* $x\in X$,

(c) $q(Sx,Sy)\preceq k(x)q(x,y)+l(x)q(x,Sx)+r(x)q(y,Sy)+t(x)[q(Sx,y)+q(Sy,x)]$ *for all* $x,y\in X$.

Then S *has a fixed point point* x^* *in* X*. Further, if* $v=Sv$ *then* $q(v,v)=\theta$*. The fixed point is unique.*

Theorem 3.3. *Let* (X,d) *be a cone metric space over a solid cone* P *and* q *is a* c*-Distance on* X*. Let* $S:X\to X$ *and* $T:X\to X$ *be two self mappings and suppose there exists mappings* $k,l,r:X\to[0,1)$ *such that the following conditions hold:*

(a) $k(Sx)\le k(Tx)$, $l(Sx)\le l(Tx)$, $r(Sx)\le r(Tx)$ *for all* $x\in X$,

(b) $(k+2l+2r)(x)<1$ *for all* $x\in X$,

(c) $q(Sx,Sy)\preceq k(Tx)q(Tx,Ty)+l(Tx)[q(Tx,Sy)+q(Ty,Sx)]+r(Tx)[q(Tx,Sx)+q(Ty,Sy)]$, *for all* $x,y\in X$.

If $S(X)\subseteq T(X)$ *and* $T(X)$ *is a complete subspace of* X*, then* S *and* T *have a coincidence point* x^* *in* X*. Further, if* $w=Tx^*=Sx^*$ *then* $q(w,w)=\theta$*. Moreover, if* S *and* T *are weakly compatible, then* S *and* T *have a unique common fixed point.*

Proof. The proof of this theorem is similar as Theorem 3.1. □

The following corollary can be obtained as consequences of Theorem 3.1 and Theorem 3.3.

Corollary 3.4. *Let* (X,d) *be a cone metric space over a solid cone* P *and* q *is a* c*-Distance on* X*. Let* $S:X\to X$ *and* $T:X\to X$ *be two self mappings and suppose there exists mappings* $k:X\to[0,1)$ *such that the following conditions hold:*

(a) $k(Sx)\le k(Tx)$ *for all* $x\in X$,

(b) $q(Sx,Sy)\preceq k(Tx)q(Tx,Ty)$ *for all* $x,y\in X$.

If $S(X)\subseteq T(X)$ *and* $T(X)$ *is a complete subspace of* X*, then* S *and* T *have a coincidence point* x^* *in* X*. Further, if* $w=Tx^*=Sx^*$ *then* $q(w,w)=\theta$*. Moreover, if* S *and* T *are weakly compatible, then* S *and* T *have a unique common fixed point.*

4. Conclusion

In this attempt, we have proved some coincidence point and common fixed point results in cone metric spaces under c-distance. These results generalizes and improves the recent results of Fadail *et al*. [8–11] and Dubey *et al*. [4, 6] in the sense that in our results, we are employing c-distance and in contractive conditions, replacing the constants with functions, which extends the further scope of our results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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