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k-Tuple Total Domination in Supergeneralized Petersen Graphs

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Abstract. Total domination number of a graph without isolated vertex is the minimum cardinality of a total dominating set, that is, a set of vertices such that every vertex of the graph is adjacent to at least one vertex of the set. Henning and Kazemi in [4] extended this definition as follows: for any positive integer k, and any graph G with minimum degree-k, a set D of vertices is a k-tuple total dominating set of G if each vertex of G is adjacent to at least k vertices in D. The k-tuple total domination number $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a k-tuple total dominating set of G. In this paper, we give some upper bounds for the k-tuple total domination number of the supergeneralized Petersen graphs. Also we calculate the exact amount of this number for some of them.

1. Introduction

Let $G=(V,E)$ be a graph with vertex set V of order n and edge set E. A cycle on n vertices is denoted by C_n. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$) or briefly by δ (resp., Δ).

The *cartesian product* $G\Box H$ of two graphs G and H is a graph with the vertex set $V(G\Box H)=\{(v,w)\mid v\in V(G), w\in V(H)\}$, and two vertices (v,w) and (v',w') are adjacent together in $G\Box H$ if either $w=w'$ and $vv'\in E(G)$ or $v=v'$ and $ww'\in E(H)$.

In [6], Saražin et al defined the supergeneralized Petersen graph that is an extending of generalized Petersen graph as follows: let $m\geq 2$, $n\geq 3$ be integers and $l_0,l_1,\dots,l_{m-1}\in Z_n-\{0\}$, where $Z_n=\{0,1,2,\dots,n-1\}$. The vertex set of the graph $P(m,n,l_0,l_1,\dots,l_{m-1})$ is $Z_m\times Z_n$ and the edges are defined by $(i,j)(i+1,j)$, $(i,j)(i,j+l_i)$, for all $i\in Z_m$ and $j\in Z_n$. The edges of type $(i,j)(i+1,j)$ will be called *horizontal*, while those of type $(i,j)(i,j+l_i)$ *vertical*. We will call such a graph *supergeneralized Petersen graph* (SGPG). Note that $\Delta(P(m,n,l_0,l_1,\dots,l_{m-1}))\leq 4$, and $P(m,n;1,\dots,1)$ is the Cartesian product $C_m\Box C_n$ of two cycles; in particular, the skeleton of the 4-dimensional hypercube is $Q_4=C_4\Box C_4=P(4,4;1,1,1,1)$.

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To avoid confusion we will pick $0 \leq l_i \leq \frac{n}{2}$ as a representative of $\{\pm l_i\}$. If n is odd, then $P = P(m,n;l_0,\ldots,l_{m-1})$ is obviously 4-*regular*, with the exception $m = 2$, when the graph is cubic. The same holds when n is even, if $l_i = n/2$, for every $i \in Z_m$. On the contrary, if $l_i = n/2$ for some i, then P is not regular, unless $l_i = n/2$ for all $i \in Z_n$. In this last case the graph P is not connected: it is formed by $n/2$ components each isomorphic to $C_m \Box K_2$.

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2], [3]. A set $S \subseteq V$ is a *dominating set* if each vertex in $V \setminus S$ is adjacent to at least one vertex of S, and the minimum cardinality of a dominating set is the *domination number* of G and denoted by $\gamma(G)$. If in the definition of dominating set we replace V with $V \setminus S$, we obtain a *total dominating set* and similarly the minimum cardinality of a total dominating set is the *total domination number* of G and denoted by $\gamma_t(G)$.

In [4], Henning and Kazemi initiated a study of k-tuple total domination in graphs. A subset S of V is a *k-tuple total dominating set* of G, abbreviated kTDS, if every vertex $v \in V$ has at least k neighbors in S. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ is the minimum cardinality of a kTDS of G. We remark that $\gamma_t(G) = \gamma_{\times 1,t}(G)$ and $\gamma_{\times 2,t}(G)$ is the same *double total domination number*. Obviously $\gamma_{\times k,t}(G) \leq \gamma_{\times (k+1),t}(G)$ for all graphs G with $\delta(G) \geq k+1$.

In this paper, we give some upper bounds for the k-tuple total domination number of the supergeneralized Petersen graphs $P(m,n,l_0,l_1,\ldots,l_{m-1})$, when $l_0 = l_1 = \ldots = l_{m-1}$, and $1 \leq k \leq 4$. Also we calculate the exact amount of this number for some of them.

2. Some Bounds

We begin with the following trivial observation about the k-tuple total domination number of a graph. The proof follow readily from the definitions and is omitted.

Observation 2.1. Let G be a graph of order n with $\delta(G) \geq k$, and let S be a kTDS in G. Then

(i) $k+1 \leq \gamma_{\times k,t}(G) \leq n$,
(ii) if G is a spanning subgraph of a graph H, then $\gamma_{\times k,t}(H) \leq \gamma_{\times k,t}(G)$,
(iii) if v is a degree-k vertex in G, then $N_G(v) \subseteq S$.

Observation 2.1(iii) concludes that $\gamma_{\times k,t}(G) = nm$, when $G = P(m,n;l_0,\ldots,l_{m-1})$ is 4-regular and $k = 4$. Therefore in continuance, we consider $1 \leq k \leq 3$. Now we give a lower bound that we stated it in [5].

Lemma 2.2. *If G is a graph of order n with $\delta(G) \geq k$, then*

$$\gamma_{\times k,t}(G) \geq \lceil kn/\Delta \rceil.$$

Proof. Let $G = (V, E)$. Let S be a $\gamma_{\times k,t}(G)$-set. Each vertex $v \in S$ is adjacent to at least k vertices in S and therefore to at most $d_G(v) - k$ vertices outside S. Hence,

$$|[S, V \setminus S]| \leq \sum_{v \in S}(d_G(v) - k) = \sum_{v \in S} d_G(v) - k|S| \leq |S|(\Delta - k).$$

Since each vertex in $V \setminus S$ is adjacent to at least k vertices in S, we note that

$$|[S, V \setminus S]| \geq k|V \setminus S| = k(n - |S|).$$

Thus, $|S|(\Delta - k) \geq k(n - |S|)$, whence $\gamma_{\times k,t}(G) = |S| \geq kn/\Delta$. □

Remark 1. Let $P(m, n; l, l, \ldots, l)$ be a SGPG. Let $n \equiv r \pmod{4l}$. If $n \equiv 0 \pmod{4l}$, then $P(m, n, l, l, \ldots, l)$ is formed by l components each isomorphic to $P\left(m, \frac{n}{l}; 1, 1, \ldots, 1\right)$ with the vertex set $\{(i, tl + j) \mid 0 \leq t \leq \frac{n}{l} - 1, 0 \leq i \leq m - 1\}$, and so

$$\gamma_{\times k,t}(P(m, n; l, l, \ldots, l)) = l\gamma_{\times k,t}\left(P\left(m, \frac{n}{l}; 1, 1, \ldots, 1\right)\right).$$

Also if $n \not\equiv 0 \pmod{4l}$, then G is formed by l' components each isomorphic to $P(m, \frac{n}{l'}; 1, 1, \ldots, 1)$ with the vertex set $\{(i, tl^{'} + j) \mid 0 \leq t \leq \frac{n}{l'} - 1, 0 \leq i \leq m - 1\}$, when l' is the greatest common divisor between n and l, and so

$$\gamma_{\times k,t}(P(m, n; l, l, \ldots, l)) = l'\gamma_{\times k,t}\left(P\left(m, \frac{n}{l'}, 1, 1, \ldots, 1\right)\right).$$

Therefore for calculating the k-tuple total domination number of $P(m, n; l, l, \ldots, l)$ it suffices to calculate the k-tuple total domination number of $P(m, n; 1, 1, \ldots, 1) = C_m \square C_n$, for every integers m and n. The next three propositions give upper bounds for the k-tuple total domination number of $C_m \square C_n$, where m and n are two arbitrary integers at least 4. Gravier in [1] gave the upper bound $\frac{(m+2)(n+2)}{4}$ for the total domination number of $C_m \square C_n$. Next proposition improves this upper bound.

Proposition 2.3. *Let $G = C_m \square C_n$, where $m \geq n \geq 4$, and let s and r be non-negative integers such that $m \equiv s \pmod 4$ and $n \equiv r \pmod 4$. Then*

$$\gamma_t(G) \leq \begin{cases} \dfrac{m(n+1)}{4} & \text{if } (s, r) = (0, 3), \\ \dfrac{m(n+r)}{4} & \text{if } (s, r) \in \{0\} \times \{0, 1, 2\}, \\ \dfrac{(m-s)(n+1)}{4} + \dfrac{n-1}{2} & \text{if } (s, r) \in \{1, 2\} \times \{1\}, \\ \dfrac{(m-3)(n+1)}{4} + \dfrac{3(n-1)}{4} + 1 & \text{if } (s, r) = (3, 1), \\ \dfrac{(m-2)(n+1)}{4} + n - 1 & \text{if } (s, r) = (2, 2), \\ \dfrac{(m+1)(n+1)}{4} & \text{if } (s, r) \in \{3\} \times \{2, 3\}. \end{cases}$$

Proof. Let

$$S_0 = \Big\{(4i,4j),(4i+3,4j),(4i+2,4j+2),(4i+1,4j+2)$$
$$\mid 0 \le i \le \frac{m-s}{4}-1, 0 \le j \le \frac{n-r}{4}-1\Big\},$$
$$S_1 = \Big\{(4i,n-1),(4i+3,n-1) \mid 0 \le i \le \frac{m-s}{4}-1\Big\},$$
$$S_2 = \Big\{(4i,n-2),(4i+3,n-2),(4i+1,n-1),(4i+2,n-1) \mid 0 \le i \le \frac{m-s}{4}-1\Big\},$$
$$S_3 = \Big\{(4i,n-3),(4i+3,n-3),(4i+1,n-1),(4i+2,n-1) \mid 0 \le i \le \frac{m-s}{4}-1\Big\},$$
$$S_{1,1} = \Big\{(m-1,4j+2),(m-1,4j+3) \mid 0 \le j \le \frac{n-1}{4}-1\Big\},$$
$$S_{2,1} = \Big\{(m-1,4j+2),(m-2,4j+2) \mid 0 \le j \le \frac{n-1}{4}-1\Big\},$$
$$S_{3,1} = \Big\{(m-1,4j),(m-2,4j+2),(m-3,4j+2) \mid$$
$$0 \le j \le \frac{n-1}{4}-1\} \cup \{(m-1,n-1)\Big\},$$
$$S_{2,2} = \Big\{(m-1,4j),(m-1,4j+3),(m-2,4j),$$
$$(m-2,4j+3) \mid 0 \le j \le \frac{n-2}{4}-1\Big\} \cup \{(m-2,n-2)\},$$
$$S_{3,2} = \Big\{(m-1,4j),(m-1,4j+2),(m-2,4j+2),(m-3,4j) \mid 0 \le j \le \frac{n-2}{4}-1\Big\}$$
$$\cup \{(m-1,n-1),(m-1,n-2),(m-3,n-2)\},$$
$$S_{3,3} = \Big\{(m-1,4j),(m-1,4j+2),(m-2,4j+2),(m-3,4j) \mid 0 \le j \le \frac{n-3}{4}-1\Big\}$$
$$\cup \{(m-1,n-1),(m-1,n-3),(m-2,n-1),(m-3,n-3)\}.$$

Since obviously $S_0 \cup S_r \cup S_{s,r}$ when $0 < r \le s \le 3$, and $S_0 \cup S_r$ when $0 \le r \le 3$ and $s = 0$, are total dominating sets of G with the wanted cardinality, then our proof will be completed. □

Proposition 2.4. *Let $G = C_m \square C_n$, where $m \ge n \ge 4$, and let s and r be non-negative integers such that $m \equiv s \pmod 4$ and $n \equiv r \pmod 4$. Then*

$$\gamma_{\times 2,t}(G) \leq \begin{cases} \dfrac{mn}{2} & \textit{if } (s,r) = (0,0) \\ \dfrac{m(n+1)}{2} & \textit{if } (s,r) \in \{0\} \times \{1,2,3\}, \\ \dfrac{(m-1)(n+1)}{2} + n - 1 & \textit{if } (s,r) = (1,1), \\ \dfrac{(m-2)(n+1)}{2} + \dfrac{3(n-r)}{2} & \textit{if } (s,r) \in \{2\} \times \{1,2\}, \\ \dfrac{(m-3)(n+1)}{2} + 2n - 2 & \textit{if } (s,r) = (3,1), \\ \dfrac{(m-3)(n+1)}{2} + \dfrac{5n-6}{2} & \textit{if } (s,r) = (3,2), \\ \dfrac{(m-3)(n+1)}{2} + \dfrac{9n-7}{4} & \textit{if } (s,r) = (3,3). \end{cases}$$

Proof. Let

$$S_0 = \Big\{(4i,4j),(4i,4j+3),(4i+1,4j+1),(4i+1,4j+2),$$
$$(4i+2,4j+1),(4i+2,4j+2),(4i+3,4j),$$
$$(4i+3,4j+3) \mid 0 \leq i \leq \frac{m-s}{4} - 1, 0 \leq j \leq \frac{n-r}{4} - 1\Big\},$$

$$S_1 = \{(i,n-1) \mid 0 \leq i \leq m-s-1\},$$

$$S_2 = \Big\{(4i,n-2),(4i+3,n-2) \mid 0 \leq i \leq \frac{m-s}{4} - 1\Big\}$$
$$\cup \{(i,n-1) \mid 0 \leq i \leq m-s-1\},$$

$$S_3 = \Big\{(4i,n-3),(4i+3,n-3),(4i+1,n-2),(4i+2,n-2) \mid 0 \leq i \leq \frac{m-s}{4} - 1\Big\}$$
$$\cup \{(i,n-1) \mid 0 \leq i \leq m-s-1\},$$

$$S_{1,1} = \{(m-1,j) \mid 0 \leq j \leq n-2\},$$

$$S_{2,1} = \{(m-1,j) \mid 0 \leq j \leq n-2\} \cup \Big\{(m-2,4j),(m-2,4j+3) \mid 0 \leq j \leq \frac{n-1}{4} - 1\Big\},$$

$$S_{3,1} = \{(m-2,j) \mid 0 \leq j \leq n-2\}$$
$$\cup \Big\{(i,4j),(i,4j+3) \mid 0 \leq j \leq \frac{n-1}{4} - 1, i \in \{m-3,m-1\}\Big\},$$

$$S_{2,2} = \{(m-1,j) \mid 0 \leq j \leq n-3\}$$
$$\cup \Big\{(m-2,4j),(m-2,4j+3) \mid 0 \leq j \leq \frac{n-2}{4} - 1\Big\},$$

$$S_{3,2} = \left\{(m-3,4j),(m-3,4j+3) \mid 0 \le j \le \frac{n-2}{4}-1\right\}$$
$$\cup \{(i,j) \mid m-2 \le i \le m-1, 0 \le j \le n-3\}$$
$$\cup \{(m-2,n-2),(m-2,n-1)\},$$
$$S_{3,3} = \left\{(m-3,4j),(m-3,4j+3) \mid 0 \le j \le \frac{n-3}{4}-1\right\} \cup$$
$$\{(m-3,n-3)\} \cup \{(m-2,j) \mid 0 \le j \le n-1\}$$
$$\cup \{(m-1,j) \mid 0 \le j \le n-1\} - \{(m-1,n-2),(m-1,n-1)\}$$
$$\cup \left\{(m-2,4j+3) \mid 0 \le j \le \frac{n-3}{4}-1\right\}.$$

Since obviously $S_0 \cup S_r \cup S_{s,r}$ when $0 < r \le s \le 3$, and $S_0 \cup S_r$ when $0 \le r \le 3$ and $s = 0$, are total dominating sets of G with the wanted cardinality, then our proof will be completed. □

Proposition 2.5. *Let $G = C_m \,\square\, C_n$, where $m \ge n \ge 4$, and let s and r be non-negative integers such that $m \equiv s \pmod 4$ and $n \equiv r \pmod 4$. Then*

$$\gamma_{\times 3,t}(G) \le \begin{cases} \dfrac{m(3n+1)}{4} & \text{if } (s,r) = (0,3), \\ \dfrac{m(3n+r)}{4} & \text{if } (s,r) \in \{0\} \times \{0,1,2\}, \\ \dfrac{(m-1)(3n+1)}{4} + \dfrac{5(n-1)}{4} & \text{if } (s,r) = (1,1), \\ \dfrac{(m-s)(3n+1)}{4} + sn - s & \text{if } (s,r) \in \{2,3\} \times \{1\}, \\ \dfrac{(m-2)(3n+1)}{4} + 2n & \text{if } (s,r) = (2,2), \\ \dfrac{(m-3)(3n+1)}{4} - \dfrac{n+6}{4} + 3n & \text{if } (s,r) = (3,2), \\ \dfrac{(m-3)(3n+1)}{4} - \dfrac{n+6}{4} + 3n & \text{if } (s,r) = (3,3), \end{cases}$$

Proof. Let

$$S_0 = \left\{(4i,4j),(4i+3,4j),(4i+2,4j+2),(4i+1,4j+2)\right.$$
$$\left. \mid 0 \le i \le \frac{m-s}{4}-1, 0 \le j \le \frac{n-r}{4}-1\right\} \cup$$
$$\left\{(i,2j+1) \mid 0 \le j \le \frac{n-r}{2}-1, 0 \le i \le m-s-1\right\},$$

$$S_1 = \{(i, n-1) \mid 0 \le i \le m-s-1\},$$

$$S_2 = \{(i, n-1), (i, n-2) \mid 0 \le i \le m-s-1\},$$

$$S_3 = \{(i, n-1), (i, n-2) \mid 0 \le i \le m-s-1\} \cup \left\{(4i, n-3), (4i+3, n-3) \mid 0 \le i \le \frac{m-s}{4} - 1\right\},$$

$$S_{1,1} = \left\{(m-1, j) \mid 0 \le j \le n-2\} \cup \{(m-2, 4j+2) \mid 0 \le j \le \frac{n-1}{4} - 1\right\},$$

$$S_{2,1} = \{(m-1, j), (m-2, j) \mid 0 \le j \le n-2\},$$

$$S_{3,1} = \{(m-1, j), (m-2, j), (m-3, j) \mid 0 \le j \le n-2\},$$

$$S_{2,2} = \{(m-1, j), (m-2, j) \mid 0 \le j \le n-1\},$$

$$S_{3,2} = \{(m-1, j), (m-2, j), (m-3, j) \mid 0 \le j \le n-1\} - \left\{(m-2, 4j+1) \mid 0 \le j \le \frac{n-2}{4} - 1\right\} \cup \{(m-3, n-2), (m-2, n-2)\},$$

$$S_{3,3} = \{(m-1, j), (m-2, j), (m-3, j) \mid 0 \le j \le n-1\} - \left\{(m-2, 4j+1) \mid 0 \le j \le \frac{n-3}{4} - 1\right\} \cup \{(m-3, n-3), (m-2, n-3)\}.$$

Since obviously $S_0 \cup S_r \cup S_{s,r}$ when $0 < r \le s \le 3$, and $S_0 \cup S_r$ when $0 \le r \le 3$ and $s = 0$, are total dominating sets of G with the wanted cardinality, then our proof will be completed. □

By the last three propositions and the previous Remark, we can conclude the next three results.

Theorem 2.6. *Let $G = P(m, n; l, l, \dots l)$ be a SGPG, where $m \ge n \ge 4$, and let s and r be non-negative integers such that $m \equiv s \pmod 4$ and $n \equiv 0 \pmod{4l}$. Then*

$$\gamma_t(G) \le \begin{cases} \dfrac{m(n+l)}{4} & \text{if } (s,r) = (0,3), \\ \dfrac{m(n+rl)}{4} & \text{if } (s,r) \in \{0\} \times \{0,1,2\}, \\ \dfrac{(m-s)(n+l)}{4} + \dfrac{n-l}{2} & \text{if } (s,r) \in \{1,2\} \times \{1\}, \\ \dfrac{(m-3)(n+l)}{4} + l & \text{if } (s,r) = (3,1), \\ \dfrac{(m-2)(n+l)}{4} + n - l & \text{if } (s,r) = (2,2), \\ \dfrac{(m+1)(n+l)}{4} & \text{if } (s,r) \in \{3\} \times \{2,3\}. \end{cases}$$

and if $n \not\equiv 0 \pmod{4l}$*, such that* l' *is the greatest common divisor between* n *and* l*, then*

$$\gamma_t(G) \leq \begin{cases} \frac{m(n+l')}{4} & \text{if } (s,r) = (0,3), \\ \frac{m(n+rl')}{4} & \text{if } (s,r) \in \{0\} \times \{0,1,2\}, \\ \frac{(m-s)(n+l')}{4} + \frac{n-l'}{2} & \text{if } (s,r) \in \{1,2\} \times \{1\}, \\ \frac{(m-3)(n+l')}{4} + l' & \text{if } (s,r) = (3,1), \\ \frac{(m-2)(n+l')}{4} + n - l' & \text{if } (s,r) = (2,2), \\ \frac{(m+1)(n+l')}{4} & \text{if } (s,r) \in \{3\} \times \{2,3\}. \end{cases}$$

Theorem 2.7. *Let* $G = P(m,n;l,l,\dots l)$ *be a SGPG, where* $m \geq n \geq 4$*, and let* s *and* r *be non-negative integers such that* $m \equiv s \pmod 4$ *and* $n \equiv 0 \pmod{4l}$*. Then*

$$\gamma_{\times 2,t}(G) \leq \begin{cases} \frac{mn}{2} & \text{if } (s,r) = (0,0), \\ \frac{m(n+l)}{2} & \text{if } (s,r) \in \{0\} \times \{1,2,3\}, \\ \frac{(m-1)(n+l)}{2} + n - l & \text{if } (s,r) = (1,1), \\ \frac{(m-2)(n+l)}{2} + \frac{3(n-rl)}{2} & \text{if } (s,r) \in \{2\} \times \{1,2\}, \\ \frac{(m-3)(n+l)}{2} + 2n - 2l & \text{if } (s,r) = (3,1), \\ \frac{(m-3)(n+l)}{2} + \frac{5n-6l}{2} & \text{if } (s,r) = (3,2), \\ \frac{(m-3)(n+l)}{2} + \frac{9n-7l}{4} & \text{if } (s,r) = (3,3). \end{cases}$$

and if $n \not\equiv 0 \pmod{4l}$*, such that* l' *is the greatest common divisor between* n *and* l*, then*

$$\gamma_{\times 2,t}(G) \leq \begin{cases} \frac{mn}{2} & \text{if } (s,r) = (0,0), \\ \frac{m(n+l')}{2} & \text{if } (s,r) \in \{0\} \times \{1,2,3\}, \\ \frac{(m-1)(n+l')}{2} + n - l' & \text{if } (s,r) = (1,1), \\ \frac{(m-2)(n+l')}{2} + \frac{3(n-rl')}{2} & \text{if } (s,r) \in \{2\} \times \{1,2\}, \\ \frac{(m-3)(n+l')}{2} + 2n - 2l' & \text{if } (s,r) = (3,1), \\ \frac{(m-3)(n+l')}{2} + \frac{5n-6l'}{2} & \text{if } (s,r) = (3,2), \\ \frac{(m-3)(n+l')}{2} + \frac{9n-7l'}{4} & \text{if } (s,r) = (3,3). \end{cases}$$

Theorem 2.8. *Let $G = P(m,n;l,l,\dots l)$ be a SGPG, where $m \geq n \geq 4$, and let s and r be non-negative integers such that $m \equiv s \pmod 4$ and $n \equiv 0 \pmod{4l}$. Then*

$$\gamma_{\times 3,t}(G) \leq \begin{cases} \dfrac{m(3n+l)}{4} & \text{if } (s,r) = (0,3), \\ \dfrac{m(3n+rl)}{4} & \text{if } (s,r) \in \{0\} \times \{0,1,2\}, \\ \dfrac{(m-1)(3n+l)}{4} + \dfrac{5n-5l}{4} & \text{if } (s,r) = (1,1), \\ \dfrac{(m-s)(3n+l)}{4} + sn - sl & \text{if } (s,r) \in \{2,3\} \times \{1\}, \\ \dfrac{(m-2)(3n+l)}{4} + 2n & \text{if } (s,r) = (2,2), \\ \dfrac{(m-3)(3n+l)}{4} - \dfrac{n+6l}{4} + 3n & \text{if } (s,r) = (3,2), \\ \dfrac{(m-3)(3n+l)}{4} - \dfrac{n+6l}{4} + 3n & \text{if } (s,r) = (3,3). \end{cases}$$

and if $n \not\equiv 0 \pmod{4l}$, such that l' is the greatest common divisor between n and l, then

$$\gamma_{\times 3,t}(G) \leq \begin{cases} \dfrac{m(3n+l')}{4} & \text{if } (s,r) = (0,3), \\ \dfrac{m(3n+rl')}{4} & \text{if } (s,r) \in \{0\} \times \{0,1,2\}, \\ \dfrac{(m-1)(3n+l')}{4} + \dfrac{5n-5l'}{4} & \text{if } (s,r) = (1,1), \\ \dfrac{(m-s)(3n+l')}{4} + sn - sl' & \text{if } (s,r) \in \{2,3\} \times \{1\}, \\ \dfrac{(m-2)(3n+l')}{4} + 2n & \text{if } (s,r) = (2,2), \\ \dfrac{(m-3)(3n+l')}{4} - \dfrac{n+6l'}{4} + 3n & \text{if } (s,r) = (3,2), \\ \dfrac{(m-3)(3n+l')}{4} - \dfrac{n+6l'}{4} + 3n & \text{if } (s,r) = (3,3). \end{cases}$$

3. **Sharp Bounds**

By Lemma 2.2 and the last three theorems and the previous Remark, we can conclude the next two results.

Theorem 3.1. *Let $1 \leq k \leq 4$. Let $G = P(m,n;l,l,\dots l)$ be a SGPG, where $m \geq n \geq 4$, and let s and r be non-negative integers such that $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod{4l}$. Then*

$$\gamma_{\times k,t}(G) = \frac{kmn}{4}.$$

Since in every cases of the following theorem $P(m, n, l_0, l_1, \ldots, l_{m-1})$ is 3-regular, then Observation 1(iii) follows the next result.

Theorem 3.2. *Let $G = P(m, n; l_0, l_1, \ldots, l_{m-1})$ be a SGPG. If either n is odd and $m = 2$ or n is even and $l_i = \frac{n}{2}$, for at least $m-2$ indices i, then*

$$\gamma_{\times 3,t}(G) = mn\,.$$

References

[1] S. Gravier, Total domination number of grid graphs, *Discrete App. Math.* **121** (2002), 119–128.

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (editors), *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc., New York, 1998.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (editors), *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[4] M.A. Henning and A.P. Kazemi, *k*-Tuple total domination in graphs, *Discrete App. Math.* **158** (2010), 1006–1011.

[5] M.A. Henning and A.P. Kazemi, *k*-Tuple total domination in cross product graphs, *Manuscript*.

[6] M.L. Saražin, W. Pacco and A. Previtali, Generalizing the generalized Petersen graphs, *Discrete Math.*, **307** (2007), 534–543.

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