



Vertex and Edge Connectivity of the Zero Divisor Graph $\Gamma[\mathbb{Z}_n]$

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Abstract. The zero divisor graph $\Gamma[R]$ of a commutative ring R is a graph with vertex set being the set of non-zero zero divisors of R and there is an edge between two vertices if their product is zero. In this paper, we prove that the vertex, edge connectivity and the minimum degree of the zero divisor graph $\Gamma[\mathbb{Z}_n]$ for any natural number n , are equal.

Keywords. Zero divisor graph; Vertex connectivity; Edge connectivity; Minimum degree

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1. Introduction

The concept of the *Zero divisor graph* of a ring R was first introduced by I. Beck [4] in 1988. Thereafter, Anderson and Livingston [3], and Akbari and Mohammadian [2] have contributed to the work of zero divisor graph.

Reza and Jahani [1] calculated the energy and Wiener index for the zero divisor graph $\Gamma[\mathbb{Z}_n]$ for $n = p^2$ and $n = pq$ where p and q are prime numbers, and then, Reddy *et al.* [6] have extended the work of Reza to $\Gamma[\mathbb{Z}_n]$ for $n = p^3$ and $n = p^2q$. The concepts of the edge and vertex connectivity of a graph [5] are new to the zero divisor graph. So, in this paper, we discuss the concepts of the edge and vertex connectivity to the zero divisor graph $\Gamma[\mathbb{Z}_n]$ for any natural number n .

The article is categorized as follows: in Section 2, we define all necessary definitions related to zero divisor graph of a commutative ring R , in Section 3, we derive the vertex connectivity of a zero divisor graph $\Gamma[\mathbb{Z}_n]$, and in Section 4, we calculate the edge connectivity of $\Gamma[\mathbb{Z}_n]$. In this section, we also find the minimum degree of the zero divisor graph $\Gamma[\mathbb{Z}_n]$ and conclude that the vertex, edge connectivity and the minimum degree of the zero divisor graph $\Gamma[\mathbb{Z}_n]$ are same.

2. Preliminaries and Notations

Definition 2.1 (Zero divisor graph [2]). Let $Z[R]$ be the set of zero divisors of the commutative ring R . Then the *zero divisor graph* $\Gamma[R]$ of R is the graph (undirected) with vertex set as the set of non-zero zero divisors $Z^*[R] = Z[R] - \{0\}$ such that two vertices $v, w \in Z^*[R]$ are adjacent if $vw = 0$.

Definition 2.2 (Vertex connectivity of a graph [5]). For a simple graph G , the *vertex connectivity* $\kappa(G)$ of G is the smallest number of vertices in G whose deletion from G leaves either a disconnected graph or K_1 .

Definition 2.3 (Edge connectivity of a graph [5]). For a simple graph G , the *edge connectivity* $\kappa_e(G)$ of G is the smallest number of edges in G whose deletion from G leaves either a disconnected graph or an empty graph.

Definition 2.4 (Minimum degree of a graph [5]). For a graph G , the *minimum degree* $\delta(G)$ of G is the minimum value among the degrees of all its vertices.

3. Vertex Connectivity of the Zero Divisor Graph $\Gamma[\mathbb{Z}_n]$

In this section we derive the vertex connectivity of $\Gamma[\mathbb{Z}_n]$ for any natural number n . We first calculate vertex connectivity of $\Gamma[\mathbb{Z}_n]$ for $n = p^2$.

Theorem 3.1. *The vertex connectivity of $\Gamma[\mathbb{Z}_{p^2}]$ is $p - 2$.*

Proof. The set of multiples of p is the vertex set of $\Gamma[\mathbb{Z}_{p^2}]$ given by $A = \{kp \mid k = 1, 2, 3, \dots, p - 1\}$ and so $|A| = (p - 1)$.

Here any two vertices are adjacent. Thus $\Gamma[\mathbb{Z}_{p^2}] = K_{p-1}$, complete graph on $(p - 1)$ vertices.

Since the graph is complete, graph will be still connected after deleting one or less than $p - 2$ vertices. But, if we delete $p - 2$ vertices then the resulting graph is K_1 .

Hence the vertex connectivity of $\Gamma[\mathbb{Z}_{p^2}]$ is $p - 2$.

Since $\kappa(\Gamma[\mathbb{Z}_{p^2}]) = p - 2$, $\Gamma[\mathbb{Z}_{p^2}]$ is $p - 2$ connected. □

Now, we consider the case for $n = p^3$ in the following theorem:

Theorem 3.2. *The vertex connectivity of $\Gamma[\mathbb{Z}_{p^3}]$ is $p - 1$.*

Proof. Here, we divide the vertices of $\Gamma[\mathbb{Z}_{p^3}]$ into two disjoint sets namely multiples of p but not p^2 ; and the multiples of p^2 , given by

$$A = \{kp \mid k = 1, 2, 3, \dots, p^2 - 1 \text{ and } k \nmid p\},$$

$$B = \{lp^2 \mid l = 1, 2, 3, \dots, p - 1\}$$

with cardinality $|A| = p(p - 1)$ and $|B| = p - 1$.

Here we can observe that no two elements of A are adjacent; each element of A is adjacent with every element of B and each element of B is adjacent with every element of A as well as of B . Even if we remove all the vertices of A , the resulting graph should be connected as it is the complete graph on the elements of B . But, if we remove all the vertices from B , then the corresponding graph with vertices from A is disconnected.

Also, if we leave even one vertex from B and remove remaining, the graph is still connected as every element of B is adjacent with A and B .

Therefore, the minimum number of vertices to be deleted from G is the number of vertices of B . Hence the vertex connectivity $\kappa(\Gamma[\mathbb{Z}_{p^3}]) = |B| = p - 1$.

Since, $\kappa(\Gamma[\mathbb{Z}_{p^3}]) = p - 1$, the graph $\Gamma[\mathbb{Z}_{p^3}]$ is $p - 1$ connected. □

With similar arguments, we prove the more general case in the following theorem:

Theorem 3.3. *The vertex connectivity of $\Gamma[\mathbb{Z}_{p^n}]$ is $p - 1 \forall n \geq 3$.*

Proof. We divide the vertices of $\Gamma[\mathbb{Z}_{p^n}]$ into $n - 1$ disjoint sets, given by

$$\begin{aligned} A_1 &= \{k_1p \mid k_1 = 1, 2, 3, \dots, p^{n-1} - 1 \text{ and } k_1 \nmid p\}, \\ A_2 &= \{k_2p^2 \mid k_2 = 1, 2, 3, \dots, p^{n-2} - 1 \text{ and } k_2 \nmid p^2\}, \\ A_i &= \{k_ip^i \mid k_i = 1, 2, 3, \dots, p^{n-i} - 1 \text{ and } k_i \nmid p^i\} \end{aligned}$$

with cardinality $|A_i| = p^{n-i} - 1$, for $i = 1, 2, \dots, n - 1$.

Among these sets, the smallest set is A_{n-1} of order $p - 1$.

Since the graph on A_{n-1} is complete, the graph is still connected even after removing all the vertices from any of the sets A_i for $i = 1, 2, \dots, n - 2$.

Where as deleting all the vertices of A_{n-1} , leaves the graph disconnected, because the elements of A_1 , which are adjacent only with the elements of A_{n-1} , become isolated.

Thus the vertex connectivity of $\Gamma[\mathbb{Z}_{p^n}]$ is $p - 1$

Since, $\kappa(\Gamma[\mathbb{Z}_{p^n}]) = p - 1$, the zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$ is $p - 1$ connected. □

Now we work on the vertex connectivity of $\Gamma[\mathbb{Z}_m]$ where $m = p^\alpha q^\beta$.

Theorem 3.4. *The vertex connectivity of $\Gamma[\mathbb{Z}_{p^\alpha q^\beta}]$ is $\min\{p - 1, q - 1\}$.*

Proof. Here also we divide the vertices of $\Gamma[\mathbb{Z}_m]$ into disjoint sets namely

$$\begin{aligned} A_{p^i} &= \{r_ip^i \mid r_i = 1, 2, 3, \dots, p^i - 1 \text{ and } r_i \nmid p^i\}, \\ A_{q^j} &= \{s_jq^j \mid s_j = 1, 2, 3, \dots, q^j - 1 \text{ and } s_j \nmid q^j\}, \\ A_{p^i q^j} &= \{t_{ij}p^i q^j \mid t_{ij} = 1, 2, 3, \dots, p^i q^j - 1 \text{ and } t_{ij} \nmid p^i \text{ and } t_{ij} \nmid q^j\}. \end{aligned}$$

Then, the smallest set among the above is either $B = A_{p^{\alpha-1}q^{\beta-1}}$ or $C = A_{p^{\alpha-1}q^\beta}$.

Suppose $p < q$.

Here, we note that every element of A_{p^i} is adjacent with each and every element of C .

So, if we delete all the elements of C then the graph becomes disconnected as the vertices of A_p becomes isolated.

Therefore, the minimum number of vertices to be deleted to make the graph disconnected is $p - 1$.

Similarly, we get that if $q < p$, then the minimum number of vertices to be deleted to make the graph disconnected is $q - 1$.

Hence the vertex connectivity of $\Gamma[\mathbb{Z}_m]$ is $\min\{p - 1, q - 1\}$. \square

The next result is about the vertex connectivity of $\Gamma[\mathbb{Z}_n]$ for any natural number n .

Theorem 3.5. *The vertex connectivity of $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is $\min\{p_1 - 1, p_2 - 1, \dots, p_k - 1\}$.*

Proof. Consider a zero divisor graph $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$.

As in the case of previous results, we divide the vertices of $\Gamma[\mathbb{Z}_n]$ into the corresponding disjoint sets like set of powers of p_j^i , set of product of powers of $p_j^i p_r^s$ and so on.

Among these sets, we consider the sets of the form

$$A_i = \{m(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i - 1} \dots p_k^{\alpha_k}) \mid m \nmid p_j \text{ for all } j\} \text{ with } |A_i| = (p_i - 1).$$

Let $r = \min\{|A_1|, \dots, |A_k|\} = \min\{p_1 - 1, \dots, p_k - 1\}$.

Suppose $r = |A_j| = p_j - 1$, for some j .

Now, consider the set $A_{p_j} = \{tp_j \mid t \nmid p_j\}$.

Since the elements of A_{p_j} are adjacent only with the vertices of A_j , the deletion of all vertices of A_j leaves the elements of A_{p_j} isolated.

Thus, the graph is disconnected if we remove all the elements of A_j .

Therefore, the minimum number of vertices to be deleted to make the graph disconnected is $r = |A_j|$.

Thus, vertex connectivity of $\Gamma[\mathbb{Z}_n]$ is $r = \min\{p_1 - 1, p_2 - 1, \dots, p_k - 1\}$. \square

4. Edge Connectivity of the Zero Divisor Graph $\Gamma[\mathbb{Z}_n]$

In this section, we discuss the edge connectivity of the zero divisor graph $\Gamma[\mathbb{Z}_n]$. We consider here several cases as we did in the previous section. To start with we consider $n = p^2$.

Theorem 4.1. *The edge connectivity of $\Gamma[\mathbb{Z}_{p^2}]$ is $p - 2$.*

Proof. The vertex set of the zero divisor graph $\Gamma[\mathbb{Z}_{p^2}]$ is

$$A = \{kp \mid k = 1, 2, 3, \dots, p - 1 \text{ and } k \nmid p\} \text{ with } |A| = (p - 1).$$

As the graph is complete, every vertex is incident with $(p - 2)$ edges.

So, removing these $(p - 2)$ edges with respect to a fixed vertex v , the graph becomes disconnected as v becomes isolated.

If we remove fewer than $(p - 2)$ edges, say $(p - 3)$, as the remaining two vertices are adjacent, the graph is still connected.

Thus, the minimum number of edges to be deleted is $p - 2$.

Hence the edge connectivity of $\Gamma[\mathbb{Z}_{p^2}]$ is $\kappa_e(\Gamma[\mathbb{Z}_{p^2}]) = p - 2$.
 Since $\kappa_e(\Gamma[\mathbb{Z}_{p^2}]) = p - 2$, the graph $\Gamma[\mathbb{Z}_{p^2}]$ is $(p - 2)$ -edge connected. □

Theorem 4.2. *The edge connectivity of $\Gamma[\mathbb{Z}_{p^n}]$ is $p - 1 \forall n \geq 3$.*

Proof. We categorize the vertices of $\Gamma[\mathbb{Z}_{p^n}]$ into $(n - 1)$ disjoint sets namely

$$A_1 = \{k_1 p \mid k_1 = 1, 2, 3, \dots, p^{n-1} - 1 \text{ and } k_1 \nmid p\},$$

$$A_2 = \{k_2 p^2 \mid k_2 = 1, 2, 3, \dots, p^{n-2} - 1 \text{ and } k_2 \nmid p^2\},$$

$$A_i = \{k_i p^i \mid k_i = 1, 2, 3, \dots, p^{n-i} - 1 \text{ and } k_i \nmid p^i\}$$

with cardinality $|A_i| = (p^{n-i} - 1)$, for $i = 1, 2, \dots, n - 1$.

Among these sets, the smallest set among the above is A_{n-1} with cardinality $p - 1$.

Here A_1 is the only set in which there exists a vertex (in fact every vertex) which is incident with only $(p - 1)$ edges because any vertex in A_i is adjacent with the elements of the sets $A_{n-i}, A_{n-i+1}, \dots, A_{n-1}$ and so with $p^i + p^{i-1} + \dots + p - i$ edges.

Here our idea is to identify a set whose elements are having edges with only A_{n-1} , and that set is A_1 .

Now, consider a vertex v in A_1 . Then there are $(p - 1)$ edges incident with v as the vertices of A_1 are adjacent with only the elements of the set A_{n-1} .

Thus by removing all these edges, we result in a disconnected graph.

If we remove fewer than $(p - 1)$ edges, the graph is still connected.

Therefore, the minimum number of edges to be removed is $p - 1$.

Thus, the edge connectivity of $\Gamma[\mathbb{Z}_{p^n}]$ is $\kappa_e(\Gamma[\mathbb{Z}_{p^n}]) = p - 1$.

Since $\kappa_e(\Gamma[\mathbb{Z}_{p^n}]) = p - 1$, the zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$ is $(p - 1)$ - edge connected. □

Now, we move to the general case: the Edge connectivity of $\Gamma[\mathbb{Z}_n]$ for any natural number n .

Theorem 4.3. *The edge connectivity of $\Gamma[\mathbb{Z}_n]$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is r , where $r = \min\{p_i - 1\}$ for $i = 1, 2, \dots, k$.*

Proof. After grouping the vertex set into suitable sets, we choose the suitable set A_j as in Theorem 3.5. Since the vertices of A_{p_j} are adjacent only with the vertices of A_j , the deletion of all edges incident with the any vertex of A_{p_j} leaves that vertex isolated and hence the graph becomes disconnected. Also, in all other possible combinations of A_{p_s} and A_s the graph is still connected.

Therefore, the minimum number of edges to be deleted to make the graph disconnected is $r = |A_j|$. Thus edge connectivity of $\Gamma[\mathbb{Z}_n]$ is

$$\kappa_e(\Gamma[\mathbb{Z}_n]) = r = \min\{p_1 - 1, p_2 - 1, \dots, p_k - 1\}. \quad \square$$

Remark 4.4. By observing all the results from Sections 3 and 4, we conclude that vertex connectivity is same as the edge connectivity of the zero divisor graph $\Gamma[\mathbb{Z}_n]$, i.e., $\kappa(\Gamma[\mathbb{Z}_n]) = \kappa_e(\Gamma[\mathbb{Z}_n])$. In fact we show that, they are equal to the minimum degree, in the next result.

Theorem 4.5. *The minimum degree of the zero divisor graph $\Gamma[\mathbb{Z}_n]$ is equal its edge connectivity. Therefore $\kappa(\Gamma[\mathbb{Z}_n]) = \kappa_e(\Gamma[\mathbb{Z}_n]) = \delta(\Gamma[\mathbb{Z}_n])$.*

Proof. Let the edge connectivity of $\Gamma[\mathbb{Z}_n]$ is r .

Then the minimum number of edges that are incident to any arbitrary vertex in the graph is r .

Also, there exists a set in which degree of each vertex is r .

Therefore, the minimum degree of $\Gamma[\mathbb{Z}_n]$ is $\delta(\Gamma[\mathbb{Z}_n]) = r$.

Hence $\kappa(\Gamma[\mathbb{Z}_n]) = \kappa_e(\Gamma[\mathbb{Z}_n]) = \delta(\Gamma[\mathbb{Z}_n])$. □

5. Conclusion

We discussed about the vertex connectivity and the edge connectivity of the graphs $\Gamma(\mathbb{Z}_n)$, also the minimum degree of the graphs $\Gamma(\mathbb{Z}_n)$ and we conclude that for the zero divisor graph $\Gamma[\mathbb{Z}_n]$, the vertex connectivity and the edge connectivity are equal and equal to Minimum degree of the graph, i.e., $\kappa(G) = \kappa_e(G) = \delta(G)$, where $G = \Gamma(\mathbb{Z}_n)$, and we can use this connectivity to find clique and chromatic number of a zero divisor graph $\Gamma(\mathbb{Z}_n)$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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