Communications in Mathematics and Applications
Volume 2 (2011), Numbers 2-3, pp. 77–85

http://www.rgnpublications.com

Initial Time Difference Quasilinearization Method in Banach Space

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Abstract. In this paper, the method of the quasilinearization technique in Banach space is applied to obtain upper and lower sequences with initial time difference in terms of the solutions of the linear differential equations that start at different initial times. It is also shown that these sequences converge to the unique solution of the nonlinear equation in Banach space uniformly and superlinearly.

1. Introduction

The most important applications of the quasilinearization method in Banach space has been to obtain a sequence of lower and upper bounds which are the solutions of linear differential equations in Banach space that converge superlinear. As a result, the method has been popular in applied areas. However, the convexity assumption that is demanded by the method of quasilinearization has been a stumbling block for further development of the theory. Recently, this method has been generalized, refined and extended in several directions so as to be applicable to a much larger class of nonlinear problems by not demanding convexity property. Moreover, other possibilities that have been explored make the method of generalized quasilinearization universally useful in applications [6]. In the investigation of initial value problems of differential equations, we have been partial to initial time all along in the sense that we only perturb the space variable and keep the initial time unchanged. However, it appears important to vary the initial time as well because it is impossible not to make errors in the starting time [4, 5, 7, 8, 9, 10, 11]. Recently, the investigations of initial value problems of differential equations where the initial time changes with each solution in addition to the change of spatial variable have been initiated [1, 12] and some results on the comparison theorems, global existence, the method of variation of parameters,

2010 *Mathematics Subject Classification.* 34A12, 34A45, 34C11.
Key words and phrases. Initial time difference; Quasilinearization in Banach space; Superlinear convergence.

the method of lower and upper solutions and the method of monotone iterative techniques [3, 4, 5, 7, 10] have been obtained.

In this paper, the generalized quasilinearization technique in Banach space is used to obtain upper and lower sequences in terms of the solutions of linear differential equations in Banach space that start at different initial times and bound the solutions of a given nonlinear differential equation in Banach space. It is also shown that these sequences converge to the unique solution of the nonlinear equation uniformly and superlinear in Banach space.

2. **Preliminaries**

In this section, we state some fundamental definitions and useful theorems for the future reference to prove the main result. First one is comparison result, the second one is existence result in terms of the upper and lower solutions with initial time difference.

Let α_0, $\beta_0 \in C^1[J,E]$ with $\alpha_0(t) \leq \beta_0(t)$ on $J = [t_0, t_0 + T]$, $t_0, T \in \mathbb{R}^+$ and

$$\Omega = \{u \in E : \alpha_0(t) \leq u \leq \beta_0(t),\ t \in J\}.$$

Let E be a Banach space and consider the following initial value problem

$$u' = N(t,u),\ u(t_0) = u_0 \quad \text{for } t \geq t_0 \tag{2.1}$$

where $N \in C[J \times \Omega, E]$ for $J = [t_0, t_0 + T]$, $t_0, T \in \mathbb{R}^+$ and $\Omega \subseteq E$.

Definition 2.1. A cone K is a subset of E such that $x, y \in K$ implies that $\lambda x + \mu y \in K$ when $\lambda, \mu \geq 0$. K is called proper if $0 \neq x \in K$ implies $-x \notin K$. When x and y are elements of E, $x \geq y$ means $x - y \in K$, so that in particular, $x \geq 0$ is equivalent to $x \in K$.

Definition 2.2. A proper subset K of a Banach space E is said to be a cone if $\lambda K \subset K$, $\lambda \geq 0$, $K + K \subset K$, $K = \bar{K}$, and $K \cap \{-K\} = 0$ where 0 denotes the null element of the Banach space E and $\bar{K}$ denotes the closure of K.

Definition 2.3. K is said to be a distance set, if, for every point $x \in E$ has the norm $\|\cdot\|$, there corresponds a point $y \in K$ such that $d(x,K) = \|x - y\|$.

In this paper we assume that K to be a distance set.

Definition 2.4. The cone K induces the order relations in E defined by

$$x \leq y \text{ if and only if } y - x \in K$$

$$x < y \text{ if and only if } y - x \in \overset{\circ}{K}$$

where $\overset{\circ}{K}$ denotes the interior of K.

Let K^* be the set of all continuous linear functionals ϕ on E such that $\phi(x) \geq 0$ for all $x \in K$ and let K_0^* be the set of all continuous linear functionals on E such that $\phi(x) > 0$ for all $x \in \overset{\circ}{K}$.

Definition 2.5. A function $f : E \to E$ is said to be quasimonotone nondecreasing if $x \le y$ and $\phi(x) = \phi(y)$ for some $\phi \in K_0^*$, then $\phi(f(x)) \le \phi(f(y))$.

Definition 2.6. Let $\alpha_0, \beta_0 \in C^1[J, E]$ be the natural lower and upper solutions of (2.1) satisfying the following inequalities

$$\alpha_0' \le N(t, \alpha_0),\ \alpha_0(t_0) \le u_0 \ \text{ for } t \ge t_0 \tag{2.2}$$

$$\beta_0' \ge N(t, \beta_0),\ \beta_0(t_0) \ge u_0 \ \text{ for } t \ge t_0 \tag{2.3}$$

respectively.

Definition 2.7. Let $f \in C[J \times E, E]$. At $x \in E$

$$f(t, x+h) = f(t,x) + L(t,x,h) + \|h\|\eta(t,x,h)$$

where $\lim\limits_{\|h\|\to 0} \|\eta(t,x,h)\| = 0$ and $L(t,x,\cdot)$ is a linear operator. $L(t,x,h)$ is called the Fr chet differential of the function f at x with increment h, $\eta(t,x,h)$ is called the remainder of the differential, and the operator $L(t,x,\cdot)$ is called the Fr chet derivative of f at x.

First we state the following Theorem 2.1, whose proof is given in [4].

Theorem 2.1. *Let K be a cone in E. Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, E]$, $N \in C[J \times E, E]$ *and* $N(t,x)$ *is quasimonotone nondecreasing in x relative to K for each $t \in J$;*
(ii) $\alpha_0'(t) \le N(t, \alpha_0(t)), N(t, \beta_0(t)) \le \beta_0'(t)$, $t \in J$;
(iii) $\|N(t,x) - N(t,y)\| \le L\|x - y\|$, $L > 0$, $x \in E - K$, $y \in \partial K$, *where ∂K denotes the boundary of K;*
(iv) *K is a distance set.*

Then $\alpha_0(t_0) \le \beta_0(t_0)$ implies $\alpha_0(t) \le \beta_0(t)$, $t \in J$.

When the lower and upper solutions do not start the same point we state also the following existence result whose proof is given in [6].

Theorem 2.2. *Assume that*

(i) *Let* $\alpha_0 \in C^1[[t_0, t_0 + T], E]$, $t_0, T > 0$, $\beta_0 \in C^1[[\tau_0, \tau_0 + T], E]$, $\tau_0 > 0$ *and* $N \in C[R^+ \times E, E]$, $\alpha_0'(t) \le N(t, \alpha_0), t_0 \le t \le t_0 + T$ *and* $\beta_0'(t) \ge N(t, \beta_0), \tau_0 \le t \le \tau_0 + T$ *with* $\alpha_0(t_0) \le \beta_0(\tau_0)$;
(ii) $t_0 \le \tau_0$, *$N(t,u)$ is nondecreasing in t for each u and* $\alpha_0(t) \le \beta_0(t + \eta)$ *for* $t_0 \le t \le t_0 + T$, $\eta = \tau_0 - t_0$.

Then there exists a solution $u(t)$ of (2.1) with $u(t_0) = u_0$, satisfying $\alpha_0(t) \le u(t) \le \beta_0(t + \eta)$ for $t_0 \le t \le t_0 + T$.

3. **Main Results**

In this section, we will prove the main theorem that gives several different conditions to apply the method of generalized quasilinearization to the nonlinear differential equations in Banach space with initial time difference and state remarks and corollaries for special cases.

Theorem 3.1. *Assume that*

(i) $N \in C[J \times E, E]$ *and there exists a constant* M *such that* $\|N(t,u)\| \leq M$ *on* $J \times \Omega$*;*

(ii) $N(t,u)$ *is quasimonotone nondecreasing in* u *relative to* K *for each* $t \in J$*, where* K *is a cone in* E *is distance set ;*

(iii) $\alpha_0 \in C^1[[t_0, t_0+T], E]$ *and* $\beta_0 \in C^1[[\tau_0, \tau_0+T], E]$ *for* $\tau_0 \geq t_0 > 0$ *and* $T > 0$*,*

$$\alpha_0' \leq N(t, \alpha_0) \quad \textit{for } t_0 \leq t \leq t_0 + T$$

$$\beta_0' \geq N(t, \beta_0) \quad \textit{for } \tau_0 \leq t \leq \tau_0 + T,$$

where $N \in C[[t_0, \tau_0+T] \times E, E]$ *and* $\alpha_0(t_0) \leq \beta_0(\tau_0)$*;*

(iv) $t_0 < s_0 < \tau_0$*,* $N(t,u)$ *is nondecreasing in* t *for each* u;
$\alpha_0, \beta_0 \in C^1[J,E]$ *such that* $\alpha_0' \leq N(t,\alpha_0)$*,* $N(t,\beta_0) \leq \beta_0'$ *and* $\alpha_0(t) \leq \beta_0(t)$*,* $t \in J$*;*

(v) *the Fr chet derivative* $N_x(t,x)$ *exists and is continuous and* $\|N_x(t,x)\| \leq L_1$ *for* $(t,x) \in J \times \Omega$*, for some* $L_1 > 0$ *and* $N(t,y) \leq N(t,x) - N_x(t,y)(x-y)$ *where* $\alpha_0(t) \leq y \leq x \leq \beta_0(t)$*,* $t \in J$*;*

(vi) $\|N_x(t,x) - N_x(t,y)\| \leq L_2\|x-y\|^\gamma$*,* $t \in J$*, where* L_2 *is a positive constant and* $0 \leq \gamma < 1$*.*

Then there exist monotone sequences $\{\widetilde{\alpha}_n(t)\}$ *and* $\{\widetilde{\beta}_n(t)\}$ *which converge uniformly to the unique solution of* (2.1) *with* $x(s_0) = x_0$ *where* s_0 *is between initial time* t_0 *and* τ_0 *and the convergence is superlinear.*

Proof. Since $\tilde{\beta}_0(t) = \beta_0(t+\eta_1)$, we get $\tilde{\beta}_0(t_0) = \beta_0(\tau_0) \geq \alpha_0(t_0) = \tilde{\alpha}_0(t_0)$ and $\tilde{\beta}_0'(t) \geq N(t+\eta_1, \tilde{\beta}_0)$ for $t_0 \leq t \leq t_0 + T$. Using the assumptions (iv) it is clear that $N(t,x)$ satisfies the Lipschitz condition in x for $(t,x) \in J \times \Omega$. Furthermore, we have the following inequalities

$$N(t,x) \geq N(t,y) + N_x(t,y)(x-y) \text{ whenever } \tilde{\alpha}_0(t) \leq y \leq x \leq \tilde{\beta}_0(t) \text{ on } J \tag{3.1}$$

and also by using (iv) we see that whenever $\tilde{\alpha}_0(t) \leq y \leq x \leq \tilde{\beta}_0(t)$

$$N(t,x) - N(t,y) \leq L(x-y) \tag{3.2}$$

for some $L > 0$.

Consider the linear initial value problems such that

$$\widetilde{\alpha}_1' = N(t+\eta_2,\tilde{\alpha}_0) + N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\alpha}_1 - \tilde{\alpha}_0), \widetilde{\alpha}_1(t_0) = u_0 \tag{3.3}$$

$$\widetilde{\beta}_1' = N(t+\eta_2,\widetilde{\beta}_0) + N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\beta}_1 - \widetilde{\beta}_0), \widetilde{\beta}_1(t_0) = u_0 \tag{3.4}$$

where $\tilde{\alpha}_0(t_0) \le u_0 \le \widetilde{\beta}_0(t_0)$. We shall show that $\tilde{\alpha}_0 \le \tilde{\alpha}_1$ on J. To do this, let $p = \tilde{\alpha}_0(t) - \tilde{\alpha}_1(t)$, so that $p(t_0) \le 0$. Then

$$\begin{aligned} p' &= \tilde{\alpha}_0' - \tilde{\alpha}_1' \\ &\le N(t+\eta_2,\tilde{\alpha}_0) - [N(t+\eta_2,\tilde{\alpha}_0) + N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\alpha}_1 - \tilde{\alpha}_0)] \\ &= N_x(t+\eta_2,\tilde{\alpha}_0)p. \end{aligned}$$

Theorem 2.1 gives $p(t) \le 0$ on J proving that $\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t)$ on J. Now set $p = \widetilde{\alpha}_1 - \widetilde{\beta}_0$ and note that $p(t_0) \le 0$. Also, using (3.1)

$$\begin{aligned} p' &= \widetilde{\alpha}_1' - \widetilde{\beta}_0' \\ &\le N(t+\eta_2,\tilde{\alpha}_0) + N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\alpha}_1 - \tilde{\alpha}_0) - N(t+\eta_1,\widetilde{\beta}_0) \\ &\le N(t+\eta_2,\widetilde{\beta}_0) - N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\beta}_0 - \tilde{\alpha}_0) \\ &\quad + N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\alpha}_1 - \tilde{\alpha}_0) - N(t+\eta_2,\widetilde{\beta}_0) \\ &\le N_x(t+\eta_2,\tilde{\alpha}_0)p, \end{aligned}$$

which again implies $\widetilde{\alpha}_1(t) \le \widetilde{\beta}_0(t)$ on J.

Similarly, we can obtain that $\tilde{\alpha}_0(t) \le \tilde{\beta}_1(t) \le \tilde{\beta}_0(t)$ on J. In order to prove that $\tilde{\alpha}_1(t) \le \tilde{\beta}_1(t)$ on J, we proceed as follows, since $\tilde{\alpha}_0 \le \widetilde{\alpha_1} \le \tilde{\beta}_0$, using (3.1), we see that

$$\tilde{\alpha}_1'(t) = N(t+\eta_2,\tilde{\alpha}_0) + N_x(t+\eta_2,\tilde{\alpha}_0)(\widetilde{\alpha}_1 - \tilde{\alpha}_0) \le N(t+\eta_2,\tilde{\alpha}_1).$$

Similarly, $N(t+\eta_2,\widetilde{\beta}_1) \le \widetilde{\beta}_1'(t)$ and therefore by Theorem 2.1 it follows that $\tilde{\alpha}_1(t) \le \widetilde{\beta}_1'(t)$ on $\mathbb{J}$ which shows that

$$\tilde{\alpha}_0(t) \le \tilde{\alpha}_1(t) \le \tilde{\beta}_1(t) \le \tilde{\beta}_0(t) \text{ on } \mathbb{J}.$$

Assume that for some $n > 1$, $\widetilde{\alpha}_n' \le N(t+\eta_2,\tilde{\alpha}_n)$, $N(t+\eta_2,\widetilde{\beta}_n) \le \widetilde{\beta}_n'$ and $\widetilde{\alpha}_n(t) \le \widetilde{\beta}_n(t)$, $t \in \mathbb{J}$.

We must show that

$$\tilde{\alpha}_n(t) \le \tilde{\alpha}_{n+1}(t) \le \tilde{\beta}_{n+1}(t) \le \tilde{\beta}_n(t) \text{ on } \mathbb{J} \tag{3.5}$$

where $\tilde{\alpha}_{n+1}(t)$ and $\tilde{\beta}_{n+1}(t)$ are the solutions of linear IVPs

$$\widetilde{\alpha}_{n+1}' = N(t+\eta_2,\tilde{\alpha}_n) + N_x(t+\eta_2,\tilde{\alpha}_n)(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n),\ \widetilde{\alpha}_{n+1}(t_0) = u_0 \tag{3.6}$$

$$\widetilde{\beta}_{n+1}' = N(t+\eta_2,\widetilde{\beta}_n) + N_x(t+\eta_2,\tilde{\alpha}_n)(\widetilde{\beta}_{n+1} - \widetilde{\beta}_n), \widetilde{\beta}_{n+1}(t_0) = u_0. \tag{3.7}$$

Hence, setting $p = \tilde{\alpha}_n(t) - \tilde{\alpha}_{n+1}(t)$, it follows as before $p' \le N_x(t+\eta_2, \tilde{\alpha}_n)p$ on J and we get $\tilde{\alpha}_n(t) \le \tilde{\alpha}_{n+1}(t) \le \tilde{\beta}_n(t)$ on J. In a similar manner, we can prove that $\tilde{\alpha}_n(t) \le \tilde{\beta}_{n+1}(t) \le \tilde{\beta}_n(t)$ on J.

Using (3.1), we obtain

$$\begin{aligned}
\alpha'_{n+1} &= N(t+\eta_2, \tilde{\alpha}_n) + N_x(t+\eta_2, \tilde{\alpha}_n)(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n) \\
&\le N(t+\eta_2, \tilde{\alpha}_{n+1}) - N_x(t+\eta_2, \tilde{\alpha}_n)(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n) \\
&\quad + N_x(t+\eta_2, \tilde{\alpha}_n)(\widetilde{\alpha}_{n+1} - \tilde{\alpha}_n) \\
&= N(t+\eta_2, \tilde{\alpha}_{n+1}).
\end{aligned}$$

Similar arguments yield $N(t+\eta_2, \widetilde{\beta}_{n+1}) \le \widetilde{\beta}'_{n+1}$and hence Theorem 2.1 shows that $\tilde{\alpha}_{n+1}(t) \le \tilde{\beta}_{n+1}(t)$ on J which proves (3.5) is true. So by using induction we obtain

$$\tilde{\alpha}_0 \le \tilde{\alpha}_1 \le \cdots \le \tilde{\alpha}_n \le \tilde{\alpha}_{n+1} \le \tilde{\beta}_{n+1} \le \tilde{\beta}_n \le \cdots \le \tilde{\beta}_1 \le \tilde{\beta}_0 \text{ on } J.$$

Now using standard arguments (Arzela-Ascoli and Dini's Theorems, see [2]), it can be shown that the sequences $\{\tilde{\alpha}_n(t)\}$ and $\{\tilde{\beta}_n(t)\}$ converge uniformly and monotonically to the unique solution of $u(t)$ of (2.1) on J.

$$\widetilde{u}'(t) = N(t+\eta_2, \widetilde{u}(t)), \quad \widetilde{u}(t_0) = u_0. \tag{3.8}$$

But letting $s = t + \eta_2$ and changing the variable, we can show that (3.8) is equivalent to

$$u'(s) = N(s, u(s)), \quad u(s_0) = u_0.$$

Finally, to prove superlinear convergence, we let

$$p_n(t) = \widetilde{u}(t) - \tilde{\alpha}_n(t) \quad \text{and} \quad q_n(t) = \tilde{\beta}_n(t) - \widetilde{u}(t).$$

Note that $p_n(t_0) = q_n(t_0) = 0$.

$$\begin{aligned}
&p'_n(t) \\
&= \widetilde{u}'(t) - \tilde{\alpha}'_n(t) \\
&= N(t+\eta_2, \widetilde{u}) - [N(t+\eta_2, \tilde{\alpha}_{n-1}) + N_x(t+\eta_2, \tilde{\alpha}_{n-1})(\widetilde{\alpha}_n - \tilde{\alpha}_{n-1})] \\
&= \int_0^1 N_x(t+\eta_2, s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1})(\widetilde{u} - \tilde{\alpha}_{n-1})ds - N_x(t+\eta_2, \tilde{\alpha}_{n-1})(\widetilde{\alpha}_n - \tilde{\alpha}_{n-1}) \\
&= \int_0^1 N_x(t+\eta_2, s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1})p_{n-1}ds - N_x(t+\eta_2, \tilde{\alpha}_{n-1})(p_{n-1} - p_n) \\
&= \int_0^1 [N_x(t+\eta_2, s\widetilde{u} + (1-s)\tilde{\alpha}_{n-1}) - N_x(t+\eta_2, \tilde{\alpha}_{n-1})]p_{n-1}ds + N_x(t+\eta_2, \tilde{\alpha}_{n-1})p_n.
\end{aligned}$$

From (iv) and (v), it follows that

$$\|p_n'(t)\| \leq \int_0^1 L_2\|s\widetilde{u}+(1-s)\tilde{\alpha}_{n-1}-\tilde{\alpha}_{n-1})\|^\gamma\|p_{n-1}\|ds+L_1\|p_n\|$$

$$\leq \int_0^1 L_2\|s\widetilde{u}-s\tilde{\alpha}_{n-1}\|^\gamma\|p_{n-1}\|ds+L_1\|p_n\|$$

$$\leq \int_0^1 L_2\|sp_{n-1}\|^\gamma\|p_{n-1}\|ds+L_1\|p_n\|$$

$$= L_2\|p_{n-1}\|^{\gamma+1}+L_1\|p_n\|.$$

Then setting $a_n = \|p_n\|$, we find,

$$a_n' \leq \|p_n'\| \leq L_2(a_{n-1})^{\gamma+1}+L_1a_n.$$

Now by using Gronwall's inequality we obtain,

$$0 \leq a_n(t) \leq L_2\int_0^t \exp[L_1(t-s)](a_n(s))^{\gamma+1}ds \text{ on } J$$

which yields the estimate

$$\max_J\|p_n(t)\| \leq L_2\frac{\exp(L_1T)}{L_1}\max_J\|p_{n-1}(t)\|^{\gamma+1}.$$

Similarly,

$$q_n'(t) = \tilde{\beta}_n'(t)-\widetilde{u}'(t)$$

$$= N(t+\eta_2,\tilde{\beta}_{n-1})+N_x(t+\eta_2,\tilde{\alpha}_{n-1})(\tilde{\beta}_n-\tilde{\beta}_{n-1})-N(t+\eta_2,\widetilde{u}(t))$$

$$= \int_0^1 N_x(t+\eta_2,s\tilde{\beta}_{n-1}+(1-s)\widetilde{u}(t))(\tilde{\beta}_{n-1}-\widetilde{u})ds+N_x(t+\eta_2,\tilde{\alpha}_{n-1})(\tilde{\beta}_n-\tilde{\beta}_{n-1})$$

$$= \int_0^1 N_x(t+\eta_2,s\tilde{\beta}_{n-1}+(1-s)\widetilde{u}(t))q_{n-1}ds+N_x(t+\eta_2,\tilde{\alpha}_{n-1})(q_n-q_{n-1})$$

$$= \int_0^1 [N_x(t+\eta_2,s\tilde{\beta}_{n-1}+(1-s)\widetilde{u}(t))-N_x(t+\eta_2,\widetilde{u}(t))]q_{n-1}ds$$

$$+[N_x(t+\eta_2,\widetilde{u}(t))-N_x(t+\eta_2,\tilde{\alpha}_{n-1})]q_{n-1}+N_x(t+\eta_2,\tilde{\alpha}_{n-1}).q_n.$$

We find, using (iv) and (v), that

$$\|q_n'(t)\| \leq \int_0^1 L_2\|s\tilde{\beta}_{n-1}+(1-s)\widetilde{u}-\widetilde{u}\|^\gamma\|q_{n-1}\|ds$$

$$+ L_2\|\widetilde{u}-\tilde{\beta}_{n-1}\|^\gamma\|q_{n-1}\|+L_1\|q_n\|$$

$$\leq L_2\|q_{n-1}\|^{\gamma+1}+L_2\|p_n\|^\gamma\|q_{n-1}\|+L_1\|q_n\|.$$

Setting $b_n = \|q_n\|$ and $a_{n-1} = \|p_{n-1}\|$, it is easily follows that

$$\begin{aligned} b_n' &\le \|q_n'\| \\ &\le L_2(b_{n-1})^{\gamma+1} + L_2(a_{n-1})^{\gamma} b_{n-1} + L_1 b_n. \end{aligned}$$

An application of Gronwall's inequality yields,

$$\begin{aligned} 0 &\le \|q_n\| \\ &\le L_2 \int_0^t \exp[L_1(t-s)][\|q_{n-1}(s)\|^{\gamma+1} + \|p_{n-1}(s)\|^{\gamma}\|q_{n-1}(s)\|]ds \text{ on } J, \end{aligned}$$

and hence

$$\max_J \|q_n(t)\| \le L_2 \frac{\exp(L_1 T)}{L_1} \left[\max_J \|q_{n-1}(t)\|^{\gamma+1} + \max_J \|p_{n-1}(t)\|^{\gamma}\|q_{n-1}(t)\|\right].$$

This completes the proof. □

Next we give the following remarks and corollaries for special cases.

Remark 3.1. Instead of assumption (v) in Theorem 3.1 if we assume that

$$\|N_x(t,x) - N_x(t,y)\| \le L_2\|x-y\|, \quad t \in J$$

where L_2 is a positive constant, then we can see that the convergence is quadratic.

Remark 3.2. Let the assumption of Remark 3.1 valid. If we assume that $N(t,x)$ is uniformly convex in x instead of condition

$$N(t,y) \le N(t,x) - N_x(t,y)(x-y) \quad \text{where } y \le x, t \in J;$$

assumption (iv) in Theorem 3.1, then by Lemma 4.5.1 in [2], the assumed inequality results. Moreover, the quasimonotonicity of $N(t,x)$ in x implies by Lemma 4.2.5 in [2] that $N_x(t,\alpha(t))x$ is also quasimonotone in x.

Corollary 3.1. *If the assumptions of the Theorem* 3.1 *hold with* $s_0 = t_0$, *then the conclusion of the theorem remains valid.*

Proof. For the proof, we let $\tilde{\beta}_0(t) = \beta(t+\eta_1)$, $\tilde{\alpha}_0(t) = \alpha(t)$ and $\tilde{u}(t) = u(t)$ and proceed, as we have done in Theorem 3.1. □

Corollary 3.2. *If the assumptions of the Theorem* 3.1 *hold with* $s_0 = \tau_0$, *then the conclusion of the theorem remains valid.*

Proof. Similarly, we let $\tilde{\alpha}_0(t) = \alpha(t-\eta_1)$, $\tilde{\beta}_0(t) = \beta(t)$ and $\tilde{u}(t) = u(t)$ and proceed, as we have done in Theorem 3.1. □

Acknowledgements

The authors wish to express their sincere gratitude to Prof. V. Lakshmikantham and Prof. S. K ksal for their helpful suggestion in improving the paper.

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Received August 8, 2011
Accepted December 26, 2011