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Research Article

All 2-potent Elements in $Hyp_G(2)$

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Abstract. A generalized hypersubstitution of type $\tau = (2)$ is a function which takes the binary operation symbol f to the term $\sigma(f)$ which does not necessarily preserve the arity. Let $Hyp_G(2)$ be the set of all these generalized hypersubstitutions of type (2). The set $Hyp_G(2)$ with a binary operation and the identity generalized hypersubstitution forms a monoid. The *index* and *period* of an element a of a finite semigroup are the smallest values of $m \geq 1$ and $r \geq 1$ such that $a^{m+r} = a^m$. An element with the index m and period 1 is called an m-potent element. In this paper we determine all 2-potent elements in $Hyp_G(2)$.

Keywords. Generalized hypersubstitution; m-potent elements; 2-potent elements

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1. Introduction

The concept of a hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [2]. They used it as the tool to study hyperidentities and solid varieties. To recall the definition of a hypersubstitution, we recall first the concept of terms. Let $n \in \mathbb{N}$ and $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-elements set. The set X_n is called an *alphabet* and its elements are called *variables*. Let $\tau = (n_i)_{i \in I}$ be a type such that the set of operation symbols $\{f_i \mid i \in I\}$ is disjoint with X_n. An *n-ary term* of type τ is defined inductively as the following steps.

(i) Every variable $x_i \in X_n$ is an *n-ary* term of type τ.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are *n-ary* terms of type τ and f_i is an n_i*-ary* operation symbol of type τ, then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an *n-ary* term of type τ.

The set $W_\tau(X_n)$ of all *n-ary* terms of type τ is the smallest set containing $x_1, x_2, \ldots, x_n$ that is closed under finite application of (2). The set of all terms of type τ over the alphabet $X := \{x_1, x_2, \ldots\}$ is defined as $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$. For any $t \in W_\tau(X)$, the set of all variables occurring in the term t is denoted by $\text{var}(t)$.

A hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ where $\sigma(f_i) \in W_\tau(X_{n_i})$. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ.

Every $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ as the following steps:
For any $t \in W_\tau(X)$, $\hat{\sigma}[t]$ is inductively defined by

(i) if $t = x \in X$ then $\hat{\sigma}[x] := x$,

(ii) if $t = f_i(t_1, \ldots, t_{n_i})$ then $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$,

where $S_n^{n_i} : W_\tau(X_{n_i}) \times W_\tau(X_n)^{n_i} \to W_\tau(X_n)$ is defined inductively as follows:

(i) $S_n^{n_i}(x_j, t_1, \ldots, t_{n_i}) := t_j$, $x_j \in X_{n_i}$, $t_1, \ldots, t_{n_i} \in W_\tau(X_n)$,

(ii) $S_n^{n_i}(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_{n_i}) := f_i(S_n^{n_i}(s_1, t_1, \ldots, t_{n_i}), \ldots, S_n^{n_i}(s_{n_i}, t_1, \ldots, t_{n_i}))$.

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp(\tau)} := (Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

In 2000, S. Leeratanavalee and K. Denecke [4] generalized the concept of a hypersubstitution to the concept of a generalized hypersubstitution. They used it as a tool to study strong hyperidentities and used strong hyperidentities to classify varieties into collections called strong hypervarieties. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called strongly solid.

Let $\tau = (n_i)_{i \in I}$ be a type with the sequence of operation symbols $(f_i)_{i \in I}$. A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. Firstly, we define inductively the concept of generalized superposition of term $S^m : W_\tau(X) \times W_\tau(X)^m \to W_\tau(X)$ by the following steps:
For each $t_1, \ldots, t_m \in W_\tau(X)$,

(i) $S^m(x_j, t_1, \ldots, t_m) := t_j$ when $1 \le j \le m$,

(ii) $S^m(x_j, t_1, \ldots, t_m) := x_j$ when $m < j$,

(iii) $S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

To define a binary operation on $Hyp_G(\tau)$, we extend a generalized hypersubstitution σ to a mappimg $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as the following steps.

(i) If $t = x \in X$ then $\hat{\sigma}[x] := x$.

(ii) If $t = f_i(t_1,\dots,t_{n_i})$ then $\hat{\sigma}[f_i(t_1,\dots,t_{n_i})] := S^{n_i}(\sigma(f_i),\hat{\sigma}[t_1],\dots,\hat{\sigma}[t_{n_i}])$.

We defined a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma \circ_G \alpha := \hat{\sigma}\circ\alpha$ where $\sigma,\alpha \in Hyp_G(\tau)$ and $\circ$ denotes the usual composition of mappings.

Proposition 1.1 ([3])**.** *For arbitrary terms $t,t_1,\dots,t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions σ,β we have*

(i) $\hat{\sigma}[S^n(t,t_1,\dots,t_n)] = S^n(\hat{\sigma}[t],\hat{\sigma}[t_1],\dots,\hat{\sigma}[t_n])$.

(ii) $(\hat{\sigma}\circ\beta)\hat{} = \hat{\sigma}\circ\hat{\beta}$.

Proposition 1.2 ([3])**.** *$\underline{Hyp_G(\tau)} := (Hyp_G(\tau),\circ_G,\sigma_{id})$ is a monoid where σ_{id} is the identity element, which map each n_i-ary operation symbol f_i to the term $f_i(x_1,\dots,x_{n_i})$.*

In this paper, we characterize all 2-potent elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

2. All 2-potent Elements in $Hyp_G(2)$

In 2008, all idempotent elements of $Hyp_G(2)$ were characterized by W. Puninagool and S. Leeratanavalee [6], based on the concept of orders of elements on a semigroup. In this section, we characterize all 2-potent elements in $Hyp_G(2)$. The concept of m-potent element of any given semigroup was introduced by G. Ayik *et al*. [1].

Definition 2.1 ([1])**.** An element a of any given semigroup is called *m-potent* if

(i) a^{m+1}=a^m,

(ii) $a,a^2,\dots,a^m$ are all distinct.

In particular, we refer to idempotent as 1-potent.

The order of an element a of a semigroup S is defined as the order of the cyclic subsemigroup $\langle a\rangle$. The order of any hypersubstitution of type $\tau = (2)$ was determined in [2]. An element a in a semigroup S is idempotent if and only if the order of a is 1. To characterize all 2-potent elements of $Hyp_G(2)$, we consider only the generalized hypersubstitutions of type $\tau = (2)$ which are not idempotent and has order 2. We do not consider elements in $Hyp_G(2)$ which its orders are infinite since they are not m-potent. We use the following theorems and propositions to obtain our results.

First, we introduce some notations which will be used throughout this paper. For $s \in W_{(2)}(X)$, we denote

leftmost(s) := the first variable (from the left) that occurs in s;

rightmost(s) := the last variable that occurs in s;

$W^G_{(2)}(\{x_1\}) := \{s \in W_{(2)}(X) \mid x_1 \in \text{var}(s),\ x_2 \notin \text{var}(s)\}$;

$W^G_{(2)}(\{x_2\}) := \{s \in W_{(2)}(X) \mid x_2 \in \text{var}(s),\ x_1 \notin \text{var}(s)\}$.

Proposition 2.2 ([6]). *Let $s \in W^G_{(2)}(\{x_1\})$ and σ_s be not idempotent. If* leftmost$(s) = x_i$ *where $x_i \in X$, $i > 2$, then the order of σ_s is 2.*

Proposition 2.3 ([6]). *Let $s \in W^G_{(2)}(\{x_2\})$ and σ_s be not idempotent. If* rightmost$(s) = x_i$ *where $x_i \in X$ and $i > 2$, then the order of σ_s is 2.*

Theorem 2.4 ([6]). *Let $\tau = (2)$ be a type. The order of any hypersubstitution of type τ is* 1,2 *or infinite.*

We have the following theorem and proposition:

Theorem 2.5. *Let S be a semigroup and $a \in S$. If a is m-potent then a is not $(m+1)$-potent.*

Proof. We proof this theorem by contradiction. Assume that a are m-potent and $(m+1)$-potent.
Since a is m-potent, $a^m = a^{m+1}$ where $a, a^2, \dots, a^m$ are all distinct.
Since a is $(m+1)$-potent, $a^{m+1} = a^{m+2}$ where $a, a^2, \dots, a^m, a^{m+1}$ are all distinct.
This is a contradiction. Therefore a is not $(m+1)$-potent. □

Proposition 2.6. *If $s = f(x_i, x_1)$ where $i > 2$, then σ_s is 2-potent.*

Proof. Let $s = f(x_i, x_1)$ where $i > 2$.
Consider

$$\begin{aligned}
\sigma_s^2(f) &= (\sigma_s \circ_G \sigma_s)(f) \\
&= (\sigma_{f(x_i,x_1))} \circ_G \sigma_{f(x_i,x_1)})(f) \\
&= (\widehat{\sigma}_{f(x_i,x_1)} \circ \ \sigma_{f(x_i,x_1)})(f) \\
&= \widehat{\sigma}_{f(x_i,x_1)}[f(x_i,x_1)] \\
&= S^2(f(x_i,x_1), x_i, x_1) \\
&= f(x_i,x_i),
\end{aligned}$$

and

$$\begin{aligned}
\sigma_s^3(f) &= (\sigma_s \circ_G \sigma_s^2)(f) \\
&= (\sigma_{f(x_i,x_1)} \circ_G \sigma^2_{f(x_i,x_1)})(f) \\
&= (\widehat{\sigma}_{f(x_i,x_1)} \circ \sigma^2_{f(x_i,x_1)})(f) \\
&= \widehat{\sigma}_{f(x_i,x_1)}[\sigma^2_{f(x_i,x_1)}(f)] \\
&= \widehat{\sigma}_{f(x_i,x_1)}[f(x_i,x_i)] \\
&= S^2(f(x_i,x_1), x_i, x_i) \\
&= f(x_i,x_i).
\end{aligned}$$

Thus $\sigma_s^2 = \sigma_s^3$. Therefore σ_s is 2-potent. □

Proposition 2.7. *If $s = f(x_m, t)$ where $m > 2$, $t \in W_{(2)}(X)$, $x_1 \in$ var(t) and $x_2 \notin$ var(t), then σ_s is 2-potent.*

Proof. Let $s = f(x_m,t)$ where $m > 2$, $t \in W_{(2)}(X)$, $x_1 \in \text{var}(t)$ and $x_2 \notin \text{var}(t)$.
Consider

$$\begin{aligned}
\sigma_s^2(f) &= (\sigma_s \circ_G \sigma_s)(f) \\
&= \widehat{\sigma}_{f(x_m,t)}[\sigma_{f(x_m,t)}(f)] \\
&= \widehat{\sigma}_{f(x_m,t)}[f(x_m,t)] \\
&= S^2(f(x_m,t), x_m, \widehat{\sigma}_{f(x_m,t)}[t]) \\
&= f(x_m,u),
\end{aligned}$$

where u is a new term derived by substituting x_1 which occurs in t by x_m. Then $\text{var}(\sigma^2_{f(x_m,u)}(f)) \cap X_2 = \emptyset$.
Consider

$$\begin{aligned}
\sigma^3_{f(x_m,t)}(f) &= (\sigma^2_{f(x_m,t)} \circ_G \sigma_{f(x_m,t)})(f) \\
&= \widehat{\sigma}^2_{f(x_m,t)}[\sigma_{f(x_m,t)}(f)] \\
&= \widehat{\sigma}^2_{f(x_m,t)}[f(x_m,u)] \\
&= S^2(\sigma^2_{f(x_m,t)}(f), \widehat{\sigma}^2_{f(x_m,t)}[x_m], \widehat{\sigma}^2_{f(x_m,t)}[t]) \\
&= S^2(\sigma^2_{f(x_m,t)}(f), x_m, \widehat{\sigma}^2_{f(x_m,t)}[t]).
\end{aligned}$$

Since $\text{var}(\sigma^2_{f(x_m,u)}(f)) \cap X_2 = \emptyset$, $\sigma_s^3 = \sigma_s^2$. Therefore σ_s is 2-potent. □

Proposition 2.8. *If $s = f(t,x_m)$ where $m \neq 2$, $x_1 \in \text{var}(t)$, $x_2 \notin \text{var}(t)$ and $\text{leftmost}(t) = x_i$, $i > 2$, then $\sigma_{f(t,x_m)}$ is 2-potent.*

Proof. Let $s = f(t,x_m)$ where $m \neq 2$, $x_1 \in \text{var}(t)$, $x_2 \notin \text{var}(t)$ and $\text{leftmost}(t) = x_i$, $i > 2$.
Case 1: $m = 1$. Then $s = f(t,x_1)$.
Consider

$$\begin{aligned}
\sigma^2_{f(t,x_1)}(f) &= (\sigma_{f(t,x_1)} \circ_G \sigma_{f(t,x_1)})(f) \\
&= \widehat{\sigma}_{f(t,x_1)}[\sigma_{f(t,x_1)}(f)] \\
&= \widehat{\sigma}_{f(t,x_1)}[f(t,x_1)] \\
&= S^2(f(t,x_1), \widehat{\sigma}_{f(t,x_1)}[t], x_1) \\
&= f(u, \widehat{\sigma}_{f(t,x_1)}[t]),
\end{aligned}$$

where u is a new term derived by substituting x_1 which occurs in t by $\widehat{\sigma}_{f(t,x_1)}[t]$. So $\sigma^2_{f(t,x_1)}(f) = f(u, \widehat{\sigma}_{f(t,x_1)}[t])$ and $\text{leftmost}(u) = x_i$. Since $x_1 \in \text{var}(t) \subseteq \text{var}(s)$ and $\text{leftmost}(t) = x_i$ and $x_2 \notin \text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(t,x_1)}[t]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$. Therefore $\text{var}(\widehat{\sigma}^2_{f(t,x_1)}(f)) \cap X_2 = \emptyset$.
Consider

$$\begin{aligned}
\sigma^3_{f(t,x_1)}(f) &= (\sigma^2_{f(t,x_1)} \circ_G \sigma_{f(t,x_1)})(f) \\
&= \widehat{\sigma}^2_{f(t,x_1)}[\sigma_{f(t,x_1)}(f)] \\
&= \widehat{\sigma}^2_{f(t,x_1)}[f(t,x_1)] \\
&= S^2(\sigma^2_{f(t,x_1)}(f), \widehat{\sigma}^2_{f(t,x_1)}[t], \widehat{\sigma}^2_{f(t,x_1)}[x_1])
\end{aligned}$$

$$= S^2(\sigma^2_{f(t,x_1)}(f), \widehat{\sigma}^2_{f(t,x_1)}[t], x_1).$$

Since $\text{var}(\sigma^2_{f(t,x_1)}(f)) \cap X_2 = \emptyset$, $\sigma^3_s(f) = \sigma^2_s(f)$.

Case 2: $m > 2$. Then $s = f(t, x_m)$.

Consider

$$\begin{aligned}
\sigma^2_{f(t,x_m)}(f) &= (\sigma_{f(t,x_m)} \circ_G \sigma_{f(t,x_m)})(f) \\
&= \widehat{\sigma}_{f(t,x_m)}[\sigma_{f(t,x_m)}(f)] \\
&= \widehat{\sigma}_{f(t,x_m)}[f(t,x_m)] \\
&= S^2(f(t,x_m), \widehat{\sigma}_{f(t,x_m)}[t], x_m) \\
&= S^2(f(u, \widehat{\sigma}_{f(t,x_m)}[t]) \\
&= f(u, x_m),
\end{aligned}$$

where u is a new term derived by substituting x_1 which occurs in t by $\widehat{\sigma}_{f(t,x_m)}[t]$. So $\sigma^2_{f(t,x_m)}(f) = f(u,x_m)$ and $\text{leftmost}(u) = x_i$. Since $x_1 \in \text{var}(t) \subseteq \text{var}(s)$ and $\text{leftmost}(t) = x_i$ and $x_2 \notin \text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(t,x_m)}[t]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$. Therefore $\text{var}(\widehat{\sigma}^2_{f(t,x_m)}(f)) \cap X_2 = \emptyset$.

Consider

$$\begin{aligned}
\sigma^3_{f(t,x_m)}(f) &= (\sigma^2_{f(t,x_m)} \circ_G \sigma_{f(t,x_m)})(f) \\
&= \widehat{\sigma}^2_{f(t,x_m)}[\sigma_{f(t,x_m)}(f)] \\
&= \widehat{\sigma}^2_{f(t,x_m)}[f(t,x_m)] \\
&= S^2(\sigma^2_{f(t,x_m)}(f), \widehat{\sigma}^2_{f(t,x_m)}[t], \widehat{\sigma}^2_{f(t,x_m)}[x_m]) \\
&= S^2(\sigma^2_{f(t,x_m)}(f), \widehat{\sigma}^2_{f(t,x_m)}[t], x_m).
\end{aligned}$$

Since $\text{var}(\sigma^2_{f(t,x_m)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(t,x_m)}(f) = \sigma^2_{f(t,x_m)}(f)$. Therefore σ_s is 2-potent. □

Proposition 2.9. *If $s = f(t_1,t_2)$ where $x_2 \notin \text{var}(t_1)$, $x_2 \notin \text{var}(t_2)$, $x_1 \in \text{var}(t_1) \cup \text{var}(t_2)$ and $\text{leftmost}(t_1) = x_i$, $i > 2$, then $\sigma_{f(t_1,t_2)}$ is 2-potent.*

Proof. Let $s = f(t_1,t_2)$ where $x_2 \notin \text{var}(t_1)$, $x_2 \notin \text{var}(t_2)$, $x_1 \in \text{var}(t_1) \cup \text{var}(t_2)$ and $\text{leftmost}(t_1) = x_i$, $i > 2$.

Case 1: $x_1 \in \text{var}(t_1)$, $x_1 \notin \text{var}(t_2)$.

Consider

$$\begin{aligned}
\sigma^2_{f(t_1,t_2)}(f) &= (\sigma_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f) \\
&= \widehat{\sigma}_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)] \\
&= \widehat{\sigma}_{f(t_1,t_2)}[f(t_1,t_2)] \\
&= S^2(f(t_1,t_2), \widehat{\sigma}_{f(t_1,t_2)}[t_1], \widehat{\sigma}_{f(t_1,t_2)}[t_2]) \\
&= f(u, t_2),
\end{aligned}$$

where u is a new term derived by substituting x_1 which occurs in t_1 by $\widehat{\sigma}_{f(t_1,t_2)}[t_1]$. So $\sigma^2_{f(t_1,t_2)}(f) = f(u,t_2)$ and $\text{leftmost}(u) = x_i$. Since $x_1 \in \text{var}(t_1) \subseteq \text{var}(s)$ and $\text{leftmost}(t_1) = x_i$ and $x_2 \notin \text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(t_1,t_2)}[t_1]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$.

Therefore $\text{var}(\widehat{\sigma}^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$.

Consider

$$\begin{aligned}\sigma^3_{f(t_1,t_2)}(f) &= (\sigma^2_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\ &= \hat{\sigma}^2_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\ &= \hat{\sigma}^2_{f(t_1,t_2)}[f(t_1,t_2)]\\ &= S^2(\sigma^2_{f(t_1,t_2)}(f), \hat{\sigma}^2_{f(t_1,t_2)}[t_1], \hat{\sigma}^2_{f(t_1,t_2)}[t_2]).\end{aligned}$$

Since $\text{var}(\sigma^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(t_1,t_2)}(f) = \sigma^2_{f(t_1,t_2)}(f)$. Therefore σ_s is 2-potent.

Case 2: $x_1 \in \text{var}(t_2)$, $x_1 \notin \text{var}(t_1)$.

Consider

$$\begin{aligned}\sigma^2_{f(t_1,t_2)}(f) &= (\sigma_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\ &= \hat{\sigma}_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\ &= \hat{\sigma}_{f(t_1,t_2)}[f(t_1,t_2)]\\ &= S^2(f(t_1,t_2), \hat{\sigma}_{f(t_1,t_2)}[t_1], \hat{\sigma}_{f(t_1,t_2)}[t_2])\\ &= f(t_1,u),\end{aligned}$$

where u is a new term derived by substituting x_1 which occurs in t_2 by $\hat{\sigma}_{f(t_1,t_2)}[t_1]$. So $\sigma^2_{f(t_1,t_2)}(f) = f(t_1,u)$ and $\text{leftmost}(t_1) = x_i$. Since $x_1 \in \text{var}(t_2) \subseteq \text{var}(s)$ and $\text{var}(t_1) \cap X_2 = \emptyset$, $\text{var}(\hat{\sigma}_{f(t_1,t_2)}[t_1]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$. Therefore $\text{var}(\hat{\sigma}^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$.

Consider

$$\begin{aligned}\sigma^3_{f(t_1,t_2)}(f) &= (\sigma^2_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\ &= \hat{\sigma}^2_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\ &= \hat{\sigma}^2_{f(t_1,t_2)}[f(t_1,t_2)]\\ &= S^2(\sigma^2_{f(t_1,t_2)}(f), \hat{\sigma}^2_{f(t_1,t_2)}[t_1], \hat{\sigma}^2_{f(t_1,t_2)}[t_2]).\end{aligned}$$

Since $\text{var}(\sigma^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(t_1,t_2)}(f) = \sigma^2_{f(t_1,t_2)}(f)$. Therefore σ_s is 2-potent.

Case 3: $x_1 \in \text{var}(t_1) \cap \text{var}(t_2)$.

Consider

$$\begin{aligned}\sigma^2_{f(t_1,t_2)}(f) &= (\sigma_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\ &= \hat{\sigma}_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\ &= \hat{\sigma}_{f(t_1,t_2)}[f(t_1,t_2)]\\ &= S^2(f(t_1,t_2), \hat{\sigma}_{f(t_1,t_2)}[t_1], \hat{\sigma}_{f(t_1,t_2)}[t_2])\\ &= f(u,v),\end{aligned}$$

where u and v are a new term derived by substituting x_1 which occurs in t_1 and t_2 respectively by $\hat{\sigma}_{f(t_1,t_2)}[t_1]$. So $\sigma^2_{f(t_1,t_2)}(f) = f(u,v)$ and $\text{leftmost}(u) = x_i$. Since $x_1 \in \text{var}(t_1) \cap \text{var}(t_2) \subseteq \text{var}(s)$ where $\text{leftmost}(t_1) = x_i$ and $x_2 \notin \text{var}(s)$, $\text{var}(\hat{\sigma}_{f(t_1,t_2)}[t_1]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$ and $\text{var}(v) \cap X_2 = \emptyset$. Therefore $\text{var}(\hat{\sigma}^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$.

Consider

$$\sigma^3_{f(t_1,t_2)}(f) = (\sigma^2_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)$$

$$
\begin{aligned}
&= \widehat{\sigma}^2_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\
&= \widehat{\sigma}^2_{f(t_1,t_2)}[f(t_1,t_2)]\\
&= S^2(\sigma^2_{f(t_1,t_2)}(f),\widehat{\sigma}^2_{f(t_1,t_2)}[t_1],\widehat{\sigma}^2_{f(t_1,t_2)}[t_2]).
\end{aligned}
$$

Since $\operatorname{var}(\sigma^2_{f(t_1,t_2)}(f))\cap X_2=\emptyset$, $\sigma^3_{f(t_1,t_2)}(f)=\sigma^2_{f(t_1,t_2)}(f)$. Therefore σ_s is 2-potent. □

Proposition 2.10. *If $s=f(x_2,x_j)$ where $j>2$, then σ_s is 2-potent.*

Proof. Let $s=f(x_2,x_j)$ where $j>2$.
Consider

$$
\begin{aligned}
\sigma^2_s(f) &= (\sigma_s\circ_G\sigma_s)(f)\\
&= (\sigma_{f(x_2,x_j))}\circ_G\sigma_{f(x_2,x_j)})(f)\\
&= (\widehat{\sigma}_{f(x_2,x_j)}\circ\ \sigma_{f(x_2,x_j)})(f)\\
&= \widehat{\sigma}_{f(x_2,x_j)}[f(x_2,x_j)]\\
&= S^2(f(x_2,x_j),x_2,x_j)\\
&= f(x_j,x_j),
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma^3_s(f) &= (\sigma_s\circ_G\sigma^2_s)(f)\\
&= (\sigma_{f(x_2,x_j)}\circ_G\sigma^2_{f(x_2,x_j)})(f)\\
&= (\widehat{\sigma}_{f(x_2,x_j)}\circ\sigma^2_{f(x_2,x_j)})(f)\\
&= \widehat{\sigma}_{f(x_2,x_j)}[\sigma^2_{f(x_2,x_j)}(f)]\\
&= \widehat{\sigma}_{f(x_2,x_j)}[f(x_j,x_j)]\\
&= S^2(f(x_2,x_j),x_j,x_j)\\
&= f(x_j,x_j).
\end{aligned}
$$

Thus $\sigma^2_s=\sigma^3_s$. Therefore σ_s is 2-potent. □

Proposition 2.11. *If $s=f(x_m,t)$ where $m\neq 1$, $x_2\in\operatorname{var}(t)$, $x_1\notin\operatorname{var}(t)$ and $\operatorname{rightmost}(t)=x_i$, $i>2$, then $\sigma_{f(x_m,t)}$ is 2-potent.*

Proof. *Case* 1: $m=2$. Then $s=f(x_2,t)$.
Consider

$$
\begin{aligned}
\sigma^2_{f(x_2,t)}(f) &= (\sigma_{f(x_2,t)}\circ_G\sigma_{f(x_2,t)})(f)\\
&= \widehat{\sigma}_{f(x_2,t)}[\sigma_{f(x_2,t)}(f)]\\
&= \widehat{\sigma}_{f(x_2,t)}[f(x_2,t)]\\
&= S^2(f(x_2,t),x_2,\widehat{\sigma}_{f(x_2,t)}[t])\\
&= f(\widehat{\sigma}_{f(x_2,t)}[t],u),
\end{aligned}
$$

where u is a new term derived by substituting x_2 which occurs in t by $\widehat{\sigma}_{f(x_2,t)}[t]$. So $\sigma^2_{f(x_2,t)}(f)=f(\widehat{\sigma}_{f(x_2,t)}[t],u)$ and $\operatorname{rightmost}(u)=x_i$. Since $x_2\in\operatorname{var}(t)\subseteq\operatorname{var}(s)$ and $\operatorname{rightmost}(t)=x_i$

and $x_1 \notin \text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(x_2,t)}[t]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$. Therefore $\text{var}(\widehat{\sigma}^2_{f(x_2,t)}(f)) \cap X_2 = \emptyset$.
Consider

$$\begin{aligned}\sigma^3_{f(x_2,t)}(f) &= (\sigma^2_{f(x_2,t)} \circ_G \sigma_{f(x_2,t)})(f)\\ &= \widehat{\sigma}^2_{f(x_2,t)}[\sigma_{f(x_2,t)}(f)]\\ &= \widehat{\sigma}^2_{f(x_2,t)}[f(x_2,t)]\\ &= S^2(\sigma^2_{f(x_2,t)}(f), \widehat{\sigma}^2_{f(x_2,t)}[x_2], \widehat{\sigma}^2_{f(x_2,t)}[t]).\end{aligned}$$

Since $\text{var}(\sigma^2_{f(x_2,t)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(x_2,t)}(f) = \sigma^2_{f(x_2,t)}(f)$.
Case 2: $m > 2$. Then $s = f(x_m, t)$.
Consider

$$\begin{aligned}\sigma^2_{f(x_m,t)}(f) &= (\sigma_{f(x_m,t)} \circ_G \sigma_{f(x_m,t)})(f)\\ &= \widehat{\sigma}_{f(x_m,t)}[\sigma_{f(x_m,t)}(f)]\\ &= \widehat{\sigma}_{f(x_m,t)}[f(x_m,t)]\\ &= S^2(f(x_m,t), x_m, \widehat{\sigma}_{f(x_m,t)}[t])\\ &= f(x_m,u),\end{aligned}$$

where u is a new term derived by substituting x_2 which occurs in t by $\widehat{\sigma}_{f(x_m,t)}[t]$. So $\text{rightmost}(u) = x_i$. Since $x_2 \in \text{var}(t) \subseteq \text{var}(s)$ and $\text{rightmost}(t) = x_i$ and $x_1 \notin \text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(x_m,t)}[t]) \cap X_2 = \emptyset$. Then $\text{var}(u) \cap X_2 = \emptyset$. Therefore $\text{var}(\widehat{\sigma}^2_{f(x_m,t)}(f)) \cap X_2 = \emptyset$.
Consider

$$\begin{aligned}\sigma^3_{f(x_m,t)}(f) &= (\sigma^2_{f(x_m,t)} \circ_G \sigma_{f(x_m,t)})(f)\\ &= \widehat{\sigma}^2_{f(x_m,t)}[\sigma_{f(x_m,t)}(f)]\\ &= \widehat{\sigma}^2_{f(x_m,t)}[f(x_m,t)]\\ &= S^2(\sigma^2_{f(x_m,t)}(f), \widehat{\sigma}^2_{f(x_m,t)}[x_m], \widehat{\sigma}^2_{f(x_m,t)}[t]).\end{aligned}$$

Since $\text{var}(\sigma^2_{f(x_m,t)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(x_m,t)}(f) = \sigma^2_{f(x_m,t)}(f)$. Therefore σ_s is 2-potent. □

Proposition 2.12. *If $s = f(t, x_m)$ where $m > 2$, $x_2 \in \text{var}(t)$ and $x_1 \notin \text{var}(t)$, then σ_s is 2-potent.*

Proof. Let $s = f(t, x_m)$ where $m > 2$, $x_2 \in \text{var}(t)$ and $x_1 \notin \text{var}(t)$.
Consider

$$\begin{aligned}\sigma^2_s(f) &= (\sigma_s \circ_G \sigma_s)(f)\\ &= \widehat{\sigma}_{f(t,x_m)}[\sigma_{f(t,x_m)}(f)]\\ &= \widehat{\sigma}_{f(t,x_m)}[f(t,x_m)]\\ &= S^2(f(t,x_m), \widehat{\sigma}_{f(t,x_m)}[t], x_m)\\ &= f(u,x_m),\end{aligned}$$

where u is a new term derived by substituting x_2 which occurs in t by x_m. Since $x_2 \in \text{var}(t) \subseteq \text{var}(s)$ and $x_1 \notin \text{var}(s)$, $\text{var}(u) \cap X_2 = \emptyset$. Then $\text{var}(\sigma^2_{f(t,x_m)}(f)) \cap X_2 = \emptyset$.

Consider

$$\begin{aligned}
\sigma^3_{f(t,x_m)}(f) &= (\sigma^2_{f(t,x_m)} \circ_G \sigma_{f(t,x_m)})(f)\\
&= \hat{\sigma}^2_{f(t,x_m)}[\sigma_{f(t,x_m)}(f)]\\
&= \hat{\sigma}^2_{f(t,x_m)}[f(t,x_m)]\\
&= S^2(\sigma^2_{f(t,x_m)}(f), \hat{\sigma}^2_{f(t,x_m)}[t], \hat{\sigma}^2_{f(t,x_m)}[x_m]).
\end{aligned}$$

Since $\mathrm{var}(\sigma^2_{f(t,x_m)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(t,x_m)}(f) = \sigma^2_{f(t,x_m)}(f)$. Therefore σ_s is 2-potent. □

Proposition 2.13. *If $s = f(t_1,t_2)$ where $x_1 \notin \mathrm{var}(t_1)$, $x_1 \notin \mathrm{var}(t_2)$, $x_2 \in \mathrm{var}(t_1) \cup \mathrm{var}(t_2)$ and $\mathrm{rightmost}(t_2) = x_i$, $i > 2$, then $\sigma_{f(t_1,t_2)}$ is 2-potent.*

Proof. Let $s = f(t_1,t_2)$ where $x_1 \notin \mathrm{var}(t_1)$, $x_1 \notin \mathrm{var}(t_2)$, $x_2 \in \mathrm{var}(t_1) \cup \mathrm{var}(t_2)$ and $\mathrm{rightmost}(t_2) = x_i$, $i > 2$.

Case 1: $x_2 \in \mathrm{var}(t_1)$, $x_2 \notin \mathrm{var}(t_2)$.

Consider

$$\begin{aligned}
\sigma^2_{f(t_1,t_2)}(f) &= (\sigma_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\
&= \hat{\sigma}_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\
&= \hat{\sigma}_{f(t_1,t_2)}[f(t_1,t_2)]\\
&= S^2(f(t_1,t_2), \hat{\sigma}_{f(t_1,t_2)}[t_1], \hat{\sigma}_{f(t_1,t_2)}[t_2])\\
&= f(u,t_2),
\end{aligned}$$

where u is a new term derived by substituting x_2 which occurs in t_1 by $\hat{\sigma}_{f(t_1,t_2)}[t_2]$. So $\sigma^2_{f(t_1,t_2)}(f) = f(u,t_2)$ and $\mathrm{rightmost}(t_2) = x_i$, $x_2 \in \mathrm{var}(t_1) \subseteq \mathrm{var}(s)$ and $\mathrm{var}(t_2) \cap X_2 = \emptyset$ and $x_1 \notin \mathrm{var}(s)$, $\mathrm{var}(\hat{\sigma}_{f(t_1,t_2)}[t_2]) \cap X_2 = \emptyset$. Then $\mathrm{var}(u) \cap X_2 = \emptyset$.

Therefore $\mathrm{var}(\hat{\sigma}^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$.

Consider

$$\begin{aligned}
\sigma^3_{f(t_1,t_2)}(f) &= (\sigma^2_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\
&= \hat{\sigma}^2_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\
&= \hat{\sigma}^2_{f(t_1,t_2)}[f(t_1,t_2)]\\
&= S^2(\sigma^2_{f(t_1,t_2)}(f), \hat{\sigma}^2_{f(t_1,t_2)}[t_1], \hat{\sigma}^2_{f(t_1,t_2)}[t_2]).
\end{aligned}$$

Since $\mathrm{var}(\sigma^2_{f(t_1,t_2)}(f)) \cap X_2 = \emptyset$, $\sigma^3_{f(t_1,t_2)}(f) = \sigma^2_{f(t_1,t_2)}(f)$.

Case 2: $x_2 \in \mathrm{var}(t_2)$, $x_2 \notin \mathrm{var}(t_1)$.

Consider

$$\begin{aligned}
\sigma^2_{f(t_1,t_2)}(f) &= (\sigma_{f(t_1,t_2)} \circ_G \sigma_{f(t_1,t_2)})(f)\\
&= \hat{\sigma}_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\
&= \hat{\sigma}_{f(t_1,t_2)}[f(t_1,t_2)]\\
&= S^2(f(t_1,t_2), \hat{\sigma}_{f(t_1,t_2)}[t_1], \hat{\sigma}_{f(t_1,t_2)}[t_2])\\
&= f(t_1,u),
\end{aligned}$$

where u is a new term derived by substituting x_2 which occurs in t_2 by $\widehat{\sigma}_{f(t_1,t_2)}[t_2]$. So $\sigma^2_{f(t_1,t_2)}(f)=f(t_1,u)$ and rightmost$(t_2)=x_i$. Since $x_2\in\text{var}(t_2)\subseteq\text{var}(s)$ and rightmost$(t_2)=x_i$ and $x_1\notin\text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(t_1,t_2)}[t_2])\cap X_2=\emptyset$. So $\text{var}(u)\cap X_2=\emptyset$. Therefore $\text{var}(\widehat{\sigma}^2_{f(t_1,t_2)}(f))\cap X_2=\emptyset$.

Consider

$$\begin{aligned}\sigma^3_{f(t_1,t_2)}(f)&=(\sigma^2_{f(t_1,t_2)}\circ_G\sigma_{f(t_1,t_2)})(f)\\&=\widehat{\sigma}^2_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\&=\widehat{\sigma}^2_{f(t_1,t_2)}[f(t_1,t_2)]\\&=S^2(\sigma^2_{f(t_1,t_2)}(f),\widehat{\sigma}^2_{f(t_1,t_2)}[t_1],\widehat{\sigma}^2_{f(t_1,t_2)}[t_2]).\end{aligned}$$

Since $\text{var}(\sigma^2_{f(t_1,t_2)}(f))\cap X_2=\emptyset$, $\sigma^3_{f(t_1,t_2)}(f)=\sigma^2_{f(t_1,t_2)}(f)$.

Case 3: $x_2\in\text{var}(t_1)\cap\text{var}(t_2)$.

Consider

$$\begin{aligned}\sigma^2_{f(t_1,t_2)}(f)&=(\sigma_{f(t_1,t_2)}\circ_G\sigma_{f(t_1,t_2)})(f)\\&=\widehat{\sigma}_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\&=\widehat{\sigma}_{f(t_1,t_2)}[f(t_1,t_2)]\\&=S^2(f(t_1,t_2),\widehat{\sigma}_{f(t_1,t_2)}[t_1],\widehat{\sigma}_{f(t_1,t_2)}[t_2])\\&=f(u,v),\end{aligned}$$

where u and v are a new term derived by substituting x_2 which occurs in t_1 and t_2 respectively by $\widehat{\sigma}_{f(t_1,t_2)}[t_2]$. So $\sigma^2_{f(t_1,t_2)}(f)=f(u,v)$ and rightmost$(v)=x_i$. Since $x_2\in\text{var}(t_1)\cap\text{var}(t_2)\subseteq\text{var}(s)$ and rightmost$(t_2)=x_i$ and $x_1\notin\text{var}(s)$, $\text{var}(\widehat{\sigma}_{f(t_1,t_2)}[t_2])\cap X_2=\emptyset$. Then $\text{var}(u)\cap X_2=\emptyset$ and $\text{var}(v)\cap X_2=\emptyset$. Therefore $\text{var}(\widehat{\sigma}^2_{f(t_1,t_2)}(f))\cap X_2=\emptyset$.

Consider

$$\begin{aligned}\sigma^3_{f(t_1,t_2)}(f)&=(\sigma^2_{f(t_1,t_2)}\circ_G\sigma_{f(t_1,t_2)})(f)\\&=\widehat{\sigma}^2_{f(t_1,t_2)}[\sigma_{f(t_1,t_2)}(f)]\\&=\widehat{\sigma}^2_{f(t_1,t_2)}[f(t_1,t_2)]\\&=S^2(\sigma^2_{f(t_1,t_2)}(f),\widehat{\sigma}^2_{f(t_1,t_2)}[t_1],\widehat{\sigma}^2_{f(t_1,t_2)}[t_2]).\end{aligned}$$

Since $\text{var}(\sigma^2_{f(t_1,t_2)}(f))\cap X_2=\emptyset$, $\sigma^3_{f(t_1,t_2)}(f)=\sigma^2_{f(t_1,t_2)}(f)$. Therefore σ_s is 2-potent. □

3. Conclusion

We use the concept of orders of elements on a semigroup to characterize all 2-potent elements of $Hyp_G(2)$. For the characterization, we consider only the generalized hypersubstitutions of type $\tau=(2)$ which are not idempotent and has order 2 because the generalized hypersubstitutions of type $\tau=(2)$ which have infinite order are not m-potent.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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