Communications in Mathematics and Applications
Vol. 11, No. 3, pp. 425–442, 2020
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v11i3.1316

Research Article

Fixed Point Theorems in Soft Multiplicative Metric Space

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Abstract. In this article, we first define the concept of soft multiplicative metric space and then establish its relation with soft metric and multiplicative metric space. This new approach enables us to obtain some fixed point results. We also provide some examples to verify and illustrate our results.

Keywords. Soft set; Soft point; Soft real number; Soft metric space; Soft multiplicative metric space

MSC. 06D72; 54E50; 54H25

Received: November 6, 2019 **Revised:** May 16, 2020 **Accepted:** May 22, 2020

1. Introduction

In 1999, Molodtsov [9] proposed the big idea of "soft set theory". This is an advanced mathematical aid for handling uncertainty in problems related to real world which has wide potential applications and is applicable where mathematical models are not defined clearly. In 2002 and 2003, Maji *et al*. [6], [7] introduced the applications of soft set in decision making problems and defined some basic terms of soft set theory. In 2009, Ali *et al*. [3] defined some new operations in this theory. In 2010, Majumdar *et al*. [8] worked on soft mappings. Further, Das and Samanta [4], [5] used the concept of soft points to investigate some basic properties of soft metric space. In 2016, Abbas *et al*. [1] worked on the fixed point theory of soft metric spaces. In 2016, Yazar *et al*. [11] gave introduction on soft continuous mappings and investigated its properties. Inspiring from all these ideas, we define the concept of soft multiplicative metric space and discuss the relationship between soft multiplicative metric space and soft metric space. Then, from the idea of [1], we establish the relation between soft multiplicative metric and multiplicative metric and prove some "fixed point theorems" in soft multiplicative metric space.

This article consists of eight sections. Section 1 begins with the recent development of soft sets and soft metric space. Section 2 contains nomenclature of usual terms used in this article. Section 3 deals with the basic concepts and definitions related to soft set and soft metric space. In Section 4, we define the concept of soft multiplicative metric space with an example. Relation between soft multiplicative metric space and soft metric space are discussed in Section 5 while Section 6 contains the relation between soft multiplicative metric space and multiplicative metric space. Then, in Section 7, we prove some fixed point theorems in soft multiplicative metric space. At the end, Section 8 contains the outline of this article.

2. Nomenclature

U — Universal set; E — non-empty parameter set; (F,E) — soft set; $\tilde{U}$ — absolute soft set; F^u_λ — soft point; $SP(\tilde{U})$ — collection of all soft points of absolute soft set; $\mathbb{R}(E)^*$ — set of all non-negative soft real numbers.

3. Preliminaries

This section contains some well-known basic definitions which are helpful while proving our main results.

Definition 3.1 ([4]). Let U be an initial universal set and E be the non-empty parameter set. Then, a pair (F,E) is called a soft set over U if F is a set valued mapping on E taking values in 2^U i.e., $F: E \to 2^U$.

Example 3.2. Suppose U is a collection of four rooms in a hotel under consideration such that $U = \{r_1, r_2, r_3, r_4\}$ and $E = \{a_1, a_2, a_3\}$ be the set of parameters where $a_j\,(j=1,2,3)$ stand for the parameters: furnished room, expensive room and modern room. Let (F,E) be the soft set which characterizes the nature of rooms such that $F(a_1) = \{r_1, r_3\}$, $F(a_2) = U$ and $F(a_3) = \{r_1\}$. By this collection, the soft set (F,E) can be viewed as $(F,E) = \{(\text{furnished}, \{r_1, r_3\}), (\text{expensive}, \{r_1, r_2, r_3\}), (\text{modern}, \{r_1\})\}$.

Definition 3.3 ([4]). A soft set (F,E) over U is said to be an absolute soft set if $F(e) = U$ for all $e \in E$.

Definition 3.4 ([4]). A soft set (F,E) over U is said to be a soft point if there is exactly one $e \in E$ such that $F(e) = \{u\}$, for some $u \in U$ and $F(\lambda) = \phi$ for all $\lambda \in E \setminus \{e\}$. Such a soft point is denoted by F^u_e.

Remark 3.5 ([4]). The collection of all soft points of a soft set (F,E) is denoted by $SP(F,E)$.

Example 3.6. Consider a universal set $U = \{r_1, r_2, r_3\}$ and parameter set $E = \{\lambda, \mu\}$, then $F_\lambda^{r_1}$ is a soft point where $F(\lambda) = \{r_1\}$ and $F(\mu) = \phi$.

Definition 3.7 ([4]). A soft point P_e^u belongs to a soft set (F,E) if $e \in E$, $u \in U$ and $P(e) = \{u\}\tilde{\subseteq}F(e)$ and we write $P_e^u\tilde{\in}(F,E)$.

Example 3.8. Consider a universal set $U = \{r_1, r_2, r_3\}$ and $E = \{\lambda, \mu\}$. Suppose a soft set (F,E) such that $F(\lambda) = \{r_1, r_2\}$ and $F(\mu) = \{r_2, r_3\}$, then a soft point $P_\lambda^{r_1}\tilde{\in}F(\lambda)$ as $P(\lambda) = \{r_1\}\tilde{\subseteq}F(\lambda)$ but $P_\mu^{r_1}\tilde{\notin}F(\mu)$ as $P(\mu) = \{r_1\}\tilde{\nsubseteq}F(\mu)$.

Definition 3.9 ([4]). Two soft points $F_\lambda^{u_1}$ and $F_\mu^{u_2}$ are said to be equal if $\lambda = \mu$ and $\mathrm{F}(\lambda) = F(\mu)$ i.e., $u_1 = u_2$.

Definition 3.10 ([5]). Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$. Then, the function $F : E \to B(\mathbb{R})$ is called a soft real set and is denoted by (F,E). If F is a single valued function on E taking values in $\mathbb{R}$, then the pair (F,E) or simply F is called a soft real number. We denote soft real number and soft constant real number by $\tilde{r}, \tilde{s}, \tilde{t}$ and $\bar{r}, \bar{s}, \bar{t}$ respectively where $\bar{r}$ will denote a particular type of soft real number such that $\bar{r}(\lambda) = r$ for all $\lambda \in E$.

Example 3.11. Consider the soft set, given in Example 3.2. If a function $F : E \to P(\mathbb{R})$ is defined as $F(a)$ =the number of rooms available in a hotel under the category a. Then, we have $F(a_1) = 2$, $F(a_2) = 3$ and $F(a_3) = 1$. Then the soft real number (F,E) can be viewed as $(F,E) = \{\text{furnished room} = 2; \text{expensive room} = 2; \text{modern room} = 1\}$.

Definition 3.12 ([4]). For two soft real numbers $\tilde{p}$ and $\tilde{q}$ and for all $e \in E$, the following conditions hold:

(1) $\tilde{p}\tilde{\leq}\tilde{q}$ if $\tilde{p}(e)\tilde{\leq}\tilde{q}(e)$,

(2) $\tilde{p}\tilde{\geq}\tilde{q}$ if $\tilde{p}(e)\tilde{\geq}\tilde{q}(e)$,

(3) $\tilde{p}\tilde{<}\tilde{q}$ if $\tilde{p}(e)\tilde{<}\tilde{q}(e)$,

(4) $\tilde{p}\tilde{>}\tilde{q}$ if $\tilde{p}(e)\tilde{>}\tilde{q}(e)$.

Definition 3.13 ([4]). A mapping $\tilde{d} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ is a soft metric on the absolute soft set $\tilde{U}$ if $\tilde{d}$ satisfies the following conditions:

(1) $\tilde{d}(F_\lambda^x, F_\mu^y)\tilde{\geq}\bar{0}$ for all $F_\lambda^x, F_\mu^y\tilde{\in}SP(\tilde{U})$,

(2) $\tilde{d}(F_\lambda^x, F_\mu^y) = \bar{0}$ if and only if $\lambda = \mu$ and $x = y$ for all $F_\lambda^x, F_\mu^y\tilde{\in}SP(\tilde{U})$,

(3) $\tilde{d}(F_\lambda^x, F_\mu^y) = \tilde{d}(F_\mu^y, F_\lambda^x)$ for all $F_\lambda^x, F_\mu^y\tilde{\in}SP(\tilde{U})$,

(4) $\tilde{d}(F_\lambda^x, F_\gamma^z)\tilde{\leq}\tilde{d}(F_\lambda^x, F_\mu^y) + \tilde{d}(F_\mu^y, F_\gamma^z)$ for all $F_\lambda^x, F_\mu^y, F_\gamma^z\tilde{\in}SP(\tilde{U})$.

The soft set $\tilde{U}$ together with soft metric $\tilde{d}$ is called a soft metric space and is denoted by $(\tilde{U}, \tilde{d}, E)$ or simply by $(\tilde{U}, \tilde{d})$.

Example 3.14. Suppose $U = \{u\}$ and $E = \{a, b\}$, then $SP(\tilde{U}) = \{F^u_a, F^u_b\}$. Let $\tilde{d}$ be a function defined as $\tilde{d} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that

$$\tilde{d}(F^u_\lambda, F^u_\lambda) = \bar{0} \quad \text{for all } \lambda \in E$$
$$\tilde{d}(F^u_a, F^u_b) = \tilde{d}(F^u_b, F^u_a) = \bar{1},$$

then the pair $(\tilde{U}, \tilde{d})$ is a soft metric space.

Definition 3.15 ([1]). Let $\{F^x_{\lambda,n}\}_n$ be a sequence in a soft metric space $(\tilde{U}, \tilde{d})$. Then, $\{F^x_{\lambda,n}\}_n$ is said to be convergent in $(\tilde{U}, \tilde{d})$ if there is a soft point $F^y_\mu \tilde{\in} \tilde{U}$ such that $\tilde{d}(F^x_{\lambda,n}, F^y_\mu) \to \bar{0}$ as n $\to \infty$.

Definition 3.16 ([1]). A sequence $\{F^x_{\lambda,n}\}_n$ of soft points in $(\tilde{U}, \tilde{d})$ is considered as a Cauchy sequence in $\tilde{U}$ if corresponding to every $\tilde{\epsilon} \tilde{\geq} \bar{0}$, there exists $m \in \mathbb{N}$ such that $\tilde{d}(F^x_{\lambda,i}, F^x_{\lambda,j}) \tilde{\leq} \tilde{\epsilon}$, for all $i, j \geq m$ i.e., $\tilde{d}(F^x_{\lambda,i}, F^x_{\lambda,j}) \to \bar{0}$ as $i, j \to \infty$.

Definition 3.17 ([1]). A soft metric space $(\tilde{U}, \tilde{d})$ is called complete if every Cauchy sequence in $\tilde{U}$ converges to some soft point of $\tilde{U}$.

Definition 3.18 ([2]). A mapping $d^* : U \times U \to \mathbb{R}^*$ is multiplicative metric if d^* satisfies the following conditions:

(1) $d^*(u_1, u_2) \geq 1$ for all $u_1, u_2 \in U$;

(2) $d^*(u_1, u_2) = 1$ if and only if $u_1 = u_2$ for all $u_1, u_2 \in U$;

(3) $d^*(u_1, u_2) = d^*(u_2, u_1)$ for all $u_1, u_2 \in U$;

(4) $d^*(u_1, u_2) \leq d^*(u_1, u_3) \cdot d^*(u_3, u_2)$ for all $u_1, u_2, u_3 \in U$,

and the pair (U, d^*) is a multiplicative metric space.

Definition 3.19 ([11]). Let $(\tilde{X}, \tilde{d}, E)$ and $(\tilde{Y}, \tilde{\rho}, E')$ be two soft metric spaces. The mapping $(T, \psi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{Y}, \tilde{\rho}, E')$ is a soft mapping where $T : X \to Y$ and $\psi : E \to E'$ are two mappings.

4. Soft Multiplicative Metric Space

Inspiring from the ideas of [2] and [4], we combine the concept of soft metric space and multiplicative metric space to generate a new space called soft multiplicative metric space. With the help of examples, we understand the concept of soft multiplicative metric space.

Definition 4.1. A function $\tilde{d}^* : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ is a soft multiplicative metric on the absolute soft set $\tilde{U}$ if $\tilde{d}^*$ meets the properties given as:

(1) $\tilde{d}^*(F^x_\lambda, F^y_\mu) \tilde{\geq} \bar{1}$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$;

(2) $\tilde{d}^*(F^x_\lambda, F^y_\mu) = \bar{1} \Leftrightarrow \lambda = \mu$ and $x = y$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$;

(3) $\tilde{d}^*(F^x_\lambda, F^y_\mu) = \tilde{d}^*(F^y_\mu, F^x_\lambda)$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$;

(4) $\tilde{d}^*(F^x_\lambda, F^z_\gamma) \tilde{\leq} \tilde{d}^*(F^x_\lambda, F^y_\mu) \cdot \tilde{d}^*(F^y_\mu, F^z_\gamma)$ for all $F^x_\lambda, F^y_\mu, F^z_\gamma \tilde{\in} SP(\tilde{U})$;

and $(\tilde{U}, \tilde{d}^*, E)$ is a soft multiplicative metric space.

Example 4.2. Let $\tilde{d}^*: SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that

$$\tilde{d}^*(F_\lambda^x, F_\mu^y) = \begin{cases} \bar{1} \text{ if } F_\lambda^x = F_\mu^y \\ \bar{2} \text{ if } F_\lambda^x \neq F_\mu^y \end{cases}$$

Clearly, $\tilde{d}^*$ meets all the properties of soft multiplicative metric. So, $\tilde{d}^*$ is a soft multiplicative metric on the absolute soft set $\tilde{U}$ and hence $(\tilde{U}, \tilde{d}^*, E)$ is a soft multiplicative metric space.

Definition 4.3. Suppose $(\tilde{U}, \tilde{d}^*)$ is a soft multiplicative metric space. Then, a sequence $\{F_{\lambda,n}^x\}_n$ in $(\tilde{U}, \tilde{d}^*)$ is multiplicative convergent to a soft point $F_\mu^y \tilde{\in} \tilde{U}$ if for given $\tilde{\epsilon} \tilde{\geq} \bar{1}$, we have a unique positive integer n_0 such that $\tilde{d}^*(F_{\lambda,n}^x, F_\mu^y) \tilde{<} \tilde{\epsilon}$ for all $n \geq n_0$ i.e., $\tilde{d}^*(F_{\lambda,n}^x, F_\mu^y) \to \bar{1}$ as $n \to \infty$.

Definition 4.4. Suppose $(\tilde{U}, \tilde{d}^*)$ is a soft multiplicative metric space. Then, a sequence $\{F_{\lambda,n}^x\}_n$ in $(\tilde{U}, \tilde{d}^*)$ is multiplicative Cauchy sequence if for given $\tilde{\epsilon} \tilde{\geq} \bar{1}$, we have a unique positive integer n_0 such that $\tilde{d}^*(F_{\lambda,m}^x, F_{\lambda,n}^x) \tilde{<} \tilde{\epsilon}$ for all $m, n \geq n_0$ i.e., $\tilde{d}^*(F_{\lambda,m}^x, F_{\lambda,n}^x) \to \bar{1}$ as $m, n \to \infty$.

Definition 4.5. A soft multiplicative metric space $(\tilde{U}, \tilde{d}^*)$ is complete if every multiplicative Cauchy sequence in $\tilde{U}$ converges to some soft point in $\tilde{U}$.

Definition 4.6. Consider a soft multiplicative metric space $(\tilde{U}, \tilde{d}^*, E)$. A function $(T, \psi): (\tilde{U}, \tilde{d}^*, E) \to (\tilde{U}, \tilde{d}^*, E)$ is said to be soft multiplicative contraction mapping if for every soft point $F_\lambda^x, F_\mu^y \tilde{\in} \tilde{U}$, there exists a soft real number $\bar{h}, \bar{0} \tilde{\leq} \bar{h} \tilde{\leq} \bar{1}$ such that

$$\tilde{d}^*\{(T,\psi)(F_\lambda^x), (T,\psi)(F_\mu^y)\} \tilde{\leq} \{\tilde{d}^*(F_\lambda^x, F_\mu^y)\}^{\bar{h}}.$$

Example 4.7. Consider $U = E = \{1/n : n \in \mathbb{N}\}$ and $\tilde{d}^*: SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that for all $F_\lambda^x, F_\mu^y \tilde{\in} \tilde{U}$, $\tilde{d}^*(F_\lambda^x, F_\mu^y) = e^{|\bar{x}-\bar{y}|+|\bar{\lambda}-\bar{\mu}|}$.

Let $(T, \psi): (\tilde{U}, \tilde{d}^*, E) \to (\tilde{U}, \tilde{d}^*, E)$ be a soft mapping such that for all $x \in U$ and $\lambda \in E$, $(T,\psi)(F_\lambda^x) = F_1^{x/5}$. Then, given any $x, y \in U$ and $\lambda, \mu \in E$, and for each $k \in E$, we have

$$\begin{aligned} \tilde{d}^*\{(T,\psi)(F_\lambda^x), (T,\psi)(F_\mu^y)\}(k) &= \tilde{d}^*(F_1^{x/5}, F_1^{y/5})(k) \\ &= e^{|\frac{x}{5}-\frac{y}{5}|} \\ &\tilde{\leq} \{\tilde{d}^*(F_\lambda^x, F_\mu^y)\}^{\frac{1}{5}} \end{aligned}$$

Thus, (T, ψ) is a soft multiplicative contraction mapping with $\bar{h} = \frac{\bar{1}}{5}$.

5. Relationship Between Soft Metric Space and Soft Multiplicative Metric Space

In this section, we show that with every soft metric space, we can define soft multiplicative metric space and vice-versa.

Theorem 5.1. *Suppose $(\tilde{U}, \tilde{d})$ is a soft metric space. Let $\tilde{d}^*$ be a function defined as $\tilde{d}^*: SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that $\tilde{d}^*(F_\lambda^x, F_\mu^y) = \bar{1} + \tilde{d}(F_\lambda^x, F_\mu^y)$. Then, $(\tilde{U}, \tilde{d}^*)$ is a soft multiplicative metric space.*

Proof. By using the conditions of soft metric space and $\tilde{d}^*(F^x_\lambda, F^y_\mu) = \bar{1} + \tilde{d}(F^x_\lambda, F^y_\mu)$, the first three properties of soft multiplicative metric space are obtained trivially. We only need to satisfy the property (1) of Definition 4.1. Now, for all F^x_λ, F^y_μ and $F^z_\gamma \tilde{\in} SP(\tilde{U})$, we have

$$\tilde{d}(F^x_\lambda, F^z_\gamma) \tilde{\le} \tilde{d}(F^x_\lambda, F^y_\mu) + \tilde{d}(F^y_\mu, F^z_\gamma) \qquad \text{(by property (4) of Definition 3.13)}$$

$$\Rightarrow \quad \bar{1} + \tilde{d}(F^x_\lambda, F^z_\gamma) \tilde{\le} \bar{1} + \tilde{d}(F^x_\lambda, F^y_\mu) + \tilde{d}(F^y_\mu, F^z_\gamma)$$

$$\Rightarrow \quad \bar{1} + \tilde{d}(F^x_\lambda, F^z_\gamma) \tilde{\le} (\bar{1} + \tilde{d}(F^x_\lambda, F^y_\mu))(\bar{1} + \tilde{d}(F^y_\mu, F^z_\gamma))$$

$$\Rightarrow \quad \tilde{d}^*(F^x_\lambda, F^z_\gamma) \tilde{\le} \tilde{d}^*(F^x_\lambda, F^y_\mu)\tilde{d}^*(F^y_\mu, F^z_\gamma).$$

Thus, $\tilde{d}^*$ is a soft multiplicative metric on the absolute set $\tilde{U}$. □

Example 5.2. Suppose $(\tilde{U}, \tilde{d})$ is a soft metric space where $\tilde{d} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that

$$\tilde{d}(F^x_\lambda, F^y_\mu) = \begin{cases} \bar{0} \text{ if } F^x_\lambda = F^y_\mu \\ \bar{1} \text{ if } F^x_\lambda \neq F^y_\mu \end{cases}$$

Let $\tilde{d}^* : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$. Then, clearly

$$\tilde{d}^*(F^x_\lambda, F^y_\mu) = \bar{1} + \tilde{d}(F^x_\lambda, F^y_\mu) = \begin{cases} \bar{1} \text{ if } F^x_\lambda = F^y_\mu \\ \bar{2} \text{ if } F^x_\lambda \neq F^y_\mu \end{cases}$$

is a soft multiplicative metric on $\tilde{U}$.

Theorem 5.3. *Suppose $(\tilde{U}, \tilde{d})$ is a soft metric space. Let $\tilde{d}^*$ be a function defined as $\tilde{d}^* : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that $\tilde{d}^*(F^x_\lambda, F^y_\mu) = e^{\tilde{d}(F^x_\lambda, F^y_\mu)}$. Then, $(\tilde{U}, \tilde{d}^*)$ is a soft multiplicative metric space.*

Proof. By using the properties of soft metric space and $\tilde{d}^*(F^x_\lambda, F^y_\mu) = e^{\tilde{d}(F^x_\lambda, F^y_\mu)}$, we prove that $\tilde{d}^*$ satisfy all the properties of soft multiplicative metric space.

(1) $\tilde{d}(F^x_\lambda, F^y_\mu) \tilde{\ge} \bar{0} \Rightarrow \tilde{d}^*(F^x_\lambda, F^y_\mu) \tilde{\ge} \bar{1}$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(2) $\tilde{d}^*(F^x_\lambda, F^y_\mu) = \bar{1} \Leftrightarrow e^{\tilde{d}(F^x_\lambda, F^y_\mu)} = \bar{1} \Leftrightarrow \tilde{d}(F^x_\lambda, F^y_\mu) = \bar{0} \Leftrightarrow F^x_\lambda = F^y_\mu$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(3) Clearly, $\tilde{d}^*(F^x_\lambda, F^y_\mu) = \tilde{d}^*(F^y_\mu, F^x_\lambda)$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(4) We have

$$\tilde{d}(F^x_\lambda, F^z_\gamma) \tilde{\le} \tilde{d}(F^x_\lambda, F^y_\mu) + \tilde{d}(F^y_\mu, F^z_\gamma)$$

$$\Rightarrow \quad e^{\tilde{d}(F^x_\lambda, F^z_\gamma)} \tilde{\le} e^{\tilde{d}(F^x_\lambda, F^y_\mu) + \tilde{d}(F^y_\mu, F^z_\gamma)}$$

$$\Rightarrow \quad e^{\tilde{d}(F^x_\lambda, F^z_\gamma)} \tilde{\le} e^{\tilde{d}(F^x_\lambda, F^y_\mu)} \cdot e^{\tilde{d}(F^y_\mu, F^z_\gamma)}$$

$$\Rightarrow \quad \tilde{d}^*(F^x_\lambda, F^z_\gamma) \tilde{\le} \tilde{d}^*(F^x_\lambda, F^y_\mu) \cdot \tilde{d}^*(F^y_\mu, F^z_\gamma) \text{ for all } F^x_\lambda, F^y_\mu, F^z_\gamma \tilde{\in} SP(\tilde{U}).$$ □

Example 5.4. Suppose U is the universal set, E is the non-empty parameter set and $(\tilde{U}, \tilde{d})$ is a soft metric space where $\tilde{d} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that

$$\tilde{d}(F^x_\lambda, F^y_\mu) = \begin{cases} \bar{0} \text{ if } F^x_\lambda = F^y_\mu \\ \bar{1} \text{ if } F^x_\lambda \neq F^y_\mu \end{cases}$$

Let $\tilde{d}^* : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$. Then, clearly

$$\tilde{d}^*(F^x_\lambda, F^y_\mu) = e^{\tilde{d}(F^x_\lambda, F^y_\mu)} = \begin{cases} \bar{1} \text{ if } F^x_\lambda = F^y_\mu \\ \bar{e} \text{ if } F^x_\lambda \neq F^y_\mu \end{cases},$$

is a soft multiplicative metric space on $\tilde{U}$.

Theorem 5.5. *Suppose $(\tilde{U}, \tilde{d}^*)$ is a soft multiplicative metric space. Let $\tilde{d}$ be a function defined as $\tilde{d} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that $\tilde{d}(F^x_\lambda, F^y_\mu) = \log\{\tilde{d}^*(F^x_\lambda, F^y_\mu)\}$. Then, $(\tilde{U}, \tilde{d})$ is a soft metric space.*

Proof. Since $(\tilde{U}, \tilde{d}^*)$ is a soft multiplicative metric space, therefore all the conditions of soft multiplicative metric space are satisfied. Now, we prove that $\tilde{d}$ satisfy all the properties of soft metric space.

(1) $\tilde{d}^*(F^x_\lambda, F^y_\mu) \tilde{\geq} \bar{1} \Rightarrow \tilde{d}(F^x_\lambda, F^y_\mu) \tilde{\geq} \bar{0}$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(2) $\tilde{d}(F^x_\lambda, F^y_\mu) = \bar{0} \Leftrightarrow \log\{\tilde{d}^*(F^x_\lambda, F^y_\mu)\} = \bar{0} \Leftrightarrow \tilde{d}^*(F^x_\lambda, F^y_\mu) = \bar{1} \Leftrightarrow F^x_\lambda = F^y_\mu$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(3) Clearly, $\tilde{d}(F^x_\lambda, F^y_\mu) = \tilde{d}(F^y_\mu, F^x_\lambda)$ for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(4) We have $\tilde{d}^*(F^x_\lambda, F^z_\gamma) \tilde{\leq} \tilde{d}^*(F^x_\lambda, F^y_\mu) \cdot \tilde{d}^*(F^y_\mu, F^z_\gamma)$

$$\begin{aligned} &\Rightarrow \quad \log\{\tilde{d}^*(F^x_\lambda, F^z_\gamma)\} \tilde{\leq} \log\{\tilde{d}^*(F^x_\lambda, F^y_\mu) \cdot \tilde{d}^*(F^y_\mu, F^z_\gamma)\} \\ &\Rightarrow \quad \log\{\tilde{d}^*(F^x_\lambda, F^z_\gamma)\} \tilde{\leq} \log\{\tilde{d}^*(F^x_\lambda, F^y_\mu)\} + \log\{\tilde{d}^*(F^y_\mu, F^z_\gamma)\} \\ &\Rightarrow \quad \tilde{d}(F^x_\lambda, F^z_\gamma) \tilde{\leq} \tilde{d}(F^x_\lambda, F^y_\mu) + \tilde{d}(F^y_\mu, F^z_\gamma) \quad \text{for all } F^x_\lambda, F^y_\mu, F^z_\gamma \tilde{\in} SP(\tilde{U}). \end{aligned}$$

□

Example 5.6. Suppose $(\tilde{U}, \tilde{d}^*, E)$ is a soft multiplicative metric space where $\tilde{d}^* : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that

$$\tilde{d}^*(F^x_\lambda, F^y_\mu) = \begin{cases} \bar{1} & \text{if } F^x_\lambda = F^y_\mu \\ \bar{2} & \text{if } F^x_\lambda \neq F^y_\mu \end{cases}$$

Let $\tilde{d} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$. Then, clearly

$$\tilde{d}(F^x_\lambda, F^y_\mu) = \log\{\tilde{d}^*(F^x_\lambda, F^y_\mu)\} = \begin{cases} \bar{0} & \text{if } F^x_\lambda = F^y_\mu \\ \overline{\log 2} & \text{if } F^x_\lambda \neq F^y_\mu \end{cases},$$

is a soft metric space on $\tilde{U}$.

6. Relationship Between Soft Multiplicative Metric Space and Multiplicative Metric Space

In this section, we construct a multiplicative metric space with the help of soft multiplicative metric space.

Theorem 6.1. *Suppose $(\tilde{U}, \tilde{d}^*, E)$ is a soft multiplicative metric space such that E is a finite set. Let us define a function $m_{\tilde{d}^*} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}^*$ as*

$$m_{\tilde{d}^*}(F^x_\lambda, F^y_\mu) = \max_{\kappa \in E} \tilde{d}^*(F^x_\lambda, F^y_\mu)(\kappa), \quad \textit{for all } F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U}).$$

Then, $m_{\tilde{d}^*}$ is a multiplicative metric on $\tilde{U}$.

Proof. Since $(\tilde{U},\tilde{d}^*)$ is a soft multiplicative metric space, therefore all the conditions of soft multiplicative metric are satisfied. Now, we prove that $m_{\tilde{d}^*}$ satisfy all the conditions of multiplicative metric space.

(1) For all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$, we have

$$\begin{aligned} & \tilde{d}^*(F^x_\lambda,F^y_\mu) \tilde{\geq} \bar{1} \\ \Rightarrow\quad & \tilde{d}^*(F^x_\lambda,F^y_\mu)(\kappa) \tilde{\geq} 1, \quad \text{for all } \kappa\in E \\ \Rightarrow\quad & \max_{\kappa\in E} \tilde{d}^*(F^x_\lambda,F^y_\mu)(\kappa) \tilde{\geq} 1 \\ \Rightarrow\quad & m^*_d(F^x_\lambda,F^y_\mu) \tilde{\geq} 1. \end{aligned}$$

(2) $m^*_d(F^x_\lambda,F^y_\mu)=1 \Leftrightarrow \max_{\kappa\in E} \tilde{d}^*(F^x_\lambda,F^y_\mu)(\kappa)=1 \Leftrightarrow \tilde{d}^*(F^x_\lambda,F^y_\mu)=\bar{1} \Leftrightarrow F^x_\lambda = F^y_\mu$, for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$.

(3) Clearly, $m^*_d(F^x_\lambda,F^y_\mu) = m^*_d(F^y_\mu,F^x_\lambda)$ by condition (3) of Definition 4.1.

(4) By condition (4) of Definition 4.1, we have

$$\begin{aligned} & d^*(F^x_\lambda,F^z_\gamma) \tilde{\leq} d^*(F^x_\lambda,F^y_\mu)\cdot d^*(F^y_\mu,F^z_\gamma) \\ \Rightarrow\quad & d^*(F^x_\lambda,F^z_\gamma)(\kappa) \tilde{\leq} d^*(F^x_\lambda,F^y_\mu)(\kappa)\cdot d^*(F^y_\mu,F^z_\gamma)(\kappa) \\ \Rightarrow\quad & \max_{\kappa\in E} \tilde{d}^*(F^x_\lambda,F^z_\gamma)(\kappa) \tilde{\leq} \max_{\kappa\in E} \tilde{d}^*(F^x_\lambda,F^y_\mu)(\kappa)\cdot \max_{\kappa\in E} \tilde{d}^*(F^y_\mu,F^z_\gamma)(\kappa) \\ \Rightarrow\quad & m_{\tilde{d}^*}(F^x_\lambda,F^z_\gamma) \tilde{\leq} m_{\tilde{d}^*}(F^x_\lambda,F^y_\mu)\cdot m_{\tilde{d}^*}(F^y_\mu,F^z_\gamma). \end{aligned}$$

Thus, $m_{\tilde{d}^*}$ is a multiplicative metric on $SP(\tilde{U})$. □

Theorem 6.2. *Suppose $(\tilde{U},\tilde{d}^*,E)$ is a soft multiplicative metric space such that E is a finite set. Let us define a function $m_{\tilde{d}^*}: SP(\tilde{U})\times SP(\tilde{U}) \to \mathbb{R}^*$ as*

$$m_{\tilde{d}^*}(F^x_\lambda,F^y_\mu) = \max_{\kappa\in E} \tilde{d}^*(F^x_\lambda,F^y_\mu)(\kappa),$$

for all $F^x_\lambda, F^y_\mu \tilde{\in} SP(\tilde{U})$. Then, $(\tilde{U},\tilde{d}^*,E)$ is complete if and only if $(SP(\tilde{U}),m_{\tilde{d}^*})$ is complete.

Proof. Suppose $(\tilde{U},\tilde{d}^*,E)$ is complete. Let $\{F^x_{\lambda,n}\}_n$ be a multiplicative Cauchy sequence in $(\tilde{U},\tilde{d}^*,E)$. Let $\bar{\epsilon} \tilde{>} \bar{1}$ be a constant soft real number. Then, there exists a natural number n_0 such that

$$\tilde{d}^*(F^x_{\lambda,m},F^x_{\lambda,n}) \tilde{<} \bar{\epsilon} \quad \text{for all } m,n\geq n_0.$$

Hence, $\tilde{d}^*(F^x_{\lambda,m},F^x_{\lambda,n})(\kappa) < \epsilon$ for all $\kappa\in E$ and $m,n\geq n_0$.

Thus, $m_{\tilde{d}^*}(F^x_{\lambda,m},F^x_{\lambda,n}) < \epsilon$ for all $m,n\geq n_0$.

Therefore, $\{F^x_{\lambda,n}\}_n$ is a multiplicative Cauchy sequence in $(m_{\tilde{d}^*},SP(\tilde{U}))$.

As $(\tilde{U},\tilde{d}^*,E)$ is complete, therefore $\{F^x_{\lambda,n}\}_n$ must converge to some soft point in $\tilde{U}$.

Let $\{F^x_{\lambda,n}\}_n \to F^y_\lambda$, $F^y_\lambda \in \tilde{U}$,

$$\begin{aligned} \Rightarrow\quad & \tilde{d}^*(F^y_\lambda,F^x_{\lambda,n}) \to \bar{1} \\ \Rightarrow\quad & \max_{\kappa\in E} \tilde{d}^*(F^y_\lambda,F^x_{\lambda,n})(\kappa) \to 1 \\ \Rightarrow\quad & m^*_{\tilde{d}}(F^y_\lambda,F^x_{\lambda,n}) \to 1 \end{aligned}$$

Therefore, $(SP(\tilde{U}),m_{\tilde{d}^*})$ is complete.

Suppose $(SP(\tilde{U}), m_{\tilde{d}^*})$ is complete. Let $\{F^x_{\lambda,n}\}_n$ be a multiplicative Cauchy sequence in $(SP(\tilde{U}), m_{\tilde{d}^*})$. If $\bar{\epsilon} \tilde{>} \bar{1}$, we can choose $\epsilon = \min_{\kappa \in E} \bar{\epsilon}(\kappa) > 1$, as E is a finite set. Then, there exists a natural number n_0 such that

$$m_{\tilde{d}^*}(F^x_{\lambda,m}, F^x_{\lambda,n}) \tilde{<} \epsilon \quad \text{for all } m,n \geq n_0$$

$$\tilde{d}^*(F^x_{\lambda,m}, F^x_{\lambda,n}) \tilde{<} \epsilon \leq \bar{\epsilon}(\kappa) \quad \text{for all } \kappa \in E \text{ and } m,n \geq n_0.$$

Therefore, $\{F^x_{\lambda,n}\}_n$ is a multiplicative Cauchy sequence in $(\tilde{U}, \tilde{d}^*, E)$.

As $(SP(\tilde{U}), m_{\tilde{d}^*})$ is complete, therefore $\{F^x_{\lambda,n}\}_n$ must converge to some soft point in $SP(\tilde{U})$.

Let $\{F^x_{\lambda,n}\}_n \to F^y_\lambda$, $F^y_\lambda \in SP(\tilde{U})$,

$$\Rightarrow \quad m_{\tilde{d}^*}(F^y_\lambda, F^x_{\lambda,n}) \to 1$$

$$\Rightarrow \quad \max_{\kappa \in E} \tilde{d}^*(F^y_\lambda, F^x_{\lambda,n})(\kappa) \to 1$$

$$\Rightarrow \quad \tilde{d}^*(F^y_\lambda, F^x_{\lambda,n}) \to \bar{1}$$

Therefore, $(\tilde{U}, \tilde{d}^*, E)$ is complete.

Thus, $(\tilde{U}, \tilde{d}^*, E)$ is complete if and only if $(SP(\tilde{U}), m_{\tilde{d}^*})$ is complete. □

Example 6.3. Suppose $U = \{1\}$ is the universal set and $E = \{\lambda, \mu\}$ is the non-empty parameter set. Then, $SP(\tilde{U}) = \{F^1_\lambda, F^1_\mu\}$.

Let us define $\tilde{d}^* : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}(E)^*$ such that

$$\tilde{d}^*(F^1_\kappa, F^1_\kappa) = \bar{1} \text{ where } \kappa \in E$$

$$\tilde{d}^*(F^1_\lambda, F^1_\mu)(\lambda) = 2 = \tilde{d}^*(F^1_\mu, F^1_\lambda)(\lambda)$$

$$\tilde{d}^*(F^1_\lambda, F^1_\mu)(\mu) = 1 = \tilde{d}^*(F^1_\mu, F^1_\lambda)(\mu).$$

Then, $\tilde{d}^*$ is a soft multiplicative metric on $\tilde{U}$.

Now, let $m_{\tilde{d}^*} : SP(\tilde{U}) \times SP(\tilde{U}) \to \mathbb{R}^*$ defined as

$$m_{\tilde{d}^*}(F^x_\lambda, F^y_\mu) = \max_{\kappa \in A} \tilde{d}^*(F^x_\lambda, F^y_\mu)(\kappa) = \begin{cases} 1 \text{ if } F^x_\lambda = F^y_\mu \\ 2 \text{ if } F^x_\lambda \neq F^y_\mu \end{cases}$$

Here, $m_{\tilde{d}^*}$ is a multiplicative metric on $\tilde{U}$.

7. Fixed Point Theorems in Soft Multiplicative Metric Space

This section contains some fixed point theorems in soft multiplicative metric space.

Theorem 7.1. *Let $(\tilde{U}, \tilde{d}^*, E)$ be a complete soft multiplicative metric space and $(T,\psi) : (\tilde{U}, \tilde{d}^*, E) \to (\tilde{U}, \tilde{d}^*, E)$ is a soft multiplicative contraction mapping. Then, there exists a unique soft point $F^x_\lambda \tilde{\in} \tilde{U}$ such that $(T,\psi)(F^x_\lambda) = F^x_\lambda$.*

Proof. Consider a soft point $F^{x_0}_{\lambda_0} \tilde{\in} \tilde{U}$. Define a sequence of soft points $\{F^x_{\lambda,n}\}_n$ in $\tilde{U}$ such that $F^{x_{n+1}}_{\lambda_{n+1}} = (T,\psi)(F^{x_n}_{\lambda_n})$.

Since (T,ψ) has a soft multiplicative contraction, therefore

$$\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^{x_n}_{\lambda_n}) \tilde{\leq} \{\tilde{d}^*(F^{x_n}_{\lambda_n}, F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{h}}$$

$$\tilde{\lesssim}\{\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}},F^{x_{n-2}}_{\lambda_{n-2}})\}^{\bar{h}^2}$$
$$\vdots$$
$$\tilde{\lesssim}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{h}^n}.$$

Let for $p>0$, we have

$$\begin{aligned}\tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}},F^{x_n}_{\lambda_n})&\tilde{\lesssim}\tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}},F^{x_{n+p-1}}_{\lambda_{n+p-1}})\cdot\tilde{d}^*(F^{x_{n+p-1}}_{\lambda_{n+p-1}},F^{x_{n+p-2}}_{\lambda_{n+p-2}})\cdots\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\\&\tilde{\lesssim}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{h}^{n+p-1}}\cdot\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{h}^{n+p-2}}\cdots\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{h}^n}\\&\tilde{\lesssim}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{h}^{n+p-1}+\bar{h}^{n+p-2}+\cdots+\bar{h}^n}\\&\tilde{\lesssim}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\frac{\bar{h}^n}{1-\bar{h}}}.\end{aligned}$$

This shows that $\{\tilde{d}^*(F^{x_{n+p-1}}_{\lambda_{n+p-1}},F^{x_n}_{\lambda_n})\}\to\bar{1}$ as $n\to\infty$. Thus, the soft sequence $F^{x_n}_{\lambda_n}=(T,\psi)^n(F^{x_0}_{\lambda_0})$ is soft multiplicative Cauchy sequence. But $(\tilde{U},\tilde{d}^*,E)$ is given to be a complete soft multiplicative metric space. Therefore, there exist a soft point $F^y_\mu\tilde{\in}\tilde{U}$ such that $F^{x_n}_{\lambda_n}\to F^y_\mu$ as $n\to\infty$.

Further,

$$\begin{aligned}\tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}&\tilde{\le}\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^y_\mu)\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^y_\mu\}\\&\tilde{\lesssim}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^y_\mu)\}^{\bar{h}}\cdot\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^y_\mu)\to\bar{1}\ \text{ as }\ n\to\infty\end{aligned}$$

$$\Rightarrow\qquad\tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}=\bar{1}$$

This shows that F^y_μ is a soft fixed point of (T,ψ) i.e., $(T,\psi)(F^y_\mu)=F^y_\mu$.

Now, we prove the uniqueness of soft fixed point of (T,ψ). For this, let F^z_γ be another soft fixed point of (T,ψ) i.e., $(T,\psi)(F^z_\gamma)=F^z_\gamma$. Then,

$$\begin{aligned}\tilde{d}^*(F^y_\mu,F^z_\gamma)&=\tilde{d}^*\{(T,\psi)(F^y_\mu),(T,\psi)(F^z_\gamma)\}\\&\tilde{\lesssim}\{\tilde{d}^*(F^y_\mu,F^z_\gamma)\}^{\bar{h}}\end{aligned}$$

Thus, $\tilde{d}^*(F^y_\mu,F^z_\gamma)=\bar{1}$ and thus $F^y_\mu=F^z_\gamma$.

This shows that F^y_μ is the unique soft fixed point of (T,ψ). □

Remark 7.2. The condition that (T,ψ) is soft multiplicative contraction mapping cannot be omitted in Theorem 7.1. For example, Let $U=\{1\}$ and $E=\{\lambda,\mu\}$, then $SP(\tilde{U})=\{F^1_\lambda,F^1_\mu\}$. Define a mapping $\tilde{d}^*:SP(\tilde{U})\times SP(\tilde{U})\to\mathbb{R}(\mathbb{E})^*$ as

$$\tilde{d}^*(F^x_\lambda,F^y_\mu)=\begin{cases}\bar{1}\text{ if }F^x_\lambda=F^y_\mu\\\bar{2}\text{ if }F^x_\lambda\neq F^y_\mu\end{cases},$$

and $(T,\psi):(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ is a mapping defined as $(T,\psi)(F^1_\lambda)=F^1_\mu$ and $(T,\psi)(F^1_\mu)=F^1_\lambda$ then (T,ψ) is not a multiplicative mapping and the mapping has no soft fixed point.

Corollary 7.3. *Suppose $(\tilde{U},\tilde{d}^*,E)$ is a complete soft multiplicative metric space and $(T,\psi):(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ is such that $(T,\psi)^N$ is a soft multiplicative contraction mapping for some positive integer N. Then, (T,ψ) has a unique soft fixed point.*

Proof. From Theorem 7.1, $(T,\psi)^N$ has a unique soft fixed point say $F^x_\lambda \tilde{\in} \tilde{U}$. But

$$(T,\psi)^N\{(T,\psi)(F^x_\lambda)\} = (T,\psi)\{(T,\psi)^N(F^x_\lambda)\} = (T,\psi)(F^x_\lambda).$$

Therefore, $(T,\psi)(F^x_\lambda)$ is also a soft fixed point of $(T,\psi)^N$. Hence, $(T,\psi)(F^x_\lambda) = F^x_\lambda$.

This shows that F^x_λ is a soft fixed point of (T,ψ). □

Theorem 7.4. *Suppose $(\tilde{U},\tilde{d}^*,E)$ is a complete soft multiplicative metric space. Suppose $(T,\psi):(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ is a soft mapping such that*

$$\tilde{d}^*\{(T,\psi)(F^x_\lambda),(T,\psi)(F^y_\mu)\} \tilde{\le} [\tilde{d}^*\{(T,\psi)(F^x_\lambda),F^x_\lambda\}\cdot \tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}]^{\bar{h}} \tag{7.1}$$

for all $F^x_\lambda, F^y_\mu \tilde{\in} \tilde{U}$ and $\bar{0} \tilde{\le} \bar{h} \tilde{<} \frac{\bar{1}}{2}$ is a soft real number. Then, the soft mapping (T,ψ) has a unique soft fixed point in $\tilde{U}$ and the soft sequence $\{(T,\psi)^n(F^x_\lambda)\}$ converges to the soft fixed point.

Proof. Consider a soft point $F^{x_0}_{\lambda_0} \tilde{\in} \tilde{U}$. Define a sequence of soft points $\{F^x_{\lambda,n}\}_n$ in $\tilde{U}$ such that $F^{x_{n+1}}_{\lambda_{n+1}} = (T,\psi)(F^{x_n}_{\lambda_n})$.

Since (T,ψ) satisfies the given contractive condition, therefore

$$\begin{aligned}
\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n}) &= \tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}})\}\\
&\tilde{\le} [\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\cdot \tilde{d}^*\{(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}}),F^{x_{n-1}}_{\lambda_{n-1}}\}]^{\bar{h}}\\
&\tilde{\le} \{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot \tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{h}}\\
\Rightarrow \quad \{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\}^{1-\bar{h}} &\tilde{\le} \tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{h}}\\
\Rightarrow \quad \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n}) &\tilde{\le} \tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{\frac{\bar{h}}{1-\bar{h}}}\\
&= \tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{k}}, \text{ where } \bar{k} = \frac{\bar{h}}{1-\bar{h}}\\
&\tilde{\le} \{\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}},F^{x_{n-2}}_{\lambda_{n-2}})\}^{\bar{k}^2}\\
&\vdots\\
&\tilde{\le} \{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{k}^n}.
\end{aligned}$$

Let for $p > 0$, we have

$$\begin{aligned}
\tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}},F^{x_n}_{\lambda_n}) &\tilde{\le} \tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}},F^{x_{n+p-1}}_{\lambda_{n+p-1}})\cdot \tilde{d}^*(F^{x_{n+p-1}}_{\lambda_{n+p-1}},F^{x_{n+p-2}}_{\lambda_{n+p-2}})\cdots \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\\
&\tilde{\le} \{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{k}^{n+p-1}}\cdot\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{k}^{n+p-2}}\cdots\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{k}^{n}}\\
&\tilde{\le} \{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\bar{k}^{n+p-1}+\bar{k}^{n+p-2}+\dots+\bar{k}^{n}}\\
&\tilde{\le} \{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\frac{\bar{k}^n}{1-\bar{k}}}.
\end{aligned}$$

This shows that $\{\tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}},F^{x_n}_{\lambda_n})\} \to \bar{1}$ as $n\to\infty$. Thus, the soft sequence $F^{x_n}_{\lambda_n} = (T,\psi)^n(F^{x_0}_{\lambda_0})$ is soft multiplicative Cauchy. But $(\tilde{U},\tilde{d}^*,E)$ is given to be a complete soft multiplicative metric space. Therefore, there exist a soft point $F^y_\mu \tilde{\in} \tilde{U}$ such that $F^{x_n}_{\lambda_n} \to F^y_\mu$ as $n\to\infty$.

Further,

$$\tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\} \tilde{\le} \tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^y_\mu)\}\cdot \tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^y_\mu\}$$

$$\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}]^{\bar{h}}\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^y_\mu\}$$

$$\Rightarrow\quad \tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}^{1-\bar{h}}\tilde{\le}\{\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\}^{\bar{h}}\cdot\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^y_\mu)$$

$$\Rightarrow\quad \tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}\tilde{\le}[\{\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\}^{\bar{h}}\cdot\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^y_\mu)]^{\frac{1}{1-\bar{h}}}$$

$$\Rightarrow\quad \tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}\to\bar{1}\ \text{ as }\ n\to\infty$$

$$\Rightarrow\quad \tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}=\bar{1}$$

This shows that F^y_μ is a soft fixed point of (T,ψ) i.e., $(T,\psi)(F^y_\mu)=F^y_\mu$.

Now, we prove the uniqueness of soft fixed point of (T,ψ). For this, let F^z_γ be another soft fixed point of (T,ψ) i.e., $(T,\psi)(F^z_\gamma)=F^z_\gamma$. Then,

$$\tilde{d}^*(F^y_\mu,F^z_\gamma)=\tilde{d}^*\{(T,\psi)(F^y_\mu),(T,\psi)(F^z_\gamma)\}$$

$$\tilde{\le}[\{\tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}\cdot\tilde{d}^*\{(T,\psi)(F^z_\gamma),F^z_\gamma\}]^{\bar{h}}\to\bar{1}$$

Thus, $\tilde{d}^*(F^y_\mu,F^z_\gamma)=\bar{1}$ and thus $F^y_\mu=F^z_\gamma$.

This shows that F^y_μ is the unique soft fixed point of (T,ψ). □

Example 7.5. The converse of Theorem 7.4 need not be true. For example, let $U=E=\{\frac{1}{n}:n\in\mathbb{N}\}$. Define a mapping $\tilde{d}^*:SP(\tilde{U})\times SP(\tilde{U})\to\mathbb{R}(E)^*$ such that

$$\tilde{d}^*(F^x_\lambda,F^y_\mu)=\begin{cases}\bar{1} \text{ if } F^x_\lambda=F^y_\mu\\ \bar{2} \text{ if } F^x_\lambda\neq F^y_\mu\end{cases}$$

and (T,ψ): $(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ is a mapping defined as

$$(T,\psi)(F^{\frac{1}{n}}_{\frac{1}{k}})=\begin{cases}F^1_1 \text{ if } n=1\\ F^1_{\frac{1}{k}} \text{ if } n\neq 1\end{cases}$$

then (T,ψ) has a unique soft point $F^1_1\tilde{\in}SP(\tilde{U})$ such that $(T,\psi)(F^1_1)=F^1_1$ but it does not meet the condition (1).

Theorem 7.6. Suppose $(\tilde{U},\tilde{d}^*,E)$ is a complete soft multiplicative metric space. Suppose $(T,\psi):(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ is a soft mapping such that

$$\tilde{d}^*\{(T,\psi)(F^x_\lambda),(T,\psi)(F^y_\mu)\}\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^x_\lambda),F^y_\mu\}\cdot\tilde{d}^*\{(T,\psi)(F^y_\mu),F^x_\lambda\}]^{\bar{h}}$$

for all $F^x_\lambda,F^y_\mu\in\tilde{U}$ *and* $\bar{0}\tilde{\le}\bar{h}\tilde{<}\frac{\bar{1}}{2}$ *is a soft real number. Then, the soft mapping* (T,ψ) *has a unique soft fixed point in* $\tilde{U}$ *and the sequence* $\{(T,\psi)^n(F^x_\lambda)\}$ *converges to the soft fixed point.*

Proof. Consider a soft point $F^{x_0}_{\lambda_0}\tilde{\in}\tilde{U}$. Define a sequence of soft points $\{F^x_{\lambda,n}\}_n$ in $\tilde{U}$ such that

$$F^{x_{n+1}}_{\lambda_{n+1}}=(T,\psi)(F^{x_n}_{\lambda_n}).$$

Since (T,ψ) satisfies the given contractive condition, therefore

$$\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})=\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}})\}$$

$$\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_{n-1}}_{\lambda_{n-1}}\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}}),F^{x_n}_{\lambda_n}\}]^{\bar{h}}$$

$$\tilde{\le}\{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_{n-1}}_{\lambda_{n-1}})\cdot\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_n}_{\lambda_n})\}^{\bar{h}}$$

$$= \{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{h}}$$
$$\tilde{\leq} \{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^{x_n}_{\lambda_n}) \cdot \tilde{d}^*(F^{x_n}_{\lambda_n}, F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{h}}$$
$$\Rightarrow \quad \{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^{x_n}_{\lambda_n})\}^{1-\bar{h}} \tilde{\leq} \tilde{d}^*(F^{x_n}_{\lambda_n}, F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{h}}$$
$$\Rightarrow \quad \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^{x_n}_{\lambda_n}) \tilde{\leq} \tilde{d}^*(F^{x_n}_{\lambda_n}, F^{x_{n-1}}_{\lambda_{n-1}})\}^{\frac{\bar{h}}{1-\bar{h}}}$$
$$= \tilde{d}^*(F^{x_n}_{\lambda_n}, F^{x_{n-1}}_{\lambda_{n-1}})\}^{\bar{k}}, \quad \text{where } \bar{k} = \frac{\bar{h}}{1-\bar{h}}$$
$$\tilde{\leq} \{\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}}, F^{x_{n-2}}_{\lambda_{n-2}})\}^{\bar{k}^2}$$
$$\vdots$$
$$\tilde{\leq} \{\tilde{d}^*(F^{x_1}_{\lambda_1}, F^{x_0}_{\lambda_0})\}^{\bar{k}^n}.$$

Let for $p > 0$, we have

$$\tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}}, F^{x_n}_{\lambda_n}) \leq \tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}}, F^{x_{n+p-1}}_{\lambda_{n+p-1}}) \cdot \tilde{d}^*(F^{x_{n+p-1}}_{\lambda_{n+p-1}}, F^{x_{n+p-2}}_{\lambda_{n+p-2}}) \cdots \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^{x_n}_{\lambda_n})$$
$$\tilde{\leq} \{\tilde{d}^*(F^{x_1}_{\lambda_1}, F^{x_0}_{\lambda_0})\}^{\bar{k}^{n+p-1}} \cdot \{\tilde{d}^*(F^{x_1}_{\lambda_1}, F^{x_0}_{\lambda_0})\}^{\bar{k}^{n+p-2}} \cdots \{\tilde{d}^*(F^{x_1}_{\lambda_1}, F^{x_0}_{\lambda_0})\}^{\bar{k}^n}$$
$$\tilde{\leq} \{\tilde{d}^*(F^{x_1}_{\lambda_1}, F^{x_0}_{\lambda_0})\}^{\bar{k}^{n+p-1} + \bar{k}^{n+p-2} + \cdot + \bar{k}^n}$$
$$\tilde{\leq} \{\tilde{d}^*(F^{x_1}_{\lambda_1}, F^{x_0}_{\lambda_0})\}^{\frac{\bar{k}^n}{1-\bar{k}}}.$$

This shows that $\{\tilde{d}^*(F^{x_{n+p}}_{\lambda_{n+p}}, F^{x_n}_{\lambda_n})\} \to \bar{1}$ as $n \to \infty$. Thus, the soft sequence $F^{x_n}_{\lambda_n} = (T,\psi)^n (F^{x_0}_{\lambda_0})$ is soft multiplicative Cauchy. But $(\tilde{U}, \tilde{d}^*, E)$ is given to be a complete soft multiplicative metric space. Therefore, there exist a soft point $F^y_\mu \tilde{\in} \tilde{U}$ such that $F^{x_n}_{\lambda_n} \to F^y_\mu$ as $n \to \infty$.

Further,

$$\tilde{d}^*\{(T,\psi)(F^y_\mu), F^y_\mu\} \tilde{\leq} \tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}), (T,\psi)(F^y_\mu)\} \cdot \tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}), F^y_\mu\}$$
$$\tilde{\leq} [\tilde{d}^*\{(T,\psi)(F^y_\mu), F^y_\mu\} \cdot \tilde{d}^*(F^{x_n}_{\lambda_n}, F^y_\mu) \cdot \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^y_\mu)]^{\bar{h}} \cdot \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^y_\mu)$$
$$\Rightarrow \quad \tilde{d}^*\{(T,\psi)(F^y_\mu), F^y_\mu\}^{1-\bar{h}} \leq [\tilde{d}^*(F^{x_n}_{\lambda_n}, F^y_\mu) \cdot \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^y_\mu)]^{\bar{h}} \cdot \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^y_\mu)$$
$$\Rightarrow \quad \tilde{d}^*\{(T,\psi)(F^y_\mu), F^y_\mu\} \tilde{\leq} [\{\tilde{d}^*(F^{x_n}_{\lambda_n}, F^y_\mu) \cdot \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^y_\mu)\}^{\bar{h}} \cdot \tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}}, F^y_\mu)]^{\frac{1}{1-\bar{h}}}$$
$$\Rightarrow \quad \tilde{d}^*\{(T,\psi)(F^y_\mu), F^y_\mu\} \to \bar{1} \quad \text{as} \quad n \to \infty$$
$$\Rightarrow \quad \tilde{d}^*\{(T,\psi)(F^y_\mu), F^y_\mu\} = \bar{1}$$

This shows that F^y_μ is a soft fixed point of (T,ψ) i.e., $(T,\psi)(F^y_\mu) = F^y_\mu$.

Now, we prove the uniqueness of soft fixed point of (T,ψ). For this, let F^z_γ be another soft fixed point of (T,ψ) i.e., $(T,\psi)(F^z_\gamma) = F^z_\gamma$. Then,

$$\tilde{d}^*(F^y_\mu, F^z_\gamma) = \tilde{d}^*\{(T,\psi)(F^y_\mu), (T,\psi)(F^z_\gamma)\}$$
$$\tilde{\leq} [\{\tilde{d}^*\{(T,\psi)(F^y_\mu), F^z_\gamma\} \cdot \tilde{d}^*\{(T,\psi)(F^z_\gamma), F^y_\mu\}]^{\bar{h}}$$
$$\to \bar{1}$$

Thus, $\tilde{d}^*(F^y_\mu, F^z_\gamma) = \bar{1}$ and thus $F^y_\mu = F^z_\gamma$.

This shows that F^y_μ is the unique soft fixed point of (T,ψ). □

Theorem 7.7. *Let $(\tilde{U},\tilde{d}^*,E)$ be a soft multiplicative metric space. If the function (T,ψ): $(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ meets the following condition:*

$$\tilde{d}^*\{(T,\psi)(F^x_\lambda),(T,\psi)(F^y_\mu)\}\tilde{\le}\{\tilde{d}^*(F^x_\lambda,F^y_\mu)\}^a\cdot[\tilde{d}^*\{(T,\psi)(F^x_\lambda),F^x_\lambda\}\cdot\tilde{d}^*\{(T,\psi)(F^y_\mu),F^y_\mu\}]^b, \tag{7.2}$$

where $a+2b\tilde{<}1$, $0\tilde{<}b\tilde{<}1$, S then we get a unique soft point $F^x_\lambda\tilde{\in}SP(\tilde{U})$ such that $(T,\psi)(F^x_\lambda)=F^x_\lambda$.

Proof. Let $F^{x_0}_\lambda$ be any soft point in $SP(\tilde{U})$.

$$\text{Fix } F^{x_1}_{\lambda_1}=(T,\psi)(F^{x_0}_\lambda)$$
$$F^{x_2}_{\lambda_2}=(T,\psi)(F^{x_1}_{\lambda_1})$$
$$\vdots$$
$$F^{x_{n+1}}_{\lambda_{n+1}}=(T,\psi)(F^{x_n}_{\lambda_n})$$

We have

$$\begin{aligned}\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})&=\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}})\}\\&\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^a\cdot[\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}}),F^{x_{n-1}}_{\lambda_{n-1}}\}]^b\\&\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^a\cdot[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})]^b\end{aligned}$$

$$\Rightarrow\quad\{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\}^{1-b}\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{a+b}$$

$$\begin{aligned}\Rightarrow\quad\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})&\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{\frac{a+b}{1-b}}\\&\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{k},\quad\text{where } k=\frac{a+b}{1-b}\\&\tilde{\le}\{\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}},F^{x_{n-2}}_{\lambda_{n-2}})\}^{k^2}\\&\vdots\\&\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^n}\end{aligned}$$

So, for $n>m$

$$\begin{aligned}\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_m}_{\lambda_m})&\tilde{\le}\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\cdot\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}},F^{x_{n-2}}_{\lambda_{n-2}})\cdots\tilde{d}^*(F^{x_{m+1}}_{\lambda_{m+1}},F^{x_m}_{\lambda_m})\\&\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^{n-1}}\cdot\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^{n-2}}\cdots\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^m}\\&\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^{n-1}+k^{n-2}+\cdots+k^m}\\&\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\frac{k^m}{1-k}}\end{aligned}$$

We get

$$\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_m}_{\lambda_m})\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\frac{k^m}{1-k}}$$

This signifies that $\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_m}_{\lambda_m})\to\bar{1}$ as $n,m\to\infty$. Hence, $\{F^{x_n}_{\lambda_n}\}$ is a multiplicative soft Cauchy sequence. Since $\tilde{U}$ is complete, therefore there exists $F^{x^*}_\lambda\tilde{\in}\tilde{U}$ such that $F^{x_n}_{\lambda_n}\to F^{x^*}_\lambda$ as $n\to\infty$.

Now,

$$\begin{aligned}\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda_n}),F^{x^*}_{\lambda_n}\}&\tilde{\le}\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^{x^*}_{\lambda_n})\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x^*}_{\lambda_n}\}\\&\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x^*}_{\lambda_n})\}^a\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\end{aligned}$$

$$\cdot[\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda_n}),F^{x^*}_{\lambda_n}\}]^b\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x^*}_{\lambda_n}\}$$
$$\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x^*}_{\lambda_n})\}^a\cdot\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot\{\tilde{d}^*(F^{x^*}_{\lambda_{n+1}},F^{x^*}_{\lambda_n})\}^b\cdot\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x^*}_{\lambda_n})$$
$$\to\bar{1}$$

This signifies that $(T,\psi)(F^{x^*}_{\lambda})=F^{x^*}_{\lambda}$. So, the soft point $F^{x^*}_{\lambda}$ is a fixed soft point of the mapping (T,ψ). Now, if $F^{y^*}_{\mu}$ is another fixed soft point of the mapping (T,ψ), then we have

$$\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})=\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda},(T,\psi)(F^{y^*}_{\mu})\}$$
$$\tilde{\le}\{\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})\}^a\cdot[\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda}),F^{x^*}_{\lambda}\}\cdot\tilde{d}^*\{(T,\psi)(F^{y^*}_{\mu}),F^{y^*}_{\mu}\}]^b$$
$$\tilde{\le}\{\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})\}^a\cdot[\tilde{d}^*\{(F^{x^*}_{\lambda},F^{x^*}_{\lambda})\cdot\tilde{d}^*(F^{y^*}_{\mu},F^{y^*}_{\mu})]^b$$
$$=\{\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})\}^a$$
$$\{\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})\}^{1-a}\tilde{\le}1$$

Since $a+2b\tilde{<}1$ and $0\tilde{<}b\tilde{<}1$, therefore $a\tilde{<}1$ and thus

$$\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})=\bar{1}\Rightarrow F^{x^*}_{\lambda}=F^{y^*}_{\mu}.$$

Thus, we obtain a unique soft fixed point of (T,ψ). □

Theorem 7.8. *Suppose $(\tilde{U},\tilde{d}^*,E)$ is a soft multiplicative metric space. If the function $(T,\psi):(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ meets the following condition:*

$$\tilde{d}^*\{(T,\psi)(F^{x}_{\lambda}),(T,\psi)(F^{y}_{\mu})\}$$
$$\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^{x}_{\lambda}),F^{x}_{\lambda}\}\cdot\tilde{d}^*\{(T,\psi)(F^{y}_{\mu}),F^{y}_{\mu}\}]^a\cdot[\tilde{d}^*\{(T,\psi)(F^{x}_{\lambda}),F^{y}_{\mu}\}\cdot\tilde{d}^*\{(T,\psi)(F^{y}_{\mu}),F^{x}_{\lambda}\}]^b,$$

where $a+b\tilde{<}\frac{1}{2}$, then we obtain a unique soft point $F^{x}_{\lambda}\tilde{\in}SP(\tilde{U})$ such that $(T,\psi)(F^{x}_{\lambda})=F^{x}_{\lambda}$.

Proof. Let $F^{x_0}_{\lambda}$ be any soft point in $SP(\tilde{U})$.

$$\text{Fix } F^{x_1}_{\lambda_1}=(T,\psi)(F^{x_0}_{\lambda})$$
$$F^{x_2}_{\lambda_2}=(T,\psi)(F^{x_1}_{\lambda_1})$$
$$F^{x_{n+1}}_{\lambda_{n+1}}=(T,\psi)(F^{x_n}_{\lambda_n})$$

We have

$$\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})=\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}})\}$$
$$\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}}),F^{x_{n-1}}_{\lambda_{n-1}}\}]^a$$
$$\cdot[\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_{n-1}}_{\lambda_{n-1}}\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_{n-1}}_{\lambda_{n-1}}),F^{x_n}_{\lambda_n}\}]^b$$
$$\tilde{\le}[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})]^a\cdot[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_{n-1}}_{\lambda_{n-1}}\cdot\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_n}_{\lambda_n})]^b$$
$$\tilde{\le}[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})]^a\cdot[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})]^b$$
$$\Rightarrow\quad\{\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\}^{1-a-b}\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{a+b}$$
$$\Rightarrow\quad\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^{\frac{a+b}{1-a-b}}$$
$$\tilde{\le}\{\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\}^k,\quad\text{where } k=\frac{a+b}{1-a-b}$$

$$\tilde{\le}\{\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}},F^{x_{n-2}}_{\lambda_{n-2}})\}^{k^2}$$
$$\vdots$$
$$\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^n}$$

So, for $n>m$

$$\begin{aligned}\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_m}_{\lambda_m}) &\tilde{\le}\,\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_{n-1}}_{\lambda_{n-1}})\cdot\tilde{d}^*(F^{x_{n-1}}_{\lambda_{n-1}},F^{x_{n-2}}_{\lambda_{n-2}})\cdots\tilde{d}^*(F^{x_{m+1}}_{\lambda_{m+1}},F^{x_m}_{\lambda_m})\\ &\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^{n-1}}\cdot\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^{n-2}}\cdots\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^m}\\ &\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{k^{n-1}+k^{n-2}+\dots+k^m}\\ &\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\frac{k^m}{1-k}}\end{aligned}$$

We get

$$\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_m}_{\lambda_m})\tilde{\le}\{\tilde{d}^*(F^{x_1}_{\lambda_1},F^{x_0}_{\lambda_0})\}^{\frac{k^m}{1-k}}$$

This signifies that $\tilde{d}^*(F^{x_n}_{\lambda_n},F^{x_m}_{\lambda_m})\to\bar{1}$ as $n,m\to\infty$. Hence, $\{F^{x_n}_{\lambda_n}\}$ is a multiplicative soft Cauchy sequence. Being the completeness of $\tilde{U}$, there exists $F^{x^*}_{\lambda}\tilde{\in}\tilde{U}$ such that $F^{x_n}_{\lambda_n}\to F^{x^*}_{\lambda}$ as $n\to\infty$. Now,

$$\begin{aligned}\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda_n}),F^{x^*}_{\lambda_n}\} &\tilde{\le}\,\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),(T,\psi)(F^{x^*}_{\lambda_n})\}\cdot\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x^*}_{\lambda_n}\}\\ &\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\cdot\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda_n}),F^{x^*}_{\lambda_n}\}]^a\\ &\quad\cdot[\tilde{d}^*\{(T,\psi)(F^{x_n}_{\lambda_n}),F^{x_n}_{\lambda_n}\}\cdot\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda_n}),F^{x_n}_{\lambda_n}\}]^b\\ &\tilde{\le}[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x_n}_{\lambda_n})\cdot\tilde{d}^*(F^{x^*}_{\lambda_{n+1}},F^{x^*}_{\lambda_n})]^a\cdot[\tilde{d}^*(F^{x_{n+1}}_{\lambda_{n+1}},F^{x^*}_{\lambda_n}\cdot\tilde{d}^*(F^{x_{n+1}}_{\lambda_n},F^{x_n}_{\lambda_n})]^b\to\bar{1}\end{aligned}$$

This signifies that $(T,\psi)(F^{x^*}_{\lambda})=F^{x^*}_{\lambda}$ and hence $F^{x^*}_{\lambda}$ is a soft fixed point of the function (T,ψ). Again, if $F^{y^*}_{\mu}$ is some other soft fixed point of the function (T,ψ), then we have

$$\begin{aligned}\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu}) &=\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda}),(T,\psi)(F^{y^*}_{\mu})\}\\ &\tilde{\le}[\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda}),F^{x^*}_{\lambda}\}\cdot\tilde{d}^*\{(T,\psi)(F^{y^*}_{\mu}),F^{y^*}_{\mu}\}]^a\cdot[\tilde{d}^*\{(T,\psi)(F^{x^*}_{\lambda}),F^{y^*}_{\mu}\}\\ &\quad\cdot\tilde{d}^*\{(T,\psi)(F^{y^*}_{\mu}),F^{x^*}_{\lambda}\}]^b\\ &\tilde{\le}[\tilde{d}^*\{(F^{x^*}_{\lambda},F^{x^*}_{\lambda})\cdot\tilde{d}^*(F^{y^*}_{\mu},F^{y^*}_{\mu})]^a\cdot[\tilde{d}^*\{(F^{x^*}_{\lambda},F^{y^*}_{\mu})\cdot\tilde{d}^*(F^{y^*}_{\mu},F^{x^*}_{\lambda})]^b\\ &=\{\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})\}^{2b}\end{aligned}$$

$$\{\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})\}^{1-2b}\tilde{\le}\bar{1}$$

Since $a+b\tilde{<}\frac{1}{2}$ implies that $2b\tilde{<}\bar{1}$, therefore

$$\tilde{d}^*(F^{x^*}_{\lambda},F^{y^*}_{\mu})=\bar{1}\Rightarrow F^{x^*}_{\lambda}=F^{y^*}_{\mu}.$$

Thus, we obtain a unique soft fixed point of (T,ψ). □

Theorem 7.9. *Suppose $(\tilde{U},\tilde{d}^*,E)$ is a soft multiplicative metric space. If the function (T,ψ): $(\tilde{U},\tilde{d}^*,E)\to(\tilde{U},\tilde{d}^*,E)$ meets the following condition:*

$$\begin{aligned}\tilde{d}^*\{(T,\psi)(F^x_{\lambda}),(T,\psi)(F^y_{\mu})\}\tilde{\le}\{\tilde{d}^*(F^x_{\lambda},F^y_{\mu})\}^a[\tilde{d}^*\{(T,\psi)(F^x_{\lambda}),F^x_{\lambda}\}\cdot\tilde{d}^*\{(T,\psi)(F^y_{\mu}),F^y_{\mu}\}]^b\\ \cdot[\tilde{d}^*\{(T,\psi)(F^x_{\lambda}),F^y_{\mu}\}\cdot\tilde{d}^*\{(T,\psi)(F^y_{\mu}),F^x_{\lambda}\}]^c,\end{aligned}$$

where $a+2b+2c\tilde{<}\bar{1}$, *then we get a unique soft point* $F^x_\lambda\tilde{\in}SP(\tilde{U})$ *such that* $(T,\psi)(F^x_\lambda)=F^x_\lambda$.

Proof. The proof follows from Theorems 7.7 and 7.8. □

8. Conclusion and Future Scope

"Soft set theory" is a wide mathematical aid for handling vagueness and uncertainty. In this paper, some basic concepts of soft set and soft metric are considered and a new concept of soft multiplicative metric space is introduced. An attempt has been made to show the existence and uniqueness of fixed point theorems in context of soft multiplicative metric space.

Inspiring from the ideas presented in this paper, one can introduce the concept of generalized soft multiplicative metric space, Fuzzy soft multiplicative metric space and so on. An attempt can be made in the direction of fixed point in these spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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