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Research Article

On Eigenvalues of Hermitian-Adjacency Matrix

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Abstract. The graph of Hermitian-adjacency matrix is a mixed graph consisting adjacency matrix of an undirected graph and skew-adjacency matrix of a digraph. In this paper we discuss eigenvalues of Hermitian-adjacency matrix. Then we use the eigenvalues to determine the possible Hamiltonian cycles of its graph.

Keywords. Eigenvalues; Hermitian-adjacency matrix; Mixed graph; Hamiltonian cycle

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1. Introduction

Liu and Li [13] introduced the concept of Hermitian-adjacency matrix of a mixed graph G of order n. The graph of Hermitian-adjacency matrix is a mixed graph without multiple edges and loops consisting adjacency matrix of an undirected graph $A(G)$ and skew-adjacency matrix of a digraph $S(G^\sigma)$ [2]. An undirected graph $G=(V,E)$ is an ordered pair consisting of a set of vertices $V=\{v_1,v_2,\dots,v_n\}$ and a set of undirected edges $E=\{e_1,e_2,\dots,e_n\}$, no loops nor multiple edges permitted [14]. A directed graph or digraph is a graph that contains only set of directed arcs with the set of vertices $V=\{v_1,v_2,\dots,v_n\}$ [9]. While a mixed graph $G=(V,E,A)$

is an ordered triple consisting of a set of vertices $V = v_1, v_2, \dots, v_n$, a set of undirected edges $E = \{e_1, e_2, \dots, e_n\}$, and a set of directed arcs A [3]. A mixed graph is the orientation of subset of the undirected graph and its edge set is the union between the sets of arcs and undirected edges [4, 19]. An orientation is an assignment of exactly one direction to each of the undirected edges of G to become a directed graph. Skew-adjacency matrix $S(G^\sigma) = (s_{i,j})$ is real skew symmetric matrix, where $s_{i,j} = 1$ and $m_{j,i} = -1$ if $i \to j$, is an arc of G^σ, otherwise $s_{i,j} = s_{j,i} = 0$ [12]. A mixed graph is said to be a mixed cycle if its underlying graph is a cycle graph [17]. A Graph G is a mixed cycle graph such that the directed edges (arcs) are not oriented in the same direction [7]. A cycle graph is a closed circular connected vertices. Thus, Hamiltonian graph is a Hamiltonian cycle (Hamiltonian circuit) which has a closed Hamiltonian path. Hamiltonian path is a path that visits each vertex exactly once in an undirected, directed or mixed graph [14]. It is difficult to find Hamiltonian cycle of a graph in a polynomial time and this results to NP-complete problem. However, exponential time algorithms can be used such as brute force search and dynamic programming.

Mixed adjacency matrix $M(G) = (m_{i,j})$ of a mixed graph G is defined as an $n \times n$ matrix indexed by the vertices $(v_1, \dots, v_n)$, where $m_{i,j} = 1$ if $v_i v_j \in E$, $m_{i,j} = -1$ if $v_i, v_j \in A$, and $m_{i,j} = 0$ otherwise [9]. Let G be a mixed graph of order n with its mixed adjacency matrix $M(G)$. Then

$$\det(M(G)) = \sum_{H \in H_n} (-1)^{n+e(H)} 2^{p_o(H)} (-2)^{p_e(H)},$$

where H_n, $e(H)$, $p_o(H)$ and $p_e(H)$ are respectively, the collection of all linear subgraphs of order n having no mixed cycles with odd number of arcs, the number of edges in H, the number of mixed cycles in H that are oddly oriented and the number of mixed cycles in H that are evenly oriented [7]. Now, let G be a mixed graph of order n with its mixed adjacency matrix $M(G)$, the characteristics polynomial of mixed adjacency matrix $M(G)$ is given as $P(G, \lambda) = det(\lambda \mathbb{I} - M(G))$. The spectrum of a mixed graph is the spectrum of its mixed adjacency matrix that consists of λ_i; $i = 1, 2, \dots, n$, where $\lambda_i(G)$ is the spectrum of a mixed graph G, see [11, 18]. The graph energy of an undirected graph has been known for many years with its concept originated from theoretical chemistry [10]. Thus, the mixed energy $\mathscr{E}_M(G)$ of a mixed graph G is the sum of absolute values of eigenvalues of the adjacency matrix of the graph defined as

$$\mathscr{E}_M(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

where $\lambda_i(G)$ is the spectrum of a mixed graph G [6]. While the concept of skew-energy of a digraph was coined by [1] defined as $\mathscr{E}_S(G) = \sum_{i=1}^{n} |\rho_i S(G)|$, where $\rho_i S(G)$ is the spectrum of a mixed graph $S(G)$. Hermitian energies, Hermitian Laplacian matrix and Hermitian-Zagreb matrix are detailed in [5, 8, 9, 16]. Our aim is to discuss the unique property of eigenvalues of Hermitian-adjacency matrix.

2. Eigenvalues of Hermitian-adjacency Matrix

Hermitian-adjacency matrix of a mixed graph G of order n, denoted as $H(G)$, and it is defined as $H(G) = (h_{ij})_n \times n$, where

$$h_{ij} = \begin{cases} 1 & \text{if } v_iv_j \text{ is an edge;} \\ i & \text{if } (v_i, v_j) \text{ is an arc;} \\ -i & \text{if } (v_j, v_i) \text{ is an arc;} \\ 0 & \text{otherwise.} \end{cases}$$

For $i = \sqrt{-1}$ [15]. If $H(G)$ coincides with $M(G)$, then the we define its mixed adjacency matrix indexed by the vertices $(v_1, v_2, \dots, v_n)$ of mixed graph G to be an $n \times n$ matrix $M(G) = (m_{i,j})_n \times n$, where

$$m_{i,j} = \begin{cases} 1 & \text{if } v_iv_j \text{ is an edge;} \\ 1 & \text{if } (v_i, v_j) \text{ is an arc;} \\ -1 & \text{if } (v_j, v_i) \text{ is an arc;} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $H(G) = A(G) + iS(G^\sigma)$, where i is an imaginary number unit [2]. The spectrum of $H(G)$ has real eigenvalues, where its singular values coincide with absolute values of its eigenvalues. The sum of eigenvalues of $H(G)$ forms its energy, denoted as $\mathscr{E}_H(G)$.

Before we proceed to our objectives, we give three examples of Hermitian-adjacency matrix with their determinants, characteristic polynomials, eigenvalues and Hermitian energies, and their graphs as depicted in Figure 1.

1. Let $H(G)$ be the Hermitian-adjacency matrix of order 3 given as

$$H(G) = \begin{pmatrix} 0 & i & -i \\ -i & 0 & 1 \\ i & 1 & 0 \end{pmatrix}$$

   The determinant of $H(G)$ is -2, characteristics polynomial $P(G,\lambda)$ is $-\lambda^3 + 3\lambda - 2$, the eigenvalues are $\{1, 1, -2\}$ and its mixed energy $\mathscr{E}_H(G)$ is 4.

2. Let $H(G)$ be the Hermitian-adjacency matrix of order 4 given as

$$H(\mathscr{G}) = \begin{pmatrix} 0 & i & i & 1 \\ -i & 0 & 1 & 0 \\ -i & 1 & 0 & i \\ 1 & 0 & -i & 0 \end{pmatrix}$$

   The determinant of $H(G)$ is 4, characteristics polynomial $P(G,\lambda)$ is $\lambda^4 - 5\lambda^2 + 4$, the eigenvalues are $\{-1, 1, -2, 2\}$ and its mixed energy $\mathscr{E}_H(G)$ is 6.

3. Let $H(G)$ be the Hermitian-adjacency matrix of order 5 given as

$$H(G) = \begin{pmatrix} 0 & i & 0 & 1 & i \\ -i & 0 & -i & i & 1 \\ 0 & i & 0 & 1 & -i \\ 1 & -i & 1 & 0 & 0 \\ -i & 1 & i & 0 & 0 \end{pmatrix}$$

The determinant of $H(G)$ is 8, characteristics polynomial $P(G,\lambda)$ is $-\lambda^5+8\lambda^3-4\lambda^2-12\lambda+8$, the eigenvalues are $\{2,\sqrt{2},-\sqrt{2},-\sqrt{3}-1,\sqrt{3}-1\}$ and its mixed energy $\mathscr{E}_H(G)$ is $4+2(\sqrt{2}+\sqrt{3})$.

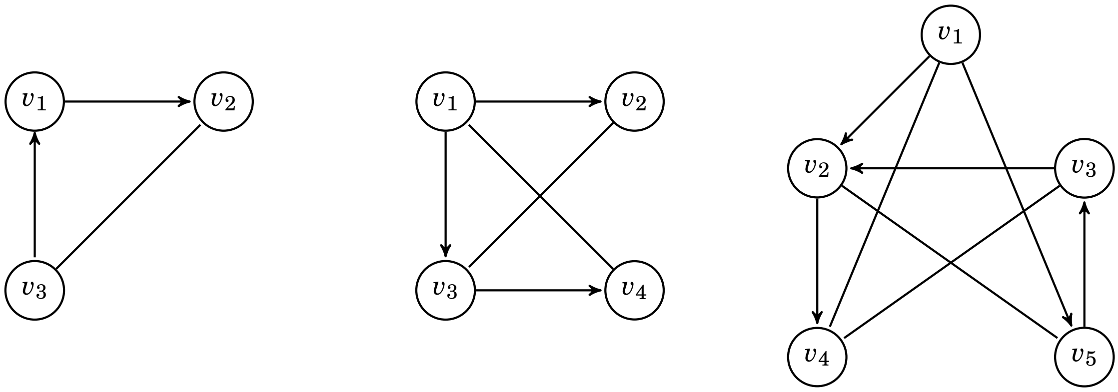

Figure 1. Hermitian graph of order 3, 4 and 5, respectively

The total number of Hamiltonian cycles in the Hermitian graph of order 3, 4 and 5 above are 1, 2 and 4.

Proposition 2.1. *Let λ_i be the eigenvalues of Hermitian graph G. If there exists Hamiltonian cycle in G, then the number of all possible Hamiltonian cycles in G is*

$$\frac{\prod_{i=1}^{n}|\lambda_i|}{2}.$$

Proof. The beauty of Hermitian graph is that it comprises undirected edges and arcs which still allows Hamiltonian cycle in the graph. Since $\lambda_i=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ are real numbers and $|\lambda_i|>0$ for $i=1,2,\dots,n$ (see [9]), it is clear that the product of all eigenvalues gives the determinant of $H(G)$. The entries in $H(G)$ determine the structure of its graph because $H(G)$ consists of 0, 1 and $\pm i$. Hence, the cycle in G may not evenly distributed to each vertex of the graph. Furthermore, no vertex should be repeated in a Hamiltonian cycle and each vertex is covered. The product of the absolute value of eigenvalues in G gives the the possible cycles, vertex repetition inclusive. To avoid vertex repetition, we divide the product of eigenvalues by 2. □

Corollary 2.2. *Let λ_i be the eigenvalues of a Hermitian graph G. and $T_{(e/a)}$ be the total number of undirected edges and arcs in G. Then*

$$T_{(e/a)}=\frac{\sum_{i=1}^{n}{\lambda_i}^2}{2},$$

where $\lambda_i=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$, for $i=1,2,\dots,n$.

Proof. Hermitian graph consists of undirected edges and arcs where the number of arcs is more than the number of undirected edges. This happens because $H(G)$ has more imaginary units

$(\pm i)$ than 1. As Proposition 2.1 posited that the entries in $H(G)$ determine the structure and the eigenvalues of G, the square of each eigenvalue is greater than or equal to 1. The sums of the squares of Hermitian eigenvalues is always even which is divisible by 2. □

Corollary 2.3. *If λ_i is the eigenvalues of a mixed Hermitian graph G, then*

$$\sum_{i=1}^{n} |\lambda_i| + \prod_{i=1}^{n} |\lambda_i| > 0.$$

Proof. Since λ_i contains at least three eigenvalues (distinct or not distinct) based on the least order of Hermitian-adjacency matrix and G has cycle, for $i = 1, 2, \dots n$. If $\lambda_i \neq 0$ then $\sum\limits_{i=1}^{n} \lambda_i = 0$ such that $\sum\limits_{i=1}^{n} |\lambda_i| \neq 0$, $\lambda_i \in \mathbb{R}$. Now, recall that determinant of Hermitian-adjacency matrix is nonsingular because $\prod\limits_{i=1}^{n} |\lambda_i| \neq 0$. It is obvious that $|\lambda_i| \geq 1$ and this makes $\sum\limits_{i=1}^{n} |\lambda_i| + \prod\limits_{i=1}^{n} |\lambda_i| \not< 0$. □

3. Conclusion

Hermitian-adjacency matrix provides unique eigenvalues for the Hermitian graph. Then we deduced the gap between eigenvalues of Hermitian-adjacency matrix and number of cycles in the graph, and the number of undirected edges and directed edges for every Hermitian graph.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Adiga, B. Rakshith and W. So, The skew energy of a digraph, *Linear Algebra Appl.* **432**(7) (2010), 1825 – 1835, DOI: 10.1016/j.laa.2009.11.034.

[2] C. Adiga, B. Rakshith and W. So, On the mixed adjacency matrix of a mixed graph, *Linear Algebra Appl.* **495** (2016), 223 – 241, DOI: 10.1016/j.laa.2016.01.033.

[3] S. Arumugam, A. Brandstädt, T. Nishizeki and K. Thulasiraman, *Handbook of Graph Theory, Combinatorial Optimization, and Algorithms*, Chapman and Hall/CRC (2016).

[4] O. Babarinsa and H. Kamarulhaili, Mixed energy of a mixed hourglass graph, *Communications in Mathematics and Applications* **10**(1) (2019), 45 – 53, DOI: 10.26713/cma.v10i1.1143.

[5] A. Bharali, On first Hermitian-Zagreb matrix and Hermitian-Zagreb energy, *International Journal of Scientific Research in Mathematical and Statistical Sciences* **5**(3) (2018), 136 – 139, `https://www.isroset.org/pub_paper/IJSRMSS/21-IJSRMSS-0910.pdf`.

[6] P. G. Bhat and S. D'Souza, Color signless Laplacian energy of graphs, *AKCE International Journal of Graphs and Combinatorics* **14**(2) (2017), 142 – 148, DOI: 10.1016/j.akcej.2017.02.003.

[7] C. Adiga, B. Rakshith and W. So, On the mixed adjacency matrix of a mixed graph, *Linear Algebra and its Applications* **495** (2016), 223 – 241, DOI: 10.1016/j.laa.2016.01.033.

[8] X. Chen, X. Li and Y. Zhang, 3-regular mixed graphs with optimum Hermitian energy, *Linear Algebra and its Applications* **496** (2016), 475 – 486, DOI: 10.1016/j.laa.2016.02.012.

[9] K. Guo and B. Mohar, Hermitian-adjacency matrix of digraphs and mixed graphs, *Journal of Graph Theory* **85**(1) (2017), 217 – 248, DOI: 10.1002/jgt.22057.

[10] I. Gutman and F. Boris, Survey of graph energies, *Mathematics Interdisciplinary Research* **2** (2017), 85 – 129, DOI: 10.22052/mir.2017.81507.1057.

[11] I. Gutman and B. Zhou, Laplacian energy of a graph, *Linear Algebra and its Applications* **414** (2006), 29 – 37, DOI: 10.1016/j.laa.2005.09.008.

[12] Y. Hou and T. Lei, Characteristic polynomials of skew-adjacency matrices of oriented graphs, *The Electronic Journal of Combinatorics* **18**(1) (2011), 156 – 167, DOI: 10.37236/643.

[13] J. Liu and X. Li, Hermitian-adjacency matrices and hermitian energies of mixed graphs, *Linear Algebra and its Applications* **466** (2015), 182 – 207, DOI: 10.1016/j.laa.2014.10.028.

[14] K. Rosen and K. Krithivasan, *Discrete Mathematics and Its Applications*, McGraw-Hill Education, Singapore (2015).

[15] F. Tian and D. Wong, Nullity of Hermitian-adjacency matrices of mixed graphs, *Journal of Mathematical Research with Applications* **38**(1) (2018), 23 – 33, `http://jmre.ijournals.cn/en/ch/reader/view_abstract.aspx?file_no=20180102&flag=1`.

[16] G. Yu and H. Qu, Hermitian laplacian matrix and positive of mixed graphs, *Applied Mathematics and Computation* **269** (2015), 70 – 76, DOI: 10.1016/j.amc.2015.07.045.

[17] G. Yu, X. Liu and H. Qu, Singularity of Hermitian (quasi-) Laplacian matrix of mixed graphs, *Applied Mathematics and Computation* **293** (2017), 287 – 292, DOI: 10.1016/j.amc.2016.08.032.

[18] J. Zhang, D. Xiao and R. Luo, The Laplacian eigenvalues of mixed graphs, *Linear Algebra and its Applications* **362** (2003), 109 – 119, DOI: 10.1016/S0024-3795(02)00509-8.

[19] J. Zhang and H. Kan, On the minimal energy of graphs, *Linear Algebra and its Applications* **453** (2014), 141 – 153, DOI: 10.1016/j.laa.2014.04.009.