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Research Article

Closed Forms of General Solutions for Rectangular Systems of Coupled Generalized Sylvester Matrix Differential Equations

Pattrawut Chansangiam

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand
pattrawut.ch@kmitl.ac.th

Abstract. We investigate a rectangular system of coupled generalized Sylvester matrix differential equations in both nonhomogeneous and homogeneous cases. In order to obtain a closed form of its general solution, we transform it to an equivalent vector differential equation. This is done by using the vector operator and the Kronecker product. An explicit form of its general solution is given in terms of matrix series concerning Mittag-Leffler functions, exponentials, and hyperbolic functions. The main system includes certain systems of coupled matrix/vector differential equations, and single matrix differential equations as special cases.

Keywords. Matrix differential equation; Vector operator; Kronecker product; Mittag-Leffler function; Matrix exponential

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1. Introduction

Theory of linear matrix differential equations, as an extension of theory of vector differential equations, is a field of attractive research nowadays. This theory can be applied widely in scientific fields especially in control theory, statistics, ecometrics, and game theory (see e.g. [7, 8, 11, 14, 18]).

Let us start with a development of single linear matrix differential equations of Sylvester type. Denote the set of n-by-n real matrices by M_n. A homogeneous Sylvester matrix differential equation takes the form

$$X'(t) = AX(t) + X(t)B, \tag{1.1}$$

where $A \in M_n$ is a given constant matrix and $X(t) \in M_n$ is an unknown matrix function. A clever idea to obtain an explicit solution for this equation is to establish a correspondence with a scalar differential equation. This is done by taking the vector operator to (1.1), so that the solution is given in terms of matrix exponentials; see [6]. In fact, this idea can be applied for several linear matrix differential equations of Sylvester type. For a nonhomogeneous linear matrix differential equation

$$X'(t) = AX(t) + U(t), \tag{1.2}$$

where $U(t) \in M_n$ is a given matrix function, a closed form of its general solution is given by a one-parameter matrix function (see [2])

$$X(t) = e^{(t-t_0)A}X(t_0) + \int_{t_0}^{t} e^{(t-s)A}U(s)\,ds. \tag{1.3}$$

For the homogeneous case $U(t) = 0$, the general solution becomes $X(t) = e^{A(t-t_0)}X(t_0)$. In practice, the matrix exponentials can be computed efficiently using appropriated methods (see e.g. [9,15]). See more information about linear matrix differential equations in [14, Chapter 3].

A general system of nonhomogeneous coupled linear matrix differential equations takes the form

$$\left.\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned}\right\} \tag{1.4}$$

Here, the constant matrices A,B,C,D,E,F,G,H, the given matrix functions $U(t)$, $V(t)$, and the unknown matrix functions $X(t), Y(t)$ are square matrices of the same size. The works including [1–5, 12, 13, 20] investigate certain special cases of the system. A main idea is to reduce the system of matrix differential equations to a vector differential equation via a certain kind of vectorizations, e.g. the vector operator or the diagonal-extraction operator. Indeed, the general solution is given in terms of Kronecker/Hadamard products, and matrix power series concerning exponentials, hyperbolic functions, or Mittag-Leffler functions. For nonhomogeneous cases, the general solution can be written in terms of the matrix convolution product (e.g. [20]). For the case that the unknown matrix functions are diagonal, we can apply the diagonal-extraction operator to reduce this system to a simple one (e.g. [1,20]). For certain types of linear matrix descriptor differential equations, their general solutions can be obtained via matrix pencil theory [17] or canonical forms [22]. Systems of coupled linear matrix differential equations show up in many application areas, such as, control theory, communication engineering, and stability of certain differential equations. In particular, the special case $B = D = F = H = I$ and $G = -A^T$ of the system (1.4) appears in an analysis of optimal control with performance index (see [16]).

In the present work, we consider a rectangular system of coupled generalized Sylvester matrix differential equations in a general form as follows:

$$X'(t) = \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t),$$
$$Y'(t) = \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t).$$

Here, the given coefficient matrices, the given matrix functions, and the unknown matrix functions are rectangular matrices, need not be square. To obtain an explicit formula of the solution, we impose an assumption on the coefficient matrices. We apply the vector operator and Kronecker products to reduce the system to a simple form so that an explicit formula of the general solution can be obtained in terms of Mittag-Leffler matrix functions. We also obtain general solutions of certain special cases of the main system including systems of coupled matrix/vector differential equations, and single matrix differential equations. In particular, our results include the results in [2, 3, 6, 12]. We also illustrate initial value problems associated with systems considered in this paper.

The rest of paper is outlined as follows. In Section 2, we setup basic notations and provide useful tools for solving linear matrix differential equations. These tools involve the Kronecker product, the Kronecker sum, the vector operator, and Mittag-Leffler functions. In Section 3, we solve a rectangular system of coupled linear matrix differential equations. Several interesting special cases of the main system are then considered in Section 4. Finally, we provide illustrative examples for our results in Section 5.

2. Preliminaries

Let us denote by $M_{m,n}$ the set of all m-by-n real matrices, and abbreviate $M_{n,n}$ to M_n. We use the notation $\mathrm{Sp}(A)$ for the set of eigenvalues of $A \in M_n$.

2.1 The Kronecker Product, the Kronecker Sum and the Vector Operator

Recall that the Kronecker product of $A = [a_{ij}] \in M_{m,n}$ and $B \in M_{p,q}$ is defined to be the mp-by-nq matrix whose the (i,j)-th block is given by $a_{ij}B$ for each $i = 1,\dots,m$ and $j = 1,\dots,n$.

Lemma 1 (e.g. [10]). *The following properties hold for matrices of appropriate sizes:*

(1) $(kA)\otimes B = k(A\otimes B) = A\otimes(kB)$ *for all* $k \in \mathbb{R}$,

(2) $(A+B)\otimes C = (A\otimes C)+(B\otimes C)$ *and* $A\otimes(B+C) = (A\otimes B)+(A\otimes C)$,

(3) $(A\otimes B)^T = A^T \otimes B^T$,

(4) $(A\otimes B)(C\otimes D) = AC\otimes BD$,

(5) $\mathrm{Sp}(A\otimes B) = \{\lambda\mu : \lambda \in \mathrm{Sp}(A), \mu \in \mathrm{Sp}(B)\}$.

The Kronecker sum of $A \in M_n$ and $B \in M_m$ is defined to be

$$A \oplus B = (A\otimes I_m)+(I_n \otimes B).$$

Let us recall a column-stacking operator, which is useful for solving linear matrix differential equations. The vector operator $\operatorname{Vec}: M_{m,n} \to \mathbb{R}^{mn}$ is defined for each $A = [a_{ij}] \in M_{m,n}$ by

$$\operatorname{Vec} A = [a_{11} \ldots a_{m1}\, a_{12} \ldots a_{m2} \ldots a_{1m} \ldots a_{mn}]^T.$$

This operator is clearly bijective, linear and continuous.

Lemma 2 (e.g. [10])**.** *For any matrices A, B, C of appropriate sizes, we have*

$$\operatorname{Vec}(ABC) = (C^T \otimes A)\operatorname{Vec} B.$$

2.2 Mittag-Leffler Functions

The *Mittag-Leffler function* with two parameters $\alpha > 0$ and $\beta > 0$ is defined for each complex number z in terms of a convergent series by

$$\mathscr{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

where Γ is the Gamma function. When $\beta = 1$, we set $\mathscr{E}_\alpha := \mathscr{E}_{\alpha,1}$. The class of Mittag-Leffler functions include the exponential function $\mathscr{E}_1(z) = e^z$ and the hyperbolic functions $\mathscr{E}_2(z^2) = \cosh z$ and $z\mathscr{E}_{2,2}(z^2) = \sinh z$. See more information in [19] and references therein. For any $A \in M_n$, we define

$$\mathscr{E}_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} A^k = I_n + \frac{1}{\Gamma(\alpha + \beta)} A + \frac{1}{\Gamma(2\alpha + \beta)} A^2 + \cdots.$$

Lemma 3 (e.g. [21])**.** *Let f be an analytic function defined on a region containing the origin and* $\operatorname{Sp}(A)$. *Then*

$$f(I \otimes A) = I \otimes f(A) \quad \textit{and} \quad f(A \otimes I) = f(A) \otimes I.$$

In particular, $\mathscr{E}_{\alpha,\beta}(A \otimes I) = \mathscr{E}_{\alpha,\beta}(A) \otimes I$ and $\mathscr{E}_{\alpha,\beta}(I \otimes A) = I \otimes \mathscr{E}_{\alpha,\beta}(A)$ for any $\alpha, \beta > 0$.

Lemma 4 (e.g. [21])**.** *The following properties hold for matrices of appropriate sizes:*

(1) *If $AB = BA$, then $e^{A+B} = e^A e^B$.*

(2) $e^{A \oplus B} = e^A \otimes e^B$.

The following explicit forms of Mittag-Leffler functions for certain block matrices are used in later discussions.

Lemma 5 (e.g. [12])**.** *For any $A, B \in M_n$, we have*

$$e^{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}} = \begin{bmatrix} e^A & 0 \\ 0 & e^B \end{bmatrix}, \quad e^{\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}} = \begin{bmatrix} \cosh A & \sinh A \\ \sinh A & \cosh A \end{bmatrix}, \quad e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} \mathscr{E}_2(AB) & [\mathscr{E}_{2,2}(AB)]A \\ [\mathscr{E}_{2,2}(BA)]B & \mathscr{E}_2(BA) \end{bmatrix}.$$

3. The Main System

In this section, we solve a rectangular system of coupled nonhomogeneous generalized Sylvester matrix differential equations. In particular, we also consider the homogeneous case of the system. The general exact solution of the system appears in terms of the vector operator, the Kronecker product, and certain Mittag-Leffler matrix functions.

Theorem 1. *Let m,n,k,l,p,q,r,s be positive integers such that $mn = kl$. For each i, let $A_i \in M_m$, $B_i \in M_n$, $C_i \in M_{m,k}$, $D_i \in M_{l,n}$, $E_i \in M_{k,m}$, $F_i \in M_{n,l}$, $G_i \in M_k$, and $H_i \in M_l$ be given constant matrices. Let $U(t) \in M_{m,n}$ and $V(t) \in M_{k,l}$ be given matrix functions. Let us denote*

$$K = \sum_{i=1}^{p} B_i^T \otimes A_i, \quad L = \sum_{i=1}^{q} D_i^T \otimes C_i, \quad M = \sum_{i=1}^{r} F_i^T \otimes E_i, \quad N = \sum_{i=1}^{s} H_i^T \otimes G_i. \tag{3.1}$$

Assume that $KL = LN$ and $NM = MK$. Then the system of coupled nonhomogeneous generalized Sylvester matrix differential equations with $X(t) \in M_{m,n}$ and $Y(t) \in M_{k,l}$ as unknown matrix functions:

$$\left.\begin{aligned} X'(t) &= \sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t), \\ Y'(t) &= \sum_{i=1}^{r} E_i X(t) F_i + \sum_{i=1}^{s} G_i Y(t) H_i + V(t), \end{aligned}\right\} \tag{3.2}$$

has the general solution given by

$$\left.\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)K}\{\mathscr{E}_2((t-t_0)^2 LM)\operatorname{Vec} X(t_0) + (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 LM)]M \operatorname{Vec} Y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)K}\{\mathscr{E}_2((t-s)^2 LM)\operatorname{Vec} U(s) + (t-s)[\mathscr{E}_{2,2}((t-s)^2 LM)]M \operatorname{Vec} V(s)\}ds, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)N}\{(t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 ML)]L \operatorname{Vec} X(t_0) + \mathscr{E}_2((t-t_0)^2 ML)\operatorname{Vec} Y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)N}\{(t-s)[\mathscr{E}_{2,2}((t-s)^2 ML)]L \operatorname{Vec} U(s) + \mathscr{E}_2((t-s)^2 ML)\operatorname{Vec} V(s)\}ds. \end{aligned}\right\} \tag{3.3}$$

In particular, the homogeneous case $U(t) = V(t) = 0$ of the system has the general solution

$$\operatorname{Vec} X(t) = e^{(t-t_0)K}\{\mathscr{E}_2((t-t_0)^2 LM)\operatorname{Vec} X(t_0) + (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 LM)]M \operatorname{Vec} Y(t_0)\},$$

$$\operatorname{Vec} Y(t) = e^{(t-t_0)N}\{(t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 ML)]L \operatorname{Vec} X(t_0) + \mathscr{E}_2((t-t_0)^2 ML)\operatorname{Vec} Y(t_0)\}.$$

Proof. Applying the vector operator to the system (3.2) and then using Lemma 2, we have

$$\begin{aligned} \operatorname{Vec} X'(t) &= \operatorname{Vec}\Big(\sum_{i=1}^{p} A_i X(t) B_i + \sum_{i=1}^{q} C_i Y(t) D_i + U(t)\Big) \\ &= \sum_{i=1}^{p} \operatorname{Vec}\Big(A_i X(t) B_i\Big) + \sum_{i=1}^{q} \operatorname{Vec}\Big(C_i Y(t) D_i\Big) + \operatorname{Vec} U(t) \\ &= \sum_{i=1}^{p} \Big(B_i^T \otimes A_i\Big)\operatorname{Vec} X(t) + \sum_{i=1}^{q} \Big(D_i^T \otimes C_i\Big)\operatorname{Vec} Y(t) + \operatorname{Vec} U(t) \\ &= K \operatorname{Vec} X(t) + L \operatorname{Vec} Y(t) + \operatorname{Vec} U(t), \end{aligned}$$

and similarly,

$$\operatorname{Vec} Y'(t) = M \operatorname{Vec} X(t) + N \operatorname{Vec} Y(t) + \operatorname{Vec} V(t).$$

Note that the matrices K,L,M,N are square matrices of the same size. Since the vector operator is bijective, the above two equations can be equivalently transformed to the following system:

$$\begin{bmatrix} \operatorname{Vec} X'(t) \\ \operatorname{Vec} Y'(t) \end{bmatrix} = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix} + \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix}.$$

We can rewrite it into a vector differential equation $z'(t) = Sz(t) + u(t)$ where

$$S = \begin{bmatrix} K & L \\ M & N \end{bmatrix}, \quad z(t) = \begin{bmatrix} \operatorname{Vec} X(t) \\ \operatorname{Vec} Y(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} \operatorname{Vec} U(t) \\ \operatorname{Vec} V(t) \end{bmatrix}.$$

It follows that

$$z(t) = e^{(t-t_0)S} z(t_0) + \int_{t_0}^{t} e^{(t-s)S} u(s)\, ds. \tag{3.4}$$

To get an explicit form of $e^{(t-t_0)S}$, let us decompose $S = P + Q$ where

$$P = \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix}.$$

Using a block-matrix multiplication, we get

$$PQ = \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix} = \begin{bmatrix} 0 & KL \\ NM & 0 \end{bmatrix}, \quad QP = \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} 0 & LN \\ MK & 0 \end{bmatrix}.$$

The hypotheses $KL = LN$ and $NM = MK$ imply that $PQ = QP$. From which it follows from Lemma 4 that

$$e^{(t-t_0)S} = e^{(t-t_0)P + (t-t_0)Q} = e^{(t-t_0)P} e^{(t-t_0)Q}.$$

Using Lemma 5, we have

$$e^{(t-t_0)P} = \begin{bmatrix} e^{(t-t_0)K} & 0 \\ 0 & e^{(t-t_0)N} \end{bmatrix},$$

$$e^{(t-t_0)Q} = \begin{bmatrix} \mathscr{E}_2((t-t_0)^2 LM) & (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 LM)]M \\ (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 ML)]L & \mathscr{E}_2((t-t_0)^2 ML) \end{bmatrix}.$$

Thus, we obtain

$$e^{(t-t_0)S} = \begin{bmatrix} \Psi_{11}(t,t_0) & \Psi_{12}(t,t_0) \\ \Psi_{21}(t,t_0) & \Psi_{22}(t,t_0) \end{bmatrix},$$

where

$$\Psi_{11}(t,t_0) = e^{(t-t_0)K} \mathscr{E}_2((t-t_0)^2 LM), \ \Psi_{12}(t,t_0) = (t-t_0) e^{(t-t_0)K} [\mathscr{E}_{2,2}((t-t_0)^2 LM)]M,$$
$$\Psi_{21}(t,t_0) = (t-t_0) e^{(t-t_0)N} [\mathscr{E}_{2,2}((t-t_0)^2 ML)]L, \ \Psi_{22}(t,t_0) = e^{(t-t_0)N} \mathscr{E}_2((t-t_0)^2 ML).$$

It follows from (3.4) via a block-matrix multiplication that

$$\begin{aligned} \operatorname{Vec} X(t) &= \Psi_{11}(t,t_0) \operatorname{Vec} X(t_0) + \Psi_{12}(t,t_0) \operatorname{Vec} Y(t_0) \\ &\quad + \int_{t_0}^{t} [\Psi_{11}(t,s) \operatorname{Vec} U(s) + \Psi_{12}(t,s) \operatorname{Vec} V(s)]\, ds, \\ \operatorname{Vec} Y(t) &= \Psi_{21}(t,t_0) \operatorname{Vec} X(t_0) + \Psi_{22}(t,t_0) \operatorname{Vec} Y(t_0) \\ &\quad + \int_{t_0}^{t} [\Psi_{21}(t,s) \operatorname{Vec} U(s) + \Psi_{22}(t,s) \operatorname{Vec} V(s)]\, ds. \end{aligned}$$

Therefore, the solution of (3.2) is given by (3.3). □

Once we get $\operatorname{Vec} X(t)$ and $\operatorname{Vec} Y(t)$, we can recover $X(t)$ and $Y(t)$ due to the injectivity of the vector operator.

4. Special Cases of the Main System

In Theorem 1, the hypothesis $KL = LN$ and $NM = MK$ is not restrictive since it includes many interesting special cases and previous works. We shall consider systems of coupled matrix/vector differential equations, and single matrix differential equations. In these cases, the general solutions are given in terms of matrix series concerning exponentials, hyperbolic functions and certain Mittag-Leffler functions.

4.1 Systems of Coupled Matrix Differential Equations

Corollary 1. *From the hypothesis and notations in Theorem 1, assume further that* $L = M$. *Then the general solution of the system* (3.2) *is given by*

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{(t-t_0)K}\{(\cosh(t-t_0)L)\operatorname{Vec} X(t_0) + (\sinh(t-t_0)L)\operatorname{Vec} Y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)K}\{(\cosh(t-s)L)\operatorname{Vec} U(s) + (\sinh(t-s)L)\operatorname{Vec} V(s)\}\,ds, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)N}\{(\sinh(t-t_0)L)\operatorname{Vec} X(t_0) + (\cosh(t-t_0)L)\operatorname{Vec} Y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)N}\{(\sinh(t-s)L)\operatorname{Vec} U(s) + (\cosh(t-s)L)\operatorname{Vec} V(s)\}\,ds.\end{aligned}$$

In particular, the homogeneous case $U(t) = V(t) = 0$ *of the system has the general solution*

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{(t-t_0)K}\{(\cosh(t-t_0)L)\operatorname{Vec} X(t_0) + (\sinh(t-t_0)L)\operatorname{Vec} Y(t_0)\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)N}\{(\sinh(t-t_0)L)\operatorname{Vec} X(t_0) + (\cosh(t-t_0)L)\operatorname{Vec} Y(t_0)\}.\end{aligned}$$

Proof. The desire formulas of $\operatorname{Vec} X(t)$ and $\operatorname{Vec} Y(t)$ follow from (3.3) in Theorem 1 together with the facts that $\mathscr{E}_2(Z^2) = \cosh Z$ and $(\mathscr{E}_{2,2}(Z^2))Z = \sinh Z$ for any square matrix Z. □

Corollary 2. *Let* m,n,k,l *be positive integers such that* $mn = kl$. *Let* $A \in M_m$, $B \in M_n$, $C \in M_{m,k}$, $D \in M_{l,n}$, $E \in M_{k,m}$, $F \in M_{n,l}$, $G \in M_k$, *and* $H \in M_l$ *be given constant matrices. Let* $U(t) \in M_{m,n}$ *and* $V(t) \in M_{k,l}$ *be given matrix functions. Denote* $K = B^T \otimes A$, $N = H^T \otimes G$, $R = (FD)^T \otimes CE$, *and* $S = (DF)^T \otimes EC$. *Assume that*

$$(DB)^T \otimes (AC) = (HD)^T \otimes (CG), \quad (FH)^T \otimes (GE) = (BF)^T \otimes (EA). \tag{4.1}$$

Then the general solution of the system:

$$\left.\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t), \end{aligned}\right\} \tag{4.2}$$

in unknown $X(t) \in M_{m,n}$ *and* $Y(t) \in M_{k,l}$ *is given by*

$$\begin{aligned}\operatorname{Vec} X(t) &= e^{(t-t_0)K}\{\mathscr{E}_2((t-t_0)^2 R)\operatorname{Vec} X(t_0) + (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 R)](F^T \otimes E)\operatorname{Vec} Y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)K}\{\mathscr{E}_2((t-s)^2 R)\operatorname{Vec} U(s)ds + (t-s)[\mathscr{E}_{2,2}((t-s)^2 R)](F^T \otimes E)\operatorname{Vec} V(s)\}ds, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)N}\{(t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2 S)](D^T \otimes C)\operatorname{Vec} X(t_0) + \mathscr{E}_2((t-t_0)^2 S)\operatorname{Vec} Y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)N}\Big\{(t-s)[\mathscr{E}_{2,2}((t-s)^2 S)](D^T \otimes C)\operatorname{Vec} U(s)ds + \mathscr{E}_2((t-s)^2 S)\operatorname{Vec} V(s)\Big\}ds.\end{aligned}$$

Proof. The desire result follows from Theorem 1 by considering the case $p = q = r = s = 1$. Note that $(D^T \otimes C)(F^T \otimes E) = R$ and $(F^T \otimes E)(D^T \otimes C) = S$ by Lemma 1. □

Corollary 2 includes the work [2, Theorem 3.7] as a special case.

Corollary 3. *Let $A \in M_m$, $D \in M_{l,n}$, and $F \in M_{n,l}$ be given constant matrices. Let $U(t) \in M_{m,n}$ and $V(t) \in M_{m,l}$ be given matrix functions. The general solution of the system*

$$X'(t) = AX(t) + Y(t)D + U(t),$$
$$Y'(t) = X(t)F + AY(t) + V(t),$$

with $X(t) \in M_{m,n}$ and $Y(t) \in M_{m,l}$ as unknown matrix functions is given by

$$X(t) = e^{(t-t_0)A}\{X(t_0)\mathscr{E}_2((t-t_0)^2FD) + (t-t_0)Y(t_0)F\mathscr{E}_{2,2}((t-t_0)^2FD)\}$$
$$+ \int_{t_0}^{t} e^{(t-s)A}\{U(s)\mathscr{E}_2((t-s)^2FD) + (t-s)V(s)F\mathscr{E}_{2,2}((t-s)^2FD)\}\,ds,$$
$$Y(t) = e^{(t-t_0)A}\{(t-t_0)X(t_0)D\mathscr{E}_{2,2}((t-t_0)^2DF) + Y(t_0)\mathscr{E}_2((t-t_0)^2DF)\}$$
$$+ \int_{t_0}^{t} e^{(t-s)A}\{(t-s)U(s)D\mathscr{E}_{2,2}((t-s)^2DF) + V(s)\mathscr{E}_2((t-s)^2DF)\}\,ds.$$

Proof. From Corollary 2, put $k = m$, $B = I_n$, $C = E = I_m$, $H = I_l$ and $G = A$. Then the condition (4.1) is satisfied. It follows that

$$\begin{aligned}\operatorname{Vec} X(t) = e^{(t-t_0)I\otimes A}\{&[\mathscr{E}_2((t-t_0)^2(FD)^T \otimes I)]\operatorname{Vec} X(t_0)\\ &+ (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2(FD)^T \otimes I)](F^T \otimes I)\operatorname{Vec} Y(t_0)\}\\ &+ \int_{t_0}^{t} e^{(t-s)I\otimes A}\{[\mathscr{E}_2((t-s)^2(FD)^T \otimes I)]\operatorname{Vec} U(s)\\ &+ (t-s)[\mathscr{E}_{2,2}((t-s)^2(FD)^T \otimes I)](F^T \otimes I)\operatorname{Vec} V(s)\}\,ds.\end{aligned}$$

Lemmas 1 and 4 together imply that $e^{(t-t_0)I\otimes A} = I \otimes e^{(t-t_0)A}$ and

$$\begin{aligned}[\mathscr{E}_{2,2}((t-t_0)^2(FD)^T \otimes I)](F^T \otimes I) &= [\mathscr{E}_{2,2}((t-t_0)^2(FD)^T) \otimes I](F^T \otimes I)\\ &= \{[\mathscr{E}_{2,2}((t-t_0)^2FD)]^T \otimes I\}(F^T \otimes I)\\ &= [F\mathscr{E}_{2,2}((t-t_0)^2FD)]^T \otimes I.\end{aligned}$$

Thus, by Lemma 2, the linearity and the continuity of $\operatorname{Vec}(\cdot)$, we have

$$\begin{aligned}\operatorname{Vec} X(t) &= [I \otimes e^{(t-t_0)A}]\Big\{\{[\mathscr{E}_2((t-t_0)^2FD)]^T \otimes I\}\operatorname{Vec} X(t_0) + (t-t_0)\{[F\mathscr{E}_{2,2}((t-t_0)^2FD)]^T \otimes I\}\operatorname{Vec} Y(t_0)\Big\}\\ &\quad + \int_{t_0}^{t} [I \otimes e^{(t-s)A}]\Big\{\{[\mathscr{E}_2((t-s)^2FD)]^T \otimes I\}\operatorname{Vec} U(s)\\ &\quad + (t-s)\{[F\mathscr{E}_{2,2}((t-s)^2FD)]^T \otimes I\}\operatorname{Vec} V(s)\Big\}\,ds\\ &= \Big\{[\mathscr{E}_2((t-t_0)^2FD)]^T \otimes e^{(t-t_0)A}\Big\}\operatorname{Vec} X(t_0) + (t-t_0)\Big\{[F\mathscr{E}_{2,2}((t-t_0)^2FD)]^T \otimes e^{(t-t_0)A}\Big\}\operatorname{Vec} Y(t_0)\\ &\quad + \int_{t_0}^{t} \Big\{\{[\mathscr{E}_2((t-s)^2FD)]^T \otimes e^{(t-s)A}\}\operatorname{Vec} U(s)\\ &\quad + (t-s)\{[F\mathscr{E}_{2,2}((t-s)^2FD)]^T \otimes e^{(t-s)A)}\}\operatorname{Vec} V(s)\Big\}\,ds\end{aligned}$$

$$= \mathrm{Vec}\{e^{(t-t_0)A}X(t_0)\mathscr{E}_2((t-t_0)^2FD)\} + (t-t_0)\mathrm{Vec}\{e^{(t-t_0)A}Y(t_0)F\mathscr{E}_{2,2}((t-t_0)^2FD)\}$$
$$+ \int_{t_0}^{t} \mathrm{Vec}\{e^{(t-s)A}U(s)\mathscr{E}_2((t-s)^2FD)\} + (t-s)\mathrm{Vec}\{e^{(t-s)A)}V(s)F\mathscr{E}_{2,2}((t-s)^2FD)\}\, ds$$
$$= \mathrm{Vec}\Big[e^{(t-t_0)A}\{X(t_0)\mathscr{E}_2((t-t_0)^2FD) + (t-t_0)Y(t_0)F\mathscr{E}_{2,2}((t-t_0)^2FD)\}$$
$$+ \int_{t_0}^{t} e^{(t-s)A}\{U(s)\mathscr{E}_2((t-s)^2FD) + (t-s)V(s)F\mathscr{E}_{2,2}((t-s)^2FD)\}\, ds\Big].$$

From the injectivity of $\mathrm{Vec}(\cdot)$, we obtain the exact formula of $X(t)$ as desire. To get the formula of $Y(t)$, apply the same process. □

The next corollary was firstly obtained in [12].

Corollary 4. *Let $A,B,C,D \in M_n$ be such that $AC = CA$ and $BD = DB$. Then the system*

$$X'(t) = AX(t)B + CY(t)D,$$
$$Y'(t) = CX(t)D + AY(t)B,$$

in unknown matrix functions $X(t), Y(t) \in M_n$ has the general solution given by

$$\mathrm{Vec}\,X(t) = e^{(t-t_0)(B^T\otimes A)}\{[\cosh(t-t_0)(D^T\otimes C)]\mathrm{Vec}\,X(t_0) + [\sinh(t-t_0)(D^T\otimes C)]\mathrm{Vec}\,Y(t_0)\},$$
$$\mathrm{Vec}\,Y(t) = e^{(t-t_0)(B^T\otimes A)}\{[\sinh(t-t_0)(D^T\otimes C)]\mathrm{Vec}\,X(t_0) + [\cosh(t-t_0)(D^T\otimes C)]\mathrm{Vec}\,Y(t_0)\}.$$

Proof. It is a special case of Corollary 1 when all given matrices are square and of the same size, $K = N = B^T \otimes A$ and $L = M = D^T \otimes C$. □

4.2 Single Matrix Differential Equations

Now, we discuss certain single matrix differential equations from the main system (3.2).

Corollary 5. *For each i, let $A_i \in M_m$, $B_i \in M_n$, and $U(t) \in M_{m,n}$. Denote $K = \sum_{i=1}^{p} B_i^T \otimes A_i$. Then the generalized Sylvester matrix differential equation*

$$X'(t) = \sum_{i=1}^{p} A_i X(t) B_i + U(t)$$

in unknown matrix function $X(t) \in M_{m,n}$ has the general solution given by

$$\mathrm{Vec}\,X(t) = e^{(t-t_0)K}\mathrm{Vec}\,X(t_0) + \int_{t_0}^{t} e^{(t-s)K}\mathrm{Vec}\,U(s)\,ds.$$

For its homogeneous case $U(t) = 0$, its general solution is reduced to $\mathrm{Vec}\,X(t) = e^{(t-t_0)K}\mathrm{Vec}\,X(t_0)$.

Proof. From Theorem 1, we can make a system of coupled equations into uncoupled equations by putting $C_i = E_i = 0$ for all i. In this case, $L = M = 0$. □

The next result was firstly obtained in [3, Theorem 1].

Corollary 6. *Let $A,B,U(t) \in M_n$ be given, and let $X(t) \in M_n$ be unknown. Then the Sylvester matrix differential equation*

$$X'(t) = AX(t) + X(t)B + U(t)$$

has the general solution given by

$$X(t) = e^{(t-t_0)A}X(t_0)e^{(t-t_0)B} + \int_{t_0}^{t} e^{(t-s)A}U(s)e^{(t-s)B}\,ds.$$

Proof. From Corollary 5, we have $K = I \otimes A + B^T \otimes I = B^T \oplus A$, the Kronecker sum of B^T and A. Applying Lemmas 1 and 4, we get

$$e^{(t-t_0)(B^T\oplus A)} = e^{(t-t_0)B^T \oplus (t-t_0)A} = e^{(t-t_0)B^T} \otimes e^{(t-t_0)A} = [e^{(t-t_0)B}]^T \otimes e^{(t-t_0)A}.$$

Now, Lemma 2 yields

$$\operatorname{Vec} X(t) = \operatorname{Vec}\{e^{(t-t_0)A}X(t_0)e^{(t-t_0)B}\} + \int_{t_0}^{t} \operatorname{Vec}\{e^{(t-s)A}U(s)e^{(t-s)B}\}\,ds.$$

Since the vector operator is linear and continuous, we obtain

$$\operatorname{Vec} X(t) = \operatorname{Vec}\{e^{(t-t_0)A}X(t_0)e^{(t-t_0)B} + \int_{t_0}^{t} e^{(t-s)A}U(s)e^{(t-s)B}\,ds\}.$$

We finally get $X(t)$ from $\operatorname{Vec} X(t)$ due to the injectivity of the vector operator. □

The explicit form of the solution for the homogeneous case $U(t) = 0$ in Corollary 6 was firstly obtained in [6].

4.3 A System of Coupled Vector Differential Equations

Corollary 7. *Let $A \in M_m$, $C \in M_{m,k}$, $E \in M_{k,m}$, $G \in M_k$ be given constant matrices such that $AC = CG$ and $GE = EA$. Let $u(t) \in \mathbb{R}^m$, $v(t) \in \mathbb{R}^k$ be given vector functions. Then the general solution of the system of coupled vector differential equations*

$$x'(t) = Ax(t) + Cy(t) + u(t),$$
$$y'(t) = Ex(t) + Gy(t) + v(t)$$

in unknown vector functions $x(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^k$ is given by

$$\begin{aligned} x(t) &= e^{(t-t_0)A}\{\mathscr{E}_2((t-t_0)^2CE)x(t_0) + (t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2CE)]Ey(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)A}\{\mathscr{E}_2((t-s)^2CE)u(s) + (t-s)[\mathscr{E}_{2,2}((t-s)^2CE)]Ev(s)\}\,ds, \\ y(t) &= e^{(t-t_0)G}\{(t-t_0)[\mathscr{E}_{2,2}((t-t_0)^2EC)]Cx(t_0) + [\mathscr{E}_2((t-t_0)^2EC)]y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)G}\{(t-s)[\mathscr{E}_{2,2}((t-s)^2EC)]Cu(s) + [\mathscr{E}_2((t-s)^2EC)]v(s)\}\,ds. \end{aligned}$$

If in addition $C = E$, then

$$\begin{aligned} x(t) &= e^{(t-t_0)A}\{(\cosh(t-t_0)C)x(t_0) + (\sinh(t-t_0)C)y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)A}\{(\cosh(t-s)C)u(s) + (\sinh(t-s)C)v(s)\}, \\ y(t) &= e^{(t-t_0)G}\{(\sinh(t-t_0)C)x(t_0) + (\cosh(t-t_0)C)y(t_0)\} \\ &\quad + \int_{t_0}^{t} e^{(t-s)N}\{(\sinh(t-s)C)u(s) + (\cosh(t-s)C)v(s)\}\,ds. \end{aligned}$$

Proof. It is a special case of Corollary 2 when $l = n = 1$ and $B = D = F = H = [1]$. □

5. Illustrative Initial Value Problems

In this section, we illustrate initial value problems associated with systems of matrix differential equations considered in the previous section. When initial conditions are imposed to such system, its solution is unique and appeared in an explicit form.

Example 1. Let $A, C \in M_n$ be such that $AC = CA$. Consider the following system of coupled matrix differential equations:

$$X'(t) = AX(t) + CY(t), \tag{5.1}$$

$$Y'(t) = CX(t) + AY(t), \tag{5.2}$$

in unknown matrix functions $X(t), Y(t) \in M_n$. From Corollary 4, we have

$$\operatorname{Vec} X(t) = e^{(t-t_0)(I\otimes A)}\{[\cosh(t-t_0)(I\otimes C)]\operatorname{Vec} X(t_0) + [\sinh(t-t_0)(I\otimes C)]\operatorname{Vec} Y(t_0)\}.$$

Using Lemma 1, we can reduce the above formula as follows:

$$\begin{aligned}\operatorname{Vec} X(t) &= (I\otimes e^{(t-t_0)A})\{[I\otimes \cosh(t-t_0)C]\operatorname{Vec} X(t_0) + I\otimes[\sinh(t-t_0)C]\operatorname{Vec} Y(t_0)\}\\ &= (I\otimes e^{(t-t_0)A}\cosh(t-t_0)C)\operatorname{Vec} X(t_0) + (I\otimes e^{(t-t_0)A}\sinh(t-t_0)C)\operatorname{Vec} Y(t_0).\end{aligned}$$

Thus, by Lemma 2, we get

$$X(t) = e^{(t-t_0)A}\left[(\cosh(t-t_0)C)X(t_0) + (\sinh(t-t_0)C)Y(t_0)\right], \tag{5.3}$$

$$Y(t) = e^{(t-t_0)A}\left[(\sinh(t-t_0)C)X(t_0) + (\cosh(t-t_0)C)Y(t_0)\right]. \tag{5.4}$$

Now, consider the system (5.1) under initial conditions $X(0) = W_1$ and $Y(0) = W_2$, where

$$A = \begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix},\quad C = \begin{bmatrix}2 & 0\\ -2 & 1\end{bmatrix},\quad W_1 = \begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix},\ \text{and } W_2 = \begin{bmatrix}1 & 0\\ -2 & 2\end{bmatrix}.$$

First, note that $AC = CA$. Now, we compute the following matrix exponentials:

$$e^{tA} = \begin{bmatrix}e^t & 0\\ 2(e^{2t}-e^t) & e^{2t}\end{bmatrix},\quad e^{tC} = \begin{bmatrix}e^{2t} & 0\\ 2(e^t-e^{2t}) & e^t\end{bmatrix},\quad e^{-tC} = \begin{bmatrix}e^{-2t} & 0\\ 2(e^{-t}-e^{-2t}) & e^{-t}\end{bmatrix}.$$

Then we have the following matrix hyperbolic functions:

$$\cosh tC = \frac{1}{2}(e^{tC} + e^{-tC}) = \begin{bmatrix}\cosh 2t & 0\\ 2\cosh t - 2\cosh 2t & \cosh t\end{bmatrix}.$$

$$\sinh tC = \frac{1}{2}(e^{tC} - e^{-tC}) = \begin{bmatrix}\sinh 2t & 0\\ 2\sinh t - 2\sinh 2t & \sinh t\end{bmatrix}.$$

From (5.3) and (5.4), we have

$$X(t) = \begin{bmatrix}e^{3t} & 0\\ -\frac{1}{2}e^{3t} + \frac{3}{2}e^t & 2e^{3t}\end{bmatrix},\quad Y(t) = \begin{bmatrix}e^{3t} & 0\\ -\frac{1}{2}e^{3t} - \frac{3}{2}e^t & 2e^{3t}\end{bmatrix}.$$

One can check that the formulas of $X(t)$ and $Y(t)$ satisfy the above initial value problem.

Example 2. Consider the following coupled matrix differential equations:

$$\left.\begin{aligned}X'(t) &= AX(t)B + CY(t)D,\\ Y'(t) &= CX(t)D + AY(t)B,\end{aligned}\right\} \tag{5.5}$$

under initial conditions $X(0)=W_1$ and $Y(0)=W_2$, where

$$A=\begin{bmatrix}1&0\\2&2\end{bmatrix},\ B=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ C=\begin{bmatrix}2&0\\-2&1\end{bmatrix},\ D=\begin{bmatrix}1&1\\1&1\end{bmatrix},\ W_1=\begin{bmatrix}1&0\\3&-1\end{bmatrix},\ W_2=\begin{bmatrix}0&-1\\1&2\end{bmatrix}.$$

First, note that $AC=CA$ and $BD=DB$. From Corollary 4, we have

$$\operatorname{Vec}X(t)=e^{tB^T\otimes A}\{[\cosh(tD^T\otimes C)]\operatorname{Vec}X(0)+[\sinh(tD^T\otimes C)]\operatorname{Vec}Y(0)\},\tag{5.6}$$

$$\operatorname{Vec}Y(t)=e^{tB^T\otimes A}\{[\sinh(tD^T\otimes C)]\operatorname{Vec}X(0)+[\cosh(tD^T\otimes C)]\operatorname{Vec}Y(0)\}.\tag{5.7}$$

Note that by Lemma 1(5), we have $\operatorname{Sp}(B^T\otimes A)=\{1,2,-1,-2\}$. So, to get an explicit formula of $e^{tB^T\otimes A}$, we apply Cayley-Hamilton theorem to write it as a matrix polynomial of degree at most 3:

$$e^{tB^T\otimes A}=\sum_{k=0}^{3}r_k(t)(B^T\otimes A)^k$$

for some $r_0(t),r_1(t),r_2(t),r_3(t)$ satisfying $e^{t\lambda}=\sum_{k=0}^3 r_k(t)\lambda^k$ for all $\lambda\in\operatorname{Sp}(B^T\otimes A)$. This leads to the Vandermonde system:

$$\begin{bmatrix}e^t\\e^{2t}\\e^{-t}\\e^{-2t}\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&2&4&8\\1&-1&1&-1\\1&-2&4&-8\end{bmatrix}\begin{bmatrix}r_0(t)\\r_1(t)\\r_2(t)\\r_3(t)\end{bmatrix}.$$

Solving this linear system to get

$$r_0(t)=\frac{1}{12}(8e^t-2e^{2t}+8e^{-t}-2e^{-2t}),\qquad r_1(t)=\frac{1}{12}(8e^t-e^{2t}-8e^{-t}+e^{-2t}),$$
$$r_2(t)=\frac{1}{12}(-2e^t+2e^{2t}-2e^{-t}+2e^{-2t}),\quad r_3(t)=\frac{1}{12}(-2e^t+e^{2t}+2e^{-t}-e^{-2t}).$$

Thus,

$$\begin{aligned}e^{tB^T\otimes A}&=r_0(t)I+r_1(t)(B^T\otimes A)+r_2(t)(B^T\otimes A)^2+r_3(t)(B^T\otimes A)^3\\&=\begin{bmatrix}r_0(t)+r_2(t)&0&r_1(t)+r_3(t)&0\\6r_2(t)&r_0(t)+4r_2(t)&2r_1(t)+14r_3(t)&2r_1(t)+8r_3(t)\\r_1(t)+r_3(t)&0&r_0(t)+r_2(t)&0\\2r_1(t)+14r_3(t)&2r_1(t)+6r_3(t)&6r_2(t)&r_0(t)+4r_2(t)\end{bmatrix}.\end{aligned}$$

Using Lemma 5 to compute the following:

$$e^{tD^T\otimes C}=e^{\begin{bmatrix}tC&tC\\tC&tC\end{bmatrix}}=e^{\begin{bmatrix}tC&0\\0&tC\end{bmatrix}}e^{\begin{bmatrix}0&tC\\tC&0\end{bmatrix}}=\frac{1}{2}\begin{bmatrix}e^{2tC}+I&e^{2tC}-I\\e^{2tC}-I&e^{2tC}+I\end{bmatrix}.$$

Now, we have

$$e^{2tC}=\begin{bmatrix}e^{4t}&0\\2(e^{2t}-e^{4t})&e^{2t}\end{bmatrix},\quad e^{tD^T\otimes C}=\begin{bmatrix}e^{4t}+1&0&e^{4t}-1&0\\2(e^{2t}-e^{4t})&e^{2t}+1&2(e^{2t}-e^{4t})&e^{2t}-1\\e^{4t}-1&0&e^{4t}+1&0\\2(e^{2t}-e^{4t})&e^{2t}-1&2(e^{2t}-e^{4t})&e^{2t}+1\end{bmatrix}.$$

Hence, one can get the solutions $X(t)$ and $Y(t)$ from (5.6) and (5.7).

6. Conclusion

We investigate certain systems of coupled generalized Sylvester matrix differential equations in both nonhomogeneous and homogeneous cases. To solve for its general solution, we apply the vector operator to the matrix systems, so that they are reduced to simple equivalent vector differential systems. We obtain explicit forms of its general solution in terms of Kronecker products, and matrix series concerning Mittag-Leffler functions, exponentials, and hyperbolic functions. Our assumptions on coefficient matrices are not too restricted anymore since they include many interesting special cases. Our results includes certain systems of coupled matrix/vector differential equations, and single matrix differential equations as special cases.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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