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Research Article

Strong Convergence Results for Continuous Hemicontractive Mappings in Hilbert Spaces

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Abstract. We use an iteration process due to Rafiq (A. Rafiq, On Mann iteration in Hilbert spaces, *Nonlinear Analysis* **66** (2007), 2230 – 2236) to approximate fixed points of continuous hemicontractive mappings in Hilbert spaces. We drop the compactness condition on the domain of the operator, imposed in [1] and [26]. Our results extend several well known results in the literature and complement the results in [1] and [26].

Keywords. Hemicontractive mappings; Continuous mappings; Convergence; Fixed points; Hilbert spaces

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1. Introduction

Let H be a real Hilbert space and $T: H \to H$, be a self map of H. We denote by $F(T) := \{x \in H : Tx = x\}$, the set of fixed points of T. Then T is called:

(i) *Nonexpansive* (see e.g. [13]) if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in H.$$

(ii) *Pseudocontractive* (see e.g. [5]) if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \text{for all } x, y \in H.$$

(iii) *Hemicontractive* (see e.g. [1]) if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \le \|x - p\|^2 + \|x - Tx\|^2, \quad \text{for all } x \in H,\ p \in F(T). \tag{1.1}$$

It is easily seen that if a pseudocontractive mapping has a nonempty fixed point set, then it is a hemicontraction. Hence the class of pseudocontracion mappings with a nonempty fixed-point set is a subclass of the class of hemicontractive mappings. In [17], Rhoades shows that this inclusion is proper.

In the recent past, many authors (see e.g. [1,2], [4–16], [18–25]) have studied existence and convergence results of fixed points of nonexpansive mappings and their generalizations, amongst which are pseudocontractions, hemicontractions and asymptotically hemicontractions. In order to obtain the existence and convergence results, authors (see e.g. [1]) have placed compactness, compactness-type and several other conditions on the domain of the operator or on the operator itself.

The construction of fixed points of nonexpansive mappings and their generalizations is achieved through iterative search techniques amonst which are the Mann, Mann-type, Ishikwa and Ishikawa-type schemes. Let X be a linear space and $T : X \to X$ be a map. The Mann iteration scheme (see e.g. [17]) is the sequence generated from an arbitrary $x_0 \in X$ by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.2}$$

where $\{\alpha_n\}$ is a real nonnegative sequence satisfying certain conditions.

In 2007, Rafiq [1] studied the convergence to fixed points of hemicontractive mappings in Hilbert spaces using a Mann-type iteration scheme generated from an arbitrary $x_0 \in H$ by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n$, where $\{\alpha_n\}$ is a real sequence in $[0,1]$, satisfying certain conditions. More precisely, the author stated and proved the following theorems:

Theorem 1 ([1]). *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ be a hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 1 ([1]). *Let H,K T be as in Theorem 1 and $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $P_K : H \to K$ be the projection operator of H onto K. Then the sequence $x_n = P_K(\alpha_n x_{n-1} + (1-\alpha_n)Tx_n)$, $n \geq 1$ converges strongly to a fixed point of T.*

We observe that the compactness condition imposed on the subset K is rather strong. It is our purpose in this paper to prove convergence results for continuous hemicontractive mappings in Hilbert spaces, using the iteration process due to Rafiq, without imposing the condition that K be compact. Furthermore, we show that if error terms are added as in [26], our results still hold, without any compactness assumption on the subset K. Our results generalize many well known results in the literature and compliment the results of Rafiq [1].

Example 1. Let R denote the reals with the usual norm. Define $T : R \to R$ by $Tx = -2x$. Observe that

$$\langle x - Tx - (y - Ty), x - y\rangle = 3|x-y|^2 \geq 0.$$

Thus, T is pseudocontractive. Since $\emptyset \neq F(T) = \{0\}$, then T is hemicontractive. It is easily seen that T is continuous.

Before we state and prove our main results, we give some lemmas which will be useful in the sequel:

Lemma 1 (see e.g. [14])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad \text{for all } n \ge 1.$$

If $\sum \delta_n < \infty$ and $\sum b_n < \infty$, then $\lim a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim a_n = 0$.

Lemma 2 (see e.g. [3, 11])**.** *Let H be a real Hilbert space. Then for all $x, y \in H$, and $\lambda \in [0,1]$ the following well-known identity holds:*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{1.3}$$

2. Main Results

We now state and prove our main results.

Lemma 3. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $T : C \subseteq H \to C$ be a hemicontractive mapping. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n, \tag{2.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying $0 < a \le \alpha_n \le b < 1$ for some real constants $a, b \in (0,1)$.
Then

(a) $\lim \|x_n - p\|$ *exists, where $p \in F(T) := \{x \in C : Tx = x\}$*

(b) $\lim \|x_n - Tx_n\| = 0$

Proof. Computing as in [1], using (1.3), (2.1) and the fact that T is hemicontractive, we have, for $p \in F(T)$

$$\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_n(x_{n-1}-p) + (1-\alpha_n)(Tx_n - p)\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - Tx_n\|^2 \\
&\le \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 + \|x_n - Tx_n\|^2] - \alpha_n(1-\alpha_n)\|x_{n-1} - Tx_n\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + {\alpha_n}^2(1-\alpha_n)\|x_{n-1} - Tx_n\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - Tx_n\|^2 \\
&= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)^2\|x_{n-1} - Tx_n\|^2.
\end{aligned}$$

This implies

$$\|x_n - p\|^2 \le \|x_{n-1}-p\|^2 - (1-\alpha_n)^2\|x_{n-1} - Tx_n\|^2. \tag{2.2}$$

Hence $\{\|x_n - p\|\}$ is a monotone deceasing sequence of positive real numbers which is bounded below, so that $\lim \|x_n - p\|$ exists.

From (2.2) and the condition $0 < a \le \alpha_n \le b < 1$, we have

$$\|x_n - p\|^2 \le \|x_{n-1}-p\|^2 - (1-b)^2\|x_{n-1} - Tx_n\|^2.$$

This implies

$$\begin{aligned}\sum(1-b)^2\|x_{n-1}-Tx_n\|^2 &\le \sum[\|x_{n-1}-p\|^2-\|x_n-p\|^2]\\ &\le \|x_0-p\|^2.\end{aligned}$$

Hence

$$\lim\|x_{n-1}-Tx_n\|=0. \tag{2.3}$$

Also,

$$\|x_n-Tx_n\|=\alpha_n\|x_{n-1}-Tx_n\|\le\|x_{n-1}-Tx_n\|\to 0 \text{ as } n\to\infty.$$ □

Remark 1. Observe that if $T:C\to C$ is a hemicontractive map, then for every fixed $u\in C$ and $t\in(0,1)$, the operator $S_t:C\to C$ defined for all $x\in C$ by

$$S_tx=tu+(1-t)Tx$$

satisfies

$$\|S_tx-S_ty\|\le(1-t)\|x-y\|,\quad \text{for all } x,y\in C.$$

Since $t\in(0,1)$, it follows that S_t is a *contraction map* and hence has a unique fixed point x_t in C. This implies that there exists a unique $x_t\in C$ such that

$$x_t=tu+(1-t)Tx_t.$$

Thus the implicit iteration process (2.1) is defined in C.

Theorem 2. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H and let $T:C\subseteq H\to C$ be a continuous hemicontractive mapping. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0\in C$ by $x_n=\alpha_nx_{n-1}+(1-\alpha_n)Tx_n$, where $\{\alpha_n\}$ is a real sequences in $(0,1)$ satisfying $0<a\le\alpha_n\le b<1$ for some real constants $a,b\in(0,1)$, converges strongly to a fixed point of T.*

Proof. Using (2.1) and (2.3), we have

$$\|x_n-x_{n-1}\|=(1-\alpha_n)\|x_{n-1}-Tx_n\|\le\|x_{n-1}-Tx_n\|\to 0 \text{ as } n\to\infty. \tag{2.4}$$

Now, using (2.1), (2.3), (2.4) and for any positive integers n and m with $m>n$, we have

$$\begin{aligned}\|x_n-x_m\| &= \|\alpha_n(x_{n-1}-x_m)+(1-\alpha_n)(Tx_n-x_m)\|\\ &\le \alpha_n\|x_{n-1}-x_m\|+(1-\alpha_n)\|Tx_n-x_m\|\\ &\le \alpha_n\|x_{n-1}-x_m\|+(1-\alpha_n)[\|Tx_n-x_{n-1}\|+\|x_{n-1}-x_m)\|]\\ &\le \|x_{n-1}-x_m\|+\|Tx_n-x_{n-1}\|\\ &\le \|x_{n-1}-x_n\|+\|x_n-x_{n+1}\|+\ldots+\|x_{m-1}-x_m\|+\|Tx_n-x_{n-1}\| \ \to 0 \text{ as } n,m\to\infty.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in C and thus $x_n\to z\in C$. Since T is continuous, we have $Tx_n\to Tz$. From Lemma 3(b), we have $0=\lim\|x_n-Tx_n\|=\|z-Tz\|$. This implies $z\in F(T)$. Setting $z=p$ in Lemma 3(a), our proof is complete. □

Corollary 2. *Let H, C and $\{\alpha_n\}$ be as in Theorem 2. Let $T:C\to C$ be a continuous pseudo-contractive mapping with a non-empty fixed point set. Then starting from an arbitrary $x_0\in C$, the sequence $\{x_n\}$ generated by (2.1) converges strongly to a fixed point of T.*

Proof. Every pseudocontractive mapping with a non empty fixed point set is a hemicontraction. Hence the proof follows from the proof of Theorem 2 above. □

Corollary 3. *Let H, C, T and $\{\alpha_n\}$ be as in Theorem 2. Let $P_C : H \to C$ be the projection operator of H onto C. Then the sequence $\{x_n\}$ defined iteratively by $x_n = P_C(\alpha_n x_{n-1} + (1-\alpha_n)Tx_n)$, $n \geq 1$, converges strongly to a fixed point of T.*

Proof. Using the fact that P_C is nonexpansive, the computations and analyses follow as in the proof of Theorem 2 above. This completes our proof. □

Remark 2. If error terms are added to (2.1) (as in [26]), the computations and analyses still follow through. We simply impose the boundedness condition on the subset C in order to obtain our results.

More precisely, the author in [26] stated and proved the following theorem:

Theorem 3 ([26])**.** *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and satisfying*

(a) $b_n \subset [\delta, 1-\delta]$ *for some* $\delta \in \left(0, \frac{1}{2}\right)$

(b) $\sum c_n < \infty$

For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by

$$x_n = a_n x_{n-1} + b_n T x_n + c_n u_n,$$

where $\{u_n\}$ is an arbitrary sequence in K. Then $\{x_n\}$ converges strongly to a fixed point of T.

We can drop the compactness assumption imposed on K as in the following theorem:

Theorem 4. *Let C be bounded, closed and convex subset of a real Hilbert space H; $T : C \to C$ be a continuous hemicontractive map. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and satisfying*

(a) $b_n \subset [\delta, 1-\delta]$ *for some* $\delta \in \left(0, \frac{1}{2}\right)$

(b) $\sum c_n < \infty$

For arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by

$$x_n = a_n x_{n-1} + b_n T x_n + c_n u_n, \tag{2.5}$$

where $\{u_n\}$ is an arbitrary sequence in C. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. Computing as in [26], we obtain the following:

(i) $\lim \|x_{n-1} - Tx_n\| = 0$

(ii) $\lim \|x_n - Tx_n\| = 0$

(iii) $\lim \|x_n - p\|$ exists (using [26, Lemma 1 and eq. (3.6)]).

Furthermore, letting $M = diam(C)$, using (i) and hypothesis (b) of the theorem, we have

$$\begin{aligned}\|x_n - x_{n-1}\| = \|b_n(Tx_n - x_{n-1}) + c_n(u_n - x_{n-1})\| &\leq b_n\|Tx_n - x_{n-1}\| + c_n\|u_n - x_{n-1}\| \\ &\leq \|Tx_n - x_{n-1}\| + Mc_n \ \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence,

$$\lim \|x_n - x_{n-1}\| = 0. \tag{2.6}$$

Next, using (i), (2.5), (2.6) and hypothesis (b) of the theorem and for any two positive integers m and n with $m > n$, we have

$$\begin{aligned}
\|x_n - x_m\| &= \|a_n(x_{n-1} - x_m) + b_n(Tx_n - x_m) + c_n(u_n - x_m)\| \\
&\le a_n\|x_{n-1} - x_m\| + b_n\|Tx_n - x_m\| + c_n\|u_n - x_m\| \\
&= a_n\|x_{n-1} - x_m\| + [1 - a_n - c_n]\|Tx_n - x_m\| + c_n\|u_n - x_m\| \\
&\le a_n\|x_{n-1} - x_m\| + [1 - a_n]\|Tx_n - x_m\| + Mc_n \\
&\le a_n\|x_{n-1} - x_m\| + [1 - a_n][\|Tx_n - x_{n-1}\| + \|x_{n-1} - x_m\|] + Mc_n \\
&\le \|x_{n-1} - x_m\| + \|Tx_n - x_{n-1}\| + Mc_n \\
&\le \|x_{n-1} - x_n\| + \|x_n - x_{n+1}\| + \ldots + \|x_{m-1} - x_m\| + \|Tx_n - x_{n-1}\| + Mc_n \ \to\ 0 \text{ as } n, m \to \infty.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in C and thus $x_n \to z \in C$. Since T is continuous, we have $Tx_n \to Tz$. From (ii), we have $0 = \lim \|x_n - Tx_n\| = \|z - Tz\|$. This implies $z \in F(T)$. Setting $z = p$ in (iii), our proof is complete. □

3. Further Research

It would be of interest if the results can be proven in arbitrary Banach spaces. Furthermore, relaxing the continuity condition placed on the operators would also be interesting.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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