Perturbation of A Nonlinear Elliptic Biological Interacting Model with Multiple Species

Joon Hyuk Kang

Abstract. In this paper, we investigate the effects of perturbations on the coexistence state of the general competition model for multiple species. Previous work by Kang, Lee and Oh (see [11]) established sufficient conditions for the uniqueness of the positive solution to the following general elliptic system for multiple competing species of animals:

$$\Delta u_i + u_i g_i(u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_N) = 0 \text{ in } \Omega, \ u_i|_{\partial\Omega} = 0$$

for $i = 1, \ldots, N$. That is, they proved that under certain conditions, the species can coexist and that the coexistence state is unique at fixed rates. In this paper, we extend their uniqueness results by perturbing functions g_i's of the above model, and applying super-sub solutions, maximum principles and spectrum estimates. Our arguments also rely on some detailed properties for the solution of logistic equations. By applying these techniques, we obtain sufficient conditions for the existence and uniqueness of a time independent coexistence state for the perturbed general competition model.

1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the competition of two or more species of animals in the same environment. Especially pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general competition model to better understand the competitive interactions between species. Specifically, we investigate the conditions needed for the coexistence of species when the factors affecting them are perturbed.

2. Literature review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple competition model, commonly known

as the Lotka-Volterra competition model. This system describes the competitive interaction of species residing in the same environment in the following manner:

$$
\begin{cases}
(u_i)_t(x,t) = \Delta u_i(x,t) + u_i(x,t)(a_i - b_i u_i(x,t) - \sum_{j=1,j\neq i}^{N} c_{ij} u_j(x,t)) \text{ in } \Omega \times R^+, \\
u_i(x,t)|_{\partial\Omega} = 0,
\end{cases}
$$

$$(2.1)$$

for $i = 1, 2, \ldots, N$, where Ω is a bounded domain in R^n. Here, $u_i(x,t)$ designate the population densities for the competing species. The positive constant coefficients in this system represent growth rates (a_is), self-limitation rates (b_is) and competition rates (c_{ij}s). Furthermore, we assume that both species are not residing on the boundary of Ω.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (2.1) (see [1], [2], [3], [4], [6], [7]). The positive steady state solution is simply the positive solution to the time-independent system

$$
\begin{cases}
\Delta u_i(x) + u_i(x)(a_i - b_i u_i(x) - \sum_{j=1,j\neq i}^{N} c_{ij} u_j(x)) = 0 \text{ in } \Omega, \\
u_i(x)|_{\partial\Omega} = 0
\end{cases}
$$

for $i = 1, 2, \ldots, N$.

One of the initial important results for the time-independent Lotka-Volterra model for $N = 2$ was obtained by Cosner and Lazer, i.e. they considered the simple model

$$
\begin{cases}
\Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\
\Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \quad \text{in } \Omega, \\
u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{cases}
$$

$$(2.2)$$

where $a, d > 0$ are growth rates, $b, f > 0$ are self-limitation rates, and $c, e > 0$ are competition rates. In 1984, they published the following sufficient conditions for the existence and uniqueness of a positive steady state solution to (2.2):

Theorem 2.1 ([4]). *Suppose*

(A) $a > \lambda_1 + \dfrac{cd}{f}$, $d > \lambda_1 + \dfrac{ae}{b}$, *where λ_1 is the smallest eigenvalue of $-\Delta$ with homogeneous boundary conditions,*

(B) $4bf > \dfrac{fc^2}{b} \sup\limits_{x\in\Omega} \left[\dfrac{\omega_a(x)}{\omega_{d-\frac{ae}{b}}(x)} \right] + 2ce + \dfrac{be^2}{f} \sup\limits_{x\in\Omega} \left[\dfrac{\omega_d(x)}{\omega_{a-\frac{cd}{f}}(x)} \right]$, *where $\omega_M(x)$ for $M > 0$ is the unique positive solution to the logistic equation as mentioned in the next section.*

Then (2.2) has a unique positive solution.

Cosner and Lazer's theorem implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, then there is a unique positive steady state solution to (2.2). In other words, the two species will coexist indefinitely at unique population densities.

In 1989, Cantrell and Cosner extended these results by proving that the reproduction and self limitation rates may vary within bounds without losing the uniqueness result, given certain conditions. Biologically, Cantrell and Cosner's theorem suggests that two species can relax ecologically and maintain a coexistence state. Their primary result is given below:

Theorem 2.2 ([3]). *If $a = d > \lambda_1$, $b = f = 1$, and $0 < c$, $e < 1$, then there is a neighborhood V of (a, a) such that if $(a_0, d_0) \in V$, then (2.2) with $(a, d) = (a_0, d_0)$ has a unique positive solution.*

In Theorem 2.2, the condition $0 < c$, $e < 1$ biologically implies that the competition rates of both species must be relatively small. This condition plays an important role in the proof of Cantrell and Cosner's theorem by implying the invertibility of the Frechet derivative (linearization) of (2.2) at a fixed reproduction rate (a, a).

The work of Lazer, Cosner, and Cantrell provides insight into the competitive interactions of species operating under the conditions described in the Lotka-Volterra model. However, their results are somewhat limited by a few key assumptions. In the Lotka-Volterra model that they studied, the rate of change of densities largely depends on constant rates of reproduction, self-limitation, and competition. The model also assumes a linear relationship of the terms affecting the rate of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple competition model. Therefore, in the last decade, significant research has been focused on the existence and uniqueness of the positive steady state solution of the general competition model for possibly multiple species, i.e. the positive solution to

$$\Delta u_i + u_i g_i(u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_N) = 0 \text{ in } \Omega, \; u_i|_{\partial \Omega} = 0 \qquad (2.3)$$

for $i = 1, \ldots, N$, where $g_i \in C^1$ designate reproduction, self-limitation and competition rates that satisfy the growth conditions (G1) and (G2) given below (see [9], [10], [11], [13], [12]).

Because of its broader applicability, the general competition model has become a more popular subject of research within the mathematical community over the past few years. In 2004, Kang, Oh, and Lee (see [11]) established the following existence, nonexistence and uniqueness result, which generalizes Theorem 2.1 obtained by Cosner and Lazer.

Theorem 2.3. *Suppose*

(G1) $\frac{\partial g_i}{\partial u_j} < 0$ *for $i, j = 1, \ldots, N$,*

(G2) *there exist constants $c_1 > 0$, $c_2 > 0, \ldots, c_N > 0$ such that $g_i(0, \ldots, 0, u_i, 0, \ldots, 0) \leq 0$ for $u_i \geq c_i$.*

(1) *If $g_i(c_1, c_2, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_N) > \lambda_1$ for $i = 1, \ldots, N$, then (2.3) has a coexistence state.*

(2) *If $g_i(0, \ldots, 0) \leq \lambda_1$ for some $i = 1, \ldots, N$, then (2.3) does not have any positive solution.*

(3) *If $g_i(c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_N) > \lambda_1$ and*

$$2\inf\left(-\frac{\partial g_i}{\partial x_i}\right) > \sum_{j=1, j\neq i}^{N}\left(\sup\left(-\frac{\partial g_i}{\partial x_j}\right) + K\sup\left(-\frac{\partial g_j}{\partial x_i}\right)\right)$$

for $i = 1, \ldots, N$, where $K = \sup\limits_{i,j\neq i}\dfrac{\theta_{g_j(0,\ldots,0,\cdot,0,\ldots,0)}}{\theta_{g_i(c_1,\ldots,c_{i-1},\cdot,c_{i+1},\ldots,c_N)}}$. Then (2.3) has a unique coexistence state.

Biologically, we can interpret the conditions in Theorem 2.3 as follows. The functions g_i, $\frac{\partial g_i}{\partial u_j}$ describe the manner in which members of each species u_i interact among themselves and with members of the other species. Hence, the conditions imply that members of each species interact strongly among themselves and weakly with members of the other species. If these conditions are fulfilled, then the species residing within the same domain will coexist indefinitely at unique densities.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the Lotka-Volterra model with constant rates(see [3], [4], [7], [12]). However, the stability of the steady state solution for the general model remains open to investigation.

The research presented in this paper therefore begins the mathematical community's discussion on the stability of the steady state solution for the general competition model. In our analysis we focus on the conditions required for the maintenance of the coexistence state of (2.3)when bounded functions g_i's are slightly perturbed. Mathematically, our results will generalize Theorem 2.2 developed by Cantrell and Cosner. Biologically, our conclusion implies that the species may slightly relax ecologically and yet continue to coexist at unique densities.

3. **Preliminaries**

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition 3.1 (Super and Sub solutions). The vector functions $(\bar{u}^1,\ldots,\bar{u}^N)$, $(\underline{u}^1,\ldots,\underline{u}^N)$ form a super/sub solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1,\ldots,u^N) = 0 & \text{in } \Omega \\ u^i = 0 & \text{on } \partial\Omega \end{cases}$$

if for $i = 1,\ldots,N$

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1,\ldots,u^{i-1},\bar{u}^i,u^{i+1},\ldots,u^N) \leq 0 \\ \Delta \underline{u}^i + g^i(u^1,\ldots,u^{i-1},\underline{u}^i,u^{i+1},\ldots,u^N) \geq 0 \\ \qquad\qquad\quad \text{in } \Omega \text{ for } \underline{u}^j \leq u^j \leq \bar{u}^j, \ j \neq i, \end{cases}$$

and

$$\underline{u}^i \leq \bar{u}^i \ \text{ on } \ \Omega$$

$$\underline{u}^i \leq 0 \leq \bar{u}^i \ \text{ on } \ \partial\Omega.$$

Lemma 3.2 ([1]). *If g^i in the Definition 3.1 are in C^1 and the system admits an super/sub solution pair $(\underline{u}^1,\ldots,\underline{u}^N),(\bar{u}^1,\ldots,\bar{u}^N)$, then there is a solution of the system in 3.1 with $\underline{u}^i \leq u^i \leq \bar{u}^i$ in $\bar{\Omega}$. If*

$$\Delta \bar{u}^i + g^i(\bar{u}^1,\ldots,\bar{u}^N) \neq 0,$$

$$\Delta \underline{u}^i + g^i(\underline{u}^1,\ldots,\underline{u}^N) \neq 0$$

in Ω for $i = 1,\ldots,N$, then $\underline{u}^i < u^i < \bar{u}^i$ in Ω.

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

Lemma 3.3 (Established in [13]). *Consider*

$$\Delta u + uf(u) = 0 \ \text{ in } \ \Omega,$$

$$u|_{\partial\Omega} = 0, u > 0,$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n.

If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary conditions. We denote this unique positive solution as θ_f.

The most important property of this positive solution, for the purposes of our research, is that θ_f is increasing as f is increasing.

We specifically note that for $a > \lambda_1$, the unique positive solution of

$$\Delta u + u(a - u) = 0 \ \text{ in } \ \Omega,$$

$$u|_{\partial\Omega} = 0, \ u > 0,$$

is denoted by $\omega_a \equiv \theta_{a-x}$. Hence, θ_a is increasing as $a > 0$ is increasing.

Having established these preliminaries, we now commence our investigation of perturbations of the general competition model.

4. **Perturbation**

We consider the model

$$\Delta u_i + u_i g_i(u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_N) = 0 \text{ in } \Omega, \ u_i|_{\partial\Omega} = 0 \tag{4.1}$$

for $i = 1, \ldots, N$. Here Ω is a smooth, bounded domain in R^n. Also, two growth conditions hold:

(P1) $g_i \in C_B^1$ for $i = 1, \ldots, N$, where C_B^1 is the set of all functions f such that f is decreasing, and all the first-order partial derivatives of f are bounded and continuous.

(P2) there are $c_i > 0$ such that $g_i(0, \ldots, 0, u_i, 0, \ldots, 0) < 0$ for $u \geq c_i$, $i = 1, \ldots, N$.

The following theorem is our main result on coexistence for the general competition model.

Theorem 4.1. *Suppose*

(A) $\lambda_1(-g_i(\theta_{g_1(\cdot,0,\ldots,0)}, \ldots, \theta_{g_{i-1}(0,\ldots,0,\cdot,0,\ldots,0)}, 0, \theta_{g_{i+1}(\cdot,0,\ldots,0)}, \ldots, \theta_{g_N(0,\ldots,0\cdot)})) < 0$ *for* $i = 1, \ldots, N$, *where in general, $\lambda_1(q)$ is the smallest eigenvalue of $-\Delta + q$ with homogeneous boundary conditions, denoted simply by λ_1 when $q \equiv 0$,*

(B) *(4.1) has a unique coexistence state $(u_1, \ldots, u_N)$,*

(C) *the Frechet derivative of (4.1) at $(u_1, \ldots, u_N)$ is invertible.*

Then there is a neighborhood V of $(g_1, \ldots, g_N)$ in $[C_B^1(R^2)]^N$ such that if $(\bar{g}_1, \ldots, \bar{g}_N) \in V$, then (4.1) with $(\bar{g}_1, \ldots, \bar{g}_N)$ has a unique coexistence state.

Biologically, the first condition in Theorem 4.1 indicates that the rates of reproduction are relatively large. Similarly, the third condition, which requires the invertibility of the Frechet derivative, signifies that the rates of self-limitation are relatively larger than the rates of competition, a relationship that will be established in Lemma 4.2. When these conditions are fulfilled, the conclusion of our theorem asserts that small perturbations of the rates do not affect the existence and uniqueness of the positive steady state. That is, the species implied can continue to coexist even if the factors determining the population densities vary slightly.

Now, at first glance, Theorem 4.1 may appear to be a consequence of the Implicit Function Theorem. However, the Implicit Function Theorem only guarantees local uniqueness. In contrast, our results in Theorem 4.1 guarantee global uniqueness. The techniques we will use in the proof of Theorem 4.1 include the Implicit Function Theorem and a priori estimates on solutions of (4.1).

Proof. Since the Frechet derivative of (4.1) at $(u_1, \ldots, u_N)$ is invertible, then, by the Implicit Function Theorem, there is a neighborhood V of $(g_1, \ldots, g_N)$ in $(C_B^1)^N$ and a neighborhood W of $(u_1, \ldots, u_N)$ in $[C_0^{2,\alpha}(\bar{\Omega})]^N$ such that for all $(\bar{g}_1, \ldots, \bar{g}_N) \in V$, there is a unique positive solution $(\bar{u}_1, \ldots, \bar{u}_N) \in W$ of (4.1). Thus, the local uniqueness of the solution is guaranteed.

To prove global uniqueness, suppose that the conclusion of Theorem 4.1 is false. Then there are sequences $(g_{1n}, g_{2n}, \ldots, g_{Nn}, u_{1n}, \ldots, u_{Nn})$, $(g_{1n}, g_{2n}, \ldots, g_{Nn}, \bar{u}_{1n}, \ldots, \bar{u}_{Nn})$ in $V \times [C_0^{2,\alpha}(\bar{\Omega})]^N$ such that $(u_{1n}, \ldots, u_{Nn})$ and $(\bar{u}_{1n}, \ldots, \bar{u}_{Nn})$ are positive solutions of (4.1) with $(g_{1n}, g_{2n}, \ldots, g_{Nn})$ and $(u_{1n}, \ldots, u_{Nn}) \neq (\bar{u}_{1n}, \ldots, \bar{u}_{Nn})$ and $(g_{1n}, g_{2n}, \ldots, g_{Nn}) \to (g_1, \ldots, g_N)$. By the Schauder's estimate in elliptic theory and the solution estimate in the proof of Theorem 2.3, there are constants $k_1 > 0, k_2 > 0, \ldots, k_N > 0$ such that

$$|u_{in}|_{2,\alpha} \le k_i \sup_{x \in \bar{\Omega}}(u_{in}(x)) \le k_i \sup_{x \in \bar{\Omega}}(\theta_{g_{in}(0,\ldots,0,\cdot,0,\ldots,0)}(x)), \quad i = 1, \ldots, N$$

for all $n = 1, 2, \ldots$.

But, by the convergence of $(g_{1n}, \ldots, g_{Nn})$ and the monotonicity of θ_f, we conclude that $|u_{in}|_{2,\alpha}$ is uniformly bounded. Therefore, there is a uniformly convergent subsequence of $(u_{1n}, \ldots, u_{Nn})$ which again will be denoted by $(u_{1n}, \ldots, u_{Nn})$.

Thus, let

$$(u_{1n}, \ldots, u_{Nn}) \to (\hat{u}_1, \ldots, \hat{u}_N),$$
$$(\bar{u}_{1n}, \ldots, \bar{u}_{Nn}) \to (\bar{u}_1, \ldots, \bar{u}_N).$$

Then $(\hat{u}_1, \ldots, \hat{u}_N), (\bar{u}_1, \ldots, \bar{u}_N) \in (C^{2,\alpha})^N$ are also solutions of (2.1) with $(g_1, \ldots, g_N)$. We claim that $\hat{u}_1 > 0, \ldots, \hat{u}_N > 0, \bar{u}_1 > 0, \ldots, \bar{u}_N > 0$. To prove this assertion, it is sufficient to show that $\hat{u}_1, \ldots, \hat{u}_N$ are not identically zero because of the Maximum Principle.

Suppose not. With no loss of generality, suppose $\hat{u}_1$ is identically zero. Let $\tilde{u}_{1n} = \dfrac{u_{1n}}{\|u_{1n}\|_\infty}$ for all $n \in N$. Then

$$\Delta \tilde{u}_{1n} + \tilde{u}_{1n} g_1(u_{1n}, \ldots, u_{Nn}) = 0$$
$$\Delta u_{2n} + u_{2n} g_2(u_{1n}, \ldots, u_{Nn}) = 0$$
$$\vdots$$
$$\Delta u_{Nn} + u_{Nn} g_N(u_{1n}, \ldots, u_{Nn}) = 0.$$

By using elliptic theory again, $\tilde{u}_{1n} \to \tilde{u}_1$ in $C^{2,\alpha}$ and

$$\Delta \tilde{u}_1 + \tilde{u}_1 g_1(0, \hat{u}_2, \ldots, \hat{u}_N) = 0$$
$$\Delta \hat{u}_2 + \hat{u}_2 g_2(\hat{u}_1, \ldots, \hat{u}_N) = 0$$
$$\vdots$$
$$\Delta \hat{u}_N + \hat{u}_N g_N(\hat{u}_1, \ldots, \hat{u}_N) = 0.$$

Hence, $\lambda_1(-g_1(0, \hat{u}_2, \ldots, \hat{u}_N)) = 0$. Let $j = 2, \ldots, N$. If $\hat{u}_j$ is identically zero, then

$$\hat{u}_j \equiv 0 \le \theta_{g_j(0,\ldots,0,\cdot,0,\ldots,0)}.$$

Suppose $\hat{u}_j$ is not identically zero. Then since

$$
\begin{aligned}
\Delta\hat{u}_j &+ \hat{u}_j g_j(0,\ldots,0,\hat{u}_j,0,\ldots,0) \\
&= \Delta\hat{u}_j + \hat{u}_j[g_j(\hat{u}_1,\ldots,\hat{u}_N) - g_j(\hat{u}_1,\ldots,\hat{u}_N) + g_j(0,\ldots,0,\hat{u}_j,0,\ldots,0)] \\
&= \hat{u}_j[g_j(0,\ldots,0,\hat{u}_j,0,\ldots,0) - g_j(\hat{u}_1,\ldots,\hat{u}_N)] \\
&\geq 0,
\end{aligned}
$$

$\hat{u}_j$ is a lower solution of

$$
\Delta Z + Z g_j(0,\ldots,0,Z,0,\ldots,0) = 0 \text{ in } \Omega, \quad Z = 0 \text{ on } \partial\Omega.
$$

Since any constant which is larger than c_j is an upper solution of

$$
\Delta Z + Z g_j(0,\ldots,0,Z,0,\ldots,0) = 0 \text{ in } \Omega, \quad Z = 0 \text{ on } \partial\Omega,
$$

by the uniqueness of positive solution,

$$
\hat{u}_j \leq \theta_{g_j(0,\ldots,0,\cdot,0,\ldots,0)}.
$$

Consequently,

$$
\begin{aligned}
\lambda_1(-g_1(0,\theta_{g_2(0,\cdot,0,\ldots,0)},\ldots,\theta_{g_N(0,\ldots,0,\cdot)})) &\geq \lambda_1(-g_1(0,\hat{u}_2,\ldots,\hat{u}_N)) \\
&= 0,
\end{aligned}
$$

by the monotonicity of g_1 and the first eigenvalue, which contradicts our assumption. Consequently, $(\hat{u}_1,\ldots,\hat{u}_N)$ and $(\bar{u}_1,\ldots,\bar{u}_N)$ are coexistence states with reproduction rate $(g_1,\ldots,g_N)$. But, since the coexistence state in this case is unique by assumption, $(\hat{u}_1,\ldots,\hat{u}_N) = (\bar{u}_1,\ldots,\bar{u}_N) = (u_1,\ldots,u_N)$, which contradicts the Implicit Function Theorem.

In biological terms, the proof of our theorem indicates that if one of the species living in the same domain becomes extinct, that is, if one species is excluded by the others, then the reproduction rates of the species must be small. In other words, the region condition of reproduction rates (A) is reasonable.

Now, the condition (C) in Theorem 4.1 requiring the invertibility of the Frechet derivative is too artificial to have any direct biological implications. We therefore turn our attention to more applicable conditions that will guarantee the invertibility of the Frechet derivative. We then obtain the following relationship:

Lemma 4.2. *Suppose* $(u_1,\ldots,u_N)$ *is a positive solution to (4.1). If*

$$
\begin{aligned}
2\inf&\left(-\frac{\partial g_i(u_1,\ldots,u_N)}{\partial u_i}\right)u_i \\
&> \sum_{j=1,j\neq i}^{N}\left(\sup\left(-\frac{\partial g_i(u_1,\ldots,u_N)}{\partial u_j}\right)u_i + \sup\left(-\frac{\partial g_j(u_1,\ldots,u_N)}{\partial u_i}\right)u_j\right)
\end{aligned}
$$

for $i = 1,\ldots,N$, *then the Frechet derivative of (4.1) at* $(u_1,\ldots,u_N)$ *is invertible.*

Proof. The Frechet derivative of (4.1) at $(u_1,\ldots,u_N)$ is $A = (A_{ij})$, where

$$
A_{ij} = \begin{cases} -\Delta - g_i(u_1,\ldots,u_N) - u_i \dfrac{\partial g_i(u_1,\ldots,u_N)}{\partial u_i}, & i = j \\[2ex] -u_i \dfrac{\partial g_i(u_1,\ldots,u_N)}{\partial u_j}, & i \neq j \end{cases}
$$

for $i, j = 1,\ldots,N$. We need to show that $N(A) = \{0\}$ by the Fredholm Alternative, where $N(A)$ is the null space of A. In fact, from the equations

$$
\int_\Omega |\nabla \phi_1|^2 - \left(g_1(u_1,\ldots,u_N) + u_1 \frac{\partial g_1(u_1,\ldots,u_N)}{\partial u_1} \right) \phi_1^2
$$
$$
- \left(\frac{\partial g_1(u_1,\ldots,u_N)}{\partial u_2} \phi_2 + \ldots + \frac{\partial g_1(u_1,\ldots,u_N)}{\partial u_N} \phi_N \right) u_1 \phi_1 \, dx = 0
$$

$$
\int_\Omega |\nabla \phi_2|^2 - \left(g_2(u_1,\ldots,u_N) + u_2 \frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_2} \right) \phi_2^2
$$
$$
- \left(\frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_1} \phi_1 + \frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_3} \phi_3 + \ldots \right.
$$
$$
\left. + \frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_N} \phi_N \right) u_2 \phi_2 \, dx = 0
$$

$$
\vdots
$$

$$
\int_\Omega |\nabla \phi_N|^2 - \left(g_N(u_1,\ldots,u_N) + u_N \frac{\partial g_N(u_1,\ldots,u_N)}{\partial u_N} \right) \phi_N^2
$$
$$
- \left(\frac{\partial g_N(u_1,\ldots,u_N)}{\partial u_1} \phi_1 + \ldots + \frac{\partial g_N(u_1,\ldots,u_N)}{\partial u_{N-1}} \phi_{N-1} \right) u_N \phi_N \, dx = 0,
$$

since $\lambda_1(-g_i(u_1,\ldots,u_N)) = 0$ for $i = 1,\ldots,N$, we see that

$$
\int_\Omega |\nabla \phi_i|^2 - g_i(u_1,\ldots,u_N) \phi_i^2 \, dx \geq 0, \quad i = 1,\ldots,N.
$$

Hence,

$$
\int_\Omega -u_1 \frac{\partial g_1(u_1,\ldots,u_N)}{\partial u_1} \phi_1^2 - \left(\frac{\partial g_1(u_1,\ldots,u_N)}{\partial u_2} \phi_2 + \ldots \right.
$$
$$
\left. + \frac{\partial g_1(u_1,\ldots,u_N)}{\partial u_N} \phi_N \right) u_1 \phi_1 \, dx \leq 0
$$

$$
\int_\Omega -u_2 \frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_2} \phi_2^2 - \left(\frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_1} \phi_1 \right.
$$
$$
\left. + \frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_3} \phi_3 + \ldots + \frac{\partial g_2(u_1,\ldots,u_N)}{\partial u_N} \phi_N \right) u_2 \phi_2 \, dx \leq 0
$$

$$
\vdots
$$

$$\int_\Omega -u_N \frac{\partial g_N(u_1,\dots,u_N)}{\partial u_N}\phi_N^2 - \left(\frac{\partial g_N(u_1,\dots,u_N)}{\partial u_1}\phi_1 + \dots\right.$$

$$\left. + \frac{\partial g_N(u_1,\dots,u_N)}{\partial u_{N-1}}\phi_{N-1}\right)u_N\phi_N\,dx \leq 0.$$

Therefore,

$$\int_\Omega -\sum_{i=1}^N u_i \frac{\partial g_i(u_1,\dots,u_N)}{\partial u_i}\phi_i^2 - \sum_{i=1}^N u_i\phi_i \sum_{j=1,j\neq i}^N \frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}\phi_j\,dx \leq 0.$$

It implies that

$$\int_\Omega \sum_{i=1}^N \left(-u_i \frac{\partial g_i(u_1,\dots,u_N)}{\partial u_i}\phi_i^2 - \sum_{j=1,j\neq i}^N \frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}u_i\phi_j\phi_i\right)dx \leq 0.$$

But

$$-\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}u_i\phi_j\phi_i \leq -\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}u_i\left(\frac{\phi_i^2}{2} + \frac{\phi_j^2}{2}\right).$$

If

$$-\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_i}u_i > -\sum_{j=1,j\neq i}^N \left(\frac{\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}u_i}{2} + \frac{\frac{\partial g_j(u_1,\dots,u_N)}{\partial u_i}u_j}{2}\right), \quad i=1,\dots,N,$$

then the integrand in above inequality is positive definite, which means $(\phi_1,\dots,\phi_N)$ is trivial. But, it holds if

$$2\inf\left(-\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_i}\right)u_i$$

$$> \sum_{j=1,j\neq i}^N \left(\sup\left(-\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}\right)u_i + \sup\left(-\frac{\partial g_j(u_1,\dots,u_N)}{\partial u_i}\right)u_j\right),$$

$$i = 1,\dots,N.$$

Combining Theorem 2.3, Theorem 4.1, and Lemma 4.2, we obtain the following corollary, which is the main result in this section. The importance of Corollary 4.3 is that it improves the results found by Kang, Oh and Lee, as described in Theorem 2.3.

Corollary 4.3. *Suppose*
(A) $g_i(c_1,\dots,c_{i-1},0,c_{i+1},\dots,c_N) > \lambda_1$, *and*
(B)

$$2\inf\left(-\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_i}\right)$$

$$> \sum_{j=1,j\neq i}^N \left(\sup\left(-\frac{\partial g_i(u_1,\dots,u_N)}{\partial u_j}\right) + K\sup\left(-\frac{\partial g_j(u_1,\dots,u_N)}{\partial u_i}\right)\right)$$

for $i=1,\dots,N$, where $K = \sup_{i\neq j} \dfrac{\theta_{g_j(0,\dots,0,\cdot,0,\dots,0)}}{\theta_{g_i(c_1,\dots,c_{i-1},\cdot,c_{i+1},\dots,c_N)}}$.

Then there is a neighborhood V of $(g_1, \ldots, g_N)$ in $(C_B^1)^N$ such that if $(\bar{g}_1, \ldots, \bar{g}_N) \in V$, then (4.1) with $(\bar{g}_1, \ldots, \bar{g}_N)$ has a unique coexistence state.

Proof. Since $\theta_{g_i(0,\ldots,0,\cdot,0,\ldots,0)} < c_i$ for $i = 1, \ldots, N$, then by the monotonicity of g_i and λ_1, we have

$$\lambda_1(-g_i(\theta_{g_1(\cdot,0,\ldots,0)}, \ldots, \theta_{g_{i-1}(0,\ldots,0,\cdot,0,\ldots,0)}, 0, \theta_{g_{i+1}(\cdot,0,\ldots,0)}, \ldots, \theta_{g_N(0,\ldots,0\cdot)}))$$

$$\leq \lambda_1(-g_i(c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_N))$$

$$= \lambda_1 - g_i(c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_N)$$

$$< 0$$

for $i = 1, \ldots, N$. By Theorem 2.3, (4.1) has a unique coexistence state $(u_1, \ldots, u_N)$. Furthermore, by the estimate of the solution in the proof of Theorem 2.3,

$$2 \inf \left(-\frac{\partial g_i(u_1, \ldots, u_N)}{\partial u_i} \right)$$

$$> \sum_{j=1, j \neq i} \left(\sup \left(-\frac{\partial g_i}{\partial u_j} \right) + \frac{\theta_{g_j(0,\ldots,0,\cdot,0,\ldots,0)}}{\theta_{g_i(c_1,\ldots,c_{i-1},\cdot,c_{i+1},\ldots,c_N)}} \sup \left(-\frac{\partial g_j}{\partial u_i} \right) \right)$$

$$\geq \sum_{j=1, j \neq i} \left(\sup \left(-\frac{\partial g_i}{\partial u_j} \right) + \frac{u_j}{u_i} \sup \left(-\frac{\partial g_j}{\partial u_i} \right) \right)$$

for $i = 1, \ldots, N$. By Lemma 4.2, this result implies that the Frechet derivative of (4.1) at $(u_1, \ldots, u_N)$ is invertible. Therefore, Corollary 4.3 follows from Theorem 4.1.

In biological terms, the results obtained in Corollary 4.3 confirm that under certain conditions, the species who relax ecologically can continue to coexist at fixed rates. The requirements given in (A) and (B) simply state that each species must interact strongly with itself and weakly with the other species.

5. perturbation of region

Consider the model

$$\Delta u_i + u_i g_i(u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_N) = 0 \text{ in } \Omega, \ u_i|_{\partial\Omega} = 0, \tag{5.1}$$

for $i = 1, \ldots, N$. Here Ω is a bounded smooth domain in R^n and $g_i \in C_B^1 = \{f : R^N \to R \in C^1 | f$ is strictly decreasing with respect to each u_i, f and all of it's first order partial derivatives are bounded$\}$.

The following theorem is the main result.

Theorem 5.1. *Suppose*

(A) *Γ is a closed, bounded, convex region in C_B^1 such that for all $(g_1, \ldots, g_N) \in \Gamma$,*
$$\lambda_1(-g_i(\theta_{g_1(\cdot,0,\ldots,0)}, \ldots, \theta_{g_{i-1}(0,\ldots,0,\cdot,0,\ldots,0)}, 0, \theta_{g_{i+1}(0,\ldots,0,\cdot,0,\ldots,0)}, \ldots, \theta_{g_N(0,\ldots,0,\cdot)})) < 0$$
for $i = 1, \ldots, N$,

(B) *there exist $c_i > 0$ such that for all $(g_1, \ldots, g_N) \in \Gamma$, $g_i(0, \ldots, 0, u_i, 0, \ldots, 0) \leq 0$ for $u_i \geq c_i, i = 1, \ldots, N$*

(C) (5.1) *has a unique positive solution for every* $(g_1, \ldots, g_N) \in \partial_L \Gamma$, *where*
$\partial_L \Gamma = \{(\lambda_{g_2, \ldots, g_N}, g_2, \ldots, g_N) \in \Gamma \ \text{for any fixed} \ g_2, \ldots, g_N, \|\lambda_{g_2, \ldots, g_N}\| = \inf\{\|g_1\| \,|\,(g_1, \ldots, g_N) \in \Gamma\}\}$,

(D) *for all* $(g_1, \ldots, g_N) \in \Gamma$, *the Frèchet derivative of* (5.1) *at every positive solution to* (5.1) *is invertible*.

Then for all $(g_1, \ldots, g_N) \in \Gamma$, (5.1) *has a unique positive solution. Furthermore, there is an open set* W *in* $[C_B^1]^N$ *such that* $\Gamma \subseteq W$ *and for every* $(g_1, \ldots, g_N) \in W$, (5.1) *has a unique positive solution.*

Theorem 5.1 goes even further than Theorem 4.1 which states uniqueness in the whole region of $(g_1, \ldots, g_N)$ whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed $g_2, \ldots, g_N$, consider $(g_1, g_2, \ldots, g_N) \in \partial_L \Gamma$ and $(\bar{g}_1, g_2, \ldots, g_N) \in \Gamma$. We need to show that for all $0 \leq t \leq 1$, (5.1) with $(1 - t)(g_1, \ldots, g_N) + t(\bar{g}_1, g_2, \ldots, g_N)$ has a unique positive solution. Since (5.1) with $(g_1, \ldots, g_N)$ has a unique positive solution $(u_1, \ldots, u_N)$ and the Frechet derivative of (5.1) at $(u_1, \ldots, u_N)$ is invertible, Theorem 4.1 implies that there is an open neighborhood V of $(g_1, \ldots, g_N)$ in $(C_B^1)^N$ such that if $(g_{10}, g_{20}, \ldots, g_{N0}) \in V$, then (5.1) with $(g_{10}, g_{20}, \ldots, g_{N0})$ has a unique positive solution.

Let $\lambda_s = \sup\{0 \leq \lambda \leq 1 \,|\, (5.1) \text{ with } (1 - t)(g_1, \ldots, g_N) + t(\bar{g}_1, g_2, \ldots, g_N)$ has a unique coexistence state for $0 \leq t \leq \lambda\}$. We need to show that $\lambda_s = 1$. Suppose $\lambda_s < 1$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and there is a sequence $(u_{1n}, u_{2n}, \ldots, u_{Nn})$ of the unique positive solution of (5.1) with $(1 - \lambda_n)(g_1, \ldots, g_N) + \lambda_n(\bar{g}_1, g_2, \ldots, g_N)$. Then by the elliptic theory, there is $(u_{10}, u_{20}, \ldots, u_{N0})$ such that $(u_{1n}, u_{2n}, \ldots, u_{Nn})$ converges to $(u_{10}, u_{20}, \ldots, u_{N0})$ uniformly and $(u_{10}, u_{20}, \ldots, u_{N0})$ is a solution of (5.1) with $(1 - \lambda_s)(g_1, \ldots, g_N) + \lambda_s(\bar{g}_1, g_2, \ldots, g_N)$. We claim that $u_{10}, u_{20}, \ldots, u_{N0}$ are not identically zero. Suppose this is false. With no loss of generality, suppose u_{10} is identically zero. Let $\tilde{u}_{1n} = \frac{u_{1n}}{\|u_{1n}\|_\infty}$ for all $n \in N$. Then

$$\begin{cases} \Delta \tilde{u}_{1n} + \tilde{u}_{1n}((1 - \lambda_n)g_1(u_{1n}, \ldots, u_{Nn}) + \lambda_n \bar{g}_1(u_{1n}, \ldots, u_{Nn})) = 0 \\ \Delta u_{2n} + u_{2n}g_2(u_{1n}, \ldots, u_{Nn}) = 0 \\ \quad \vdots \\ \Delta u_{Nn} + u_{Nn}g_N(u_{1n}, \ldots, u_{Nn}) = 0. \end{cases}$$

By using the elliptic theory again, $\tilde{u}_{1n} \to \tilde{u}_1$ uniformly in Ω and

$$\begin{cases} \Delta \tilde{u}_1 + \tilde{u}_1((1 - \lambda_s)g_1(0, u_{20}, \ldots, u_{N0}) + \lambda_s \bar{g}_1(0, u_{20}, \ldots, u_{N0})) = 0 \\ \Delta u_{20} + u_{20}g_2(0, u_{20}, \ldots, u_{N0}) = 0 \\ \quad \vdots \\ \Delta u_{N0} + u_{N0}g_N(0, u_{20}, \ldots, u_{N0}) = 0. \end{cases}$$

It implies that $\lambda_1(-((1-\lambda_s)g_1(0, u_{20}, \ldots, u_{N0}) + \lambda_s \bar{g}_1(0, u_{20}, \ldots, u_{N0}))) = 0$. Let $j = 2, \ldots, N$. If u_{j0} is identically zero, then $u_{j0} \equiv 0 \leq \theta_{g_j(0, \ldots, 0, \cdot, 0, \ldots, 0)}$. Suppose u_{j0} is not identically zero. Then since

$$\Delta u_{j0} + u_{j0} g_j(0, \ldots, 0, u_{j0}, 0, \ldots, 0)$$
$$= \Delta u_{j0} + u_{j0}[g_j(u_{10}, \ldots, u_{N0}) - g_j(u_{10}, \ldots, u_{N0}) + g_j(0, \ldots, 0, u_{j0}, 0, \ldots, 0)]$$
$$= u_{j0}[g_j(0, \ldots, 0, u_{j0}, 0, \ldots, 0) - g_j(u_{10}, \ldots, u_{N0})]$$
$$\geq 0,$$

u_{j0} is a lower solution of

$$\Delta Z + Z g_j(0, \ldots, 0, Z, 0, \ldots, 0) = 0 \ \text{in} \ \Omega, Z = 0 \ \text{in} \ \partial\Omega.$$

Since any constant which is larger than c_j is an upper solution of

$$\Delta Z + Z g_j(0, \ldots, 0, Z, 0, \ldots, 0) = 0 \ \text{in} \ \Omega, Z = 0 \ \text{in} \ \partial\Omega,$$

by the uniqueness of positive solution,

$$u_{j0} \leq \theta_{g_j(0, \ldots, 0, \cdot, 0, \ldots, 0)}.$$

Consequently,

$$\lambda_1(-((1-\lambda_s)g_1(0, \theta_{g_2(0, \cdot, 0, \ldots, 0)}, \ldots, \theta_{g_N(0, \ldots, 0, \cdot)}) + \lambda_s \bar{g}_1(0, \theta_{g_2(0, \cdot, 0, \ldots, 0)}, \ldots, \theta_{g_N(0, \ldots, 0, \cdot)})))$$
$$\geq \lambda_1(-((1-\lambda_s)g_1(0, u_{20}, \ldots, u_{N0}) + \lambda_s \bar{g}_1(0, u_{20}, \ldots, u_{N0})))$$
$$= 0,$$

which is a contradiction to our assumption since $(1-\lambda_s)g_1 + \lambda_s \bar{g}_1 \in \Gamma$.

Thus $u_{10}, \ldots, u_{N0}$ are not identically zero. We claim that (5.1) has a unique coexistence state with $(1-\lambda_s)(g_1, \ldots, g_N) + \lambda_s(\bar{g}_1, g_2, \ldots, g_N)$. In fact, if not, assume that $(u_{10}^-, \ldots, u_{N0}^-) \neq (u_{10}, \ldots, u_{N0})$ is another coexistence state. By the Implicit Function Theorem, there exists $0 < \tilde{a} < \lambda_s$ and very close to λ_s, (5.1) with $(1-\tilde{a})(g_1, \ldots, g_N) + \tilde{a}(\bar{g}_1, g_2, \ldots, g_N)$ has a coexistence state very close to $(u_{10}^-, \ldots, u_{N0}^-)$ which means that (5.1) with $(1-\tilde{a})(g_1, \ldots, g_N) + \tilde{a}(\bar{g}_1, g_2, \ldots, g_N)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (5.1) with $(1-\lambda_s)(g_1, \ldots, g_N) + \lambda_s(\bar{g}_1, g_2, \ldots, g_N)$ has a unique coexistence state and the Frechet derivative is invertible, Theorem 4.1 concluded that λ_s can not be as defined. Therefore, for each $(g_1, \ldots, g_N) \in \Gamma$, (5.1) with $(g_1, \ldots, g_N)$ has a unique coexistence state $(u_1, \ldots, u_N)$. Furthermore, by the assumption, for each $(g_1, \ldots, g_N) \in \Gamma$, the Frechet derivative of (5.1) with $(g_1, \ldots, g_N)$ at the unique solution $(u_1, \ldots, u_N)$ is invertible. Hence, Theorem 4.1 concluded that there is an open neighborhood $V_{(g_1, \ldots, g_N)}$ of $(g_1, \ldots, g_N)$ in $(C_B^1)^N$ such that if $(\bar{g}_1, \ldots, \bar{g}_N) \in V_{(g_1, \ldots, g_N)}$, then (5.1) with $(\bar{g}_1, \ldots, \bar{g}_N)$ has a unique coexistence state. Let $W = \bigcup_{(g_1, \ldots, g_N) \in \Gamma} V_{(g_1, \ldots, g_N)}$. Then W is an open set in $(C_B^1)^N$ such that $\Gamma \subseteq W$ and for each $(\bar{g}_1, \ldots, \bar{g}_N) \in W$, (5.1) with $(\bar{g}_1, \ldots, \bar{g}_N)$ has a unique coexistence state.

Apparently, Theorem 5.1 generalizes Theorem 4.1 and consequently, we have the following result which is actually the main conclusion in this section.

Corollary 5.2. *Suppose*

(A) Γ *is a closed, bounded, convex region in* $[C_B^1]^N$,

(B) *there exist* $c_i > 0$ *such that for all* $(g_1,\ldots,g_N) \in \Gamma$, $g_i(c_1,\ldots,c_{i-1},0,c_{i+1},\ldots,c_N)$
$> \lambda_1$, $g_i(0,\ldots,0,u_i,0,\ldots,0) \le 0$ *for* $u_i \ge c_i$, $i = 1,\ldots,N$,

(C) *for all* $(g_1,\ldots,g_N) \in \Gamma$,

$$2\inf\left(-\frac{\partial g_i}{\partial x_i}\right) > \sum_{j=1,j\neq i}^{N}\left(\sup\left(-\frac{\partial g_i}{\partial x_j}\right) + K\sup\left(-\frac{\partial g_j}{\partial x_i}\right)\right)$$

for $i = 1,\ldots,N$, *where* $K = \sup_{i,j,i\neq j}\dfrac{\theta_{g_j(0,\ldots,0,\cdot,0,\ldots,0)}}{\theta_{g_i(c_1,\ldots,c_{i-1},\cdot,c_{i+1},\ldots,c_N)}}$.

Then there is an open set W *in* $[C_B^1]^N$ *such that* $\Gamma \subseteq W$ *and for every* $(g_1,\ldots,g_N) \in W$, *(5.1) with* $(g_1,\ldots,g_N)$ *has a unique positive solution.*

Proof. From $\theta_{g_i(0,\ldots,0,\cdot,0,\ldots,0)} \le c_i$ for $i = 1,\ldots,N$ and the monotonicity, we have

$$\lambda_1(-g_i(\theta_{g_1(\cdot,0,\ldots,0)},\ldots,\theta_{g_{i-1}(0,\ldots,0,\cdot,0,\ldots,0)},0,\theta_{g_{i+1}(0,\ldots,0,\cdot,0,\ldots,0)},\ldots,\theta_{g_N(0,\ldots,0,\cdot)}))$$
$$< \lambda_1(-g_i(c_1,\ldots,c_{i-1},0,c_{i+1},\ldots,c_N))$$
$$= \lambda_1 - g_i(c_1,\ldots,c_{i-1},0,c_{i+1},\ldots,c_N)$$
$$< 0$$

for all $(g_1,\ldots,g_N) \in \Gamma$, $i = 1,\ldots,N$. By the condition (C) and the Theorem 2.3, (5.1) has a unique positive solution for all $(g_1,\ldots,g_N) \in \partial_L\Gamma$. Furthermore, by the estimate of the solution in the proof of Theorem 2.3, if $(u_1,\ldots,u_N)$ is a positive solution of (5.1) with $(g_1,\ldots,g_N) \in \Gamma$, then

$$2\inf\left(-\frac{\partial g_i}{\partial x_i}\right)u_i > \sum_{j=1,j\neq i}^{N}\left(\sup\left(-\frac{\partial g_i}{\partial x_j}\right)u_i + \sup\left(-\frac{\partial g_j}{\partial x_i}\right)u_j\right)$$

for $i = 1,\ldots,N$. Hence, by the Lemma 4.2, if $(u_1,\ldots,u_N)$ is a positive solution of (5.1) for $(g_1,\ldots,g_N) \in \Gamma$, then the Frechet derivative of (5.1) at $(u_1,\ldots,u_N)$ is invertible. Therefore, the theorem follows from the Theorem 5.1.

6. **Conclusions**

In this paper, our investigation of the effects of perturbations on the general competition model resulted in the development and proof of Theorem 4.1, Lemma 4.2, and Corollary 4.3, as detailed above. The three together assert that given the existence of a unique solution $(u_1,\ldots,u_N)$ to the system (4.1), perturbations of the functions $g_1,\ldots,g_N$, within a specified neighborhood, will maintain the existence and uniqueness of the positive steady state. Indeed, our results specifically outline conditions sufficient to maintain the positive, steady state solution when the general competition model is perturbed within some region.

Applying this mathematical result to real world situations, our results establish that the species residing in the same environment can vary their interactions,

within certain bounds, and continue to survive together indefinitely at unique densities. The conditions necessary for coexistence, as described in the theorem, simply require that members of each species interact strongly with themselves and weakly with members of the other species.

The research presented in this paper has a number of strengths, which confirm both the validity and the applicability of the project. First, the mathematical conditions required in Corollary 4.3 are identical to those required in Theorem 2.3 developed by Kang, Oh, and Lee. However, they used these conditions to prove the existence and uniqueness of the positive steady state solution for the general competition model. In contrast, our theorem employs the same conditions to establish that the existence and uniqueness of this solution is maintained when the model is perturbed within some neighborhood. Thus, our findings extend and improve established mathematical theory.

Secondly, perturbations of the general model render its implications more applicable both mathematically and biologically. Because our theorem extends the steady state to any function $(\bar{g}_1, \ldots, \bar{g}_N)$ within some neighborhood of $(g_1, \ldots, g_N)$, results for the general model pertain to a far wider variety of functions. Biologically, perturbations extend the model's description to species affected by factors that vary slightly yet erratically. Thus, the description of competitive interactions given by the model becomes a closer approximation of real-world population dynamics.

While our research therefore represents a progression in the field, the results obtained have an important limitation. Theorem 4.1, Lemma 4.2, and Corollary 4.3 establish that a region of perturbation exists within which the coexistence state is maintained for the general competition model. However, the exact extent of that region remains unknown.

Therefore, the results presented in this paper may serve as a platform for research of the question given above. Mathematicians should now attempt to establish the exact extent of the perturbation region in which coexistence is maintained for the general model. Such information would prove very useful not only mathematically but also biologically. Specifically, knowledge of the extent of the region would imply exactly how far the species can relax and yet continue to coexist. Thus, the results achieved through our research will enable the field to continue the development of theory on competitive interaction of populations.

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Joon Hyuk Kang, *Department of Mathematics, Andrews University, Berrien Springs, MI. 49104.*
E-mail: kang@andrews.edu