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Research Article

Some Algebraic Properties of Regular Tree Transformations

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Abstract. A regular generalized hypersubstitutions is a mapping from $\{f_i \mid i \in I\}$ to $W_\tau(X)$ such that for every $i \in I$, each of the variables $x_1, x_2, \ldots, x_{n_i}$ occur in $\hat{\sigma}[f_i(x_1, x_2, \ldots, x_{n_i})]$. We use the extension of regular generalized hypersubstitutions to define tree transformations which is useful for abstract data type specifications in Theoretical Computer Science. In this paper, we study some algebraic properties of tree transformations.

Keywords. Generalized hypersubstitution; V-generalized transformation; Regular tree transformations

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1. Introduction

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called variables. We refer to these variables as letters, to X as an alphabet, and refer to the set $X_n =: \{x_1, x_2, \ldots, x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i-ary operation symbol, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The function τ, on the values of τ written as $(n_i)_{i\in I}$ is called a type.

An n-ary term of type τ is defined inductively as follows:

(i) The variables $x_1, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

We denote by $W_\tau(X_n)$ the smallest set which contains $x_1,\dots,x_n$ and is closed under finite number of applications of (ii). Then the set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ. An *equation* of type τ is a pair (s,t) where s and t are from $W_\tau(X)$; such pairs are commonly written as $s \approx t$. An equation $s \approx t$ is an *identity* of an algebra $\mathbf{A}$, denoted by $\mathbf{A} \models s \approx t$ if $s^{\underline{A}} = t^{\underline{A}}$ where $s^{\underline{A}}$ and $t^{\underline{A}}$ are the corresponding term functions on $\mathbf{A}$. A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which does not necessarily preserve arities.

We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. We define first the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j$, $1 \le j \le m$, then $S^m(x_j, t_1, \dots, t_m) := t_j$,

(ii) if $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \dots, t_m) := x_j$,

(iii) if $t = f_i(s_1,\dots,s_{n_i})$, then $S^m(t, t_1,\dots,t_m) := f_i(S^m(s_1,t_1,\dots,t_m),\dots,S^m(s_{n_i},t_1,\dots,t_m))$.

Then the generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ defined by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1,\dots,t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1],\dots,\hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \le j \le n_i$ are already defined.

In 2000, Leeratanavalee and Denecke [5] introduced a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution mapping which maps each n_i-ary operation symbol f_i to the term $f_i(x_1,\dots,x_{n_i})$. It turns out that $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

If $\underline{M}$ is a submonoid of $\underline{Hyp_G(\tau)}$ and V is a variety, then an identity $s \approx t$ of V is called an *M-strong hyperidentity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V for every $\sigma \in M$. A variety V is called *M-strongly solid* if every identity satisfies an M-strong hyperidentity. In case of $M = Hyp_G(\tau)$ we will call *strong hyperidentity* and *strongly solid*, respectively.

Let $\mathbf{A} = (A; (f_i^A)_{i \in I})$ be an algebra of type τ and $\sigma \in Hyp_G(\tau)$. We let $\sigma[\mathbf{A}] := (A; (\sigma(f_i)^{\mathbf{A}})_{i \in I})$ which is called *generalized derived algebra* of type τ, where $\sigma(f_i)^{\mathbf{A}}$ is the term operation induced by the term $\sigma(f_i)$ on the algebra $\mathbf{A}$.

2. *V*-Proper Generalized Hypersubstitutions and Normal Forms

Let V be a variety of algebras of type τ then to test whether an identity $s \approx t$ of V is a strong hyperidentity of V, our definition requires that we check, for each generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ that $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V. In practice we restrict our testing to certain special generalized hypersubstitutions σ, those which correspond to V-normal form generalized hypersubstitutions.

Definition 2.1 ([6]). Let V be a variety of algebras of type τ. A generalized hypersubstitution σ of type τ is called a *V-proper generalized hypersubstitution* if for every $s \approx t \in IdV$ one gets $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Definition 2.2 ([7])**.** Let V be a variety of algebras of type τ. Two generalized hypersubstitutions σ_1 and σ_2 of type τ are called *V-generalized equivalent* if $\sigma_1(f_i) \approx \sigma_2(f_i)$ are identities in V for all $i \in I$. In this case we write $\sigma_1 \sim_{VG} \sigma_2$.

Theorem 2.3 ([7])**.** *Let V be a variety of algebras of type τ, and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following statements are equivalent:*

(i) $\sigma_1 \sim_{VG} \sigma_2$.

(ii) *For all $t \in W_\tau(X)$, the equations $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in V.*

(iii) *For all $\mathbf{A} \in V$, $\sigma_1[\mathbf{A}] = \sigma_2[\mathbf{A}]$ where $\sigma_k[\mathbf{A}] = (A; (\sigma_k(f_i)^{\mathbf{A}})_{i \in I})$, for $k = 1, 2$.*

Proposition 2.4 ([7])**.** *Let V be a variety of algebras of type τ. Then the following statements hold:*

(i) *For all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then σ_1 is a* V*-proper generalized hypersubstitution iff σ_2 is a* V*-proper generalized hypersubstitution.*

(ii) *For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V.*

The relation $\sim_{VG}$ is an equivalence relation on $Hyp_G(\tau)$, but it is not necessarily a congruence relation. Since $\sim_{VG}$ is not always a congruence, the structure obtained by factoring $Hyp_G(\tau)$ by this relation is not necessarily going to be a monoid. Recall that the quotient set gives a monoid if and only if the equivalence relation used to factor it is a congruence. We factorize $Hyp_G(\tau)$ by $\sim_{VG}$ and consider the submonoid $\underline{P_G(V)}$ of $\underline{Hyp_G(\tau)}$ is the union of equivalence classes of the relation $\sim_{VG}$. This may also be done for a submonoid $\underline{M}$ of $\underline{Hyp_G(\tau)}$ and the relation $\sim_{VG}|_M$.

Lemma 2.5 ([7])**.** *Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)}$ and let V be a variety of type τ. Then the monoid $P_G(V) \cap M$ is the union of all equivalence classes of the restricted relation $\sim_{VG}|_M$.*

Definition 2.6 ([7])**.** Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ, and let V be a variety of type τ. Let ϕ be a choice function which chooses from M one generalized hypersubstitution from each equivalence class of the relation $\sim_{VG}|_M$, and let $N_\phi^M(V)$ be the set of generalized hypersubstitutions which are chosen. Thus $N_\phi^M(V)$ is a set of distinguished generalized hypersubstitutions from M, which we might call *V-normal form generalized hypersubstitutions*. We will say that the variety V is $N_\phi^M(V)$*-strongly solid* if for every identity $s \approx t \in IdV$ and for every generalized hypersubstitution $\sigma \in N_\phi^M(V)$, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Theorem 2.7 ([7])**.** *Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ and let V be a variety of type τ. For any choice function ϕ, V is M-strongly solid if and only if V is $N_\phi^M(V)$-strongly solid.*

Definition 2.8 ([8])**.** A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called a *regular generalized hypersubstitution* if for every $i \in I$, each of the variables $x_1, x_2, \dots, x_{n_i}$ occur in $\hat{\sigma}[f_i(x_1, x_2, \dots, x_{n_i})]$.

Let $Reg_G(\tau)$ be the set of all regular generalized hypersubstitutions of type τ. Then we have $Reg_G(\tau) \subseteq Hyp_G(\tau)$.

Proposition 2.9 ([8])**.** *For any type τ, $\underline{Reg_G(\tau)}$ is a submonoid of $\underline{Hyp_G(\tau)}$.*

3. Tree Transformations defined by Regular Generalized Hypersubstitution

In 2014, Busaman [3] studied concept of tree transformations defined by the regular hypersubstitution σ and then we generalize this concept to tree transformations defined by the regular generalized hypersubstitution. In this section we study some properties of VRG-tree transformations as the following definition.

Definition 3.1. Let σ be a regular generalized hypersubstitution. Then $T_\sigma^{RG} := \{(t, \hat{\sigma}[t]) \mid t \in W_\tau(X)\}$ is called a *tree transformation* defined by the regular generalized hypersubstitution σ.

We denote by $T_{\sigma_1}^{RG} \circ T_{\sigma_2}^{RG}$ the composition of the tree transformation $T_{\sigma_1}^{RG}$ and $T_{\sigma_2}^{RG}$. Let $T_{Reg_G(\tau)} = \{T_\sigma^{RG} \mid \sigma \in Reg_G(\tau)\}$ and prove that

Theorem 3.2. *$(T_{Reg_G(\tau)}; \circ, T_{\sigma_{id}}^{RG})$ is a monoid which is isomorphic to the monoid $Reg_G(\tau)$ of all regular generalized hypersubstitutions of type τ.*

Proof. Let $\sigma \in Reg_G(\tau)$. Then we define a mapping $\varphi : Reg_G(\tau) \to T_{Reg_G(\tau)}$ by $\sigma \mapsto T_\sigma^{RG}$. It clearly φ is well-define and surjective. Next, we will show that $\varphi(\sigma_1 \circ_G \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$, i.e. $T_{\sigma_1}^{RG} \circ T_{\sigma_2}^{RG} = T_{\sigma_1 \circ_G \sigma_2}^{RG}$, we have

$$
\begin{aligned}
(s,t) \in T_{\sigma_1}^{RG} \circ T_{\sigma_2}^{RG} &\Leftrightarrow \exists\, p((s,p) \in T_{\sigma_2}^{RG} \text{ and } (p,t) \in T_{\sigma_1}^{RG}) \\
&\Leftrightarrow p = \hat{\sigma}_2[s] \text{ and } t = \hat{\sigma}_1[p] \\
&\Leftrightarrow t = \hat{\sigma_1}[\hat{\sigma_2}[s]] \\
&\Leftrightarrow t = (\sigma_1 \circ_G \sigma_2)\hat{}[t] \\
&\Leftrightarrow (s,t) \in T_{\sigma_1 \circ_G \sigma_2}^{RG}.
\end{aligned}
$$

This shows that $T_{Reg_G(\tau)}$ is closed under composition and that φ preserves the operation. Next, we will show φ is one-to-one. Assume that $T_{\sigma_1}^{RG} = T_{\sigma_2}^{RG}$. Then for all $t \in W_\tau(X)$ we get $\hat{\sigma_1}[t] = \hat{\sigma_2}[t]$. Thus for all operation f_i we have $\hat{\sigma_1}[f_i(x_1, \ldots, x_{n_i})] = \sigma_1(f_i) = \sigma_2(f_i) = \hat{\sigma_2}[f_i(x_1, \ldots, x_{n_i})]$ and then $\sigma_1 = \sigma_2$. Finally, since $T_{\sigma_1}^{RG} \circ T_{\sigma_2}^{RG} = T_{\sigma_1 \circ_G \sigma_2}^{RG}$, the tree transformation $T_{\sigma_{id}}^{RG}$ is an identity element with respect to the composition $\circ$. □

Theorem 3.3. *Let $\sigma \in Hyp_G(\tau)$ be a regular generalized hypersubstitution of type τ and let T_σ^{RG} be the corresponding tree transformation. Then*

(i) *T_σ^{RG} is transitive iff σ is idempotent,*

(ii) *T_σ^{RG} is reflexive iff $\sigma = \sigma_{id}$,*

(iii) *T_σ^{RG} is symmetric iff $\sigma \circ_G \sigma = \sigma_{id}$.*

Proof. (i): Assume that σ is idempotent. Then $T^{RG}_{\sigma\circ_G\sigma} = T^{RG}_\sigma \circ T^{RG}_\sigma = T^{RG}_\sigma$ by Theorem 2.2 and so T^{RG}_σ is transitive. Conversely, assume that T^{RG}_σ is transitive, we have $T^{RG}_\sigma \circ T^{RG}_\sigma \subseteq T^{RG}_\sigma$ so that $T^{RG}_{\sigma\circ_G\sigma} \subseteq T^{RG}_\sigma$. Then $(t,(\sigma\circ_G\sigma)\hat{}[t]) \in T^{RG}_{\sigma\circ_G\sigma} \Rightarrow (t,(\sigma\circ_G\sigma)\hat{}[t]) \in T^{RG}_\sigma \Rightarrow (\sigma\circ_G\sigma)\hat{}[t] = \hat{\sigma}[t]$, for all $t \in W_\tau(X)$. So σ is idempotent.

(ii): Assume that T^{RG}_σ is reflexive, so $T^{RG}_{\sigma_{id}} = \triangle_{W_\tau(X)} \subseteq T^{RG}_\sigma$. Therefore $(t,t) \in T^{RG}_\sigma$ for all $t \in W_\tau(X)$ and then $\hat{\sigma}[t] = t$ for all $t \in W_\tau(X)$ so we set $\sigma = \sigma_{id}$. Conversely, assume that $\sigma = \sigma_{id}$. Then $T^{RG}_{\sigma_{id}} = \{(t,\hat{\sigma}[t]) \mid t \in W_\tau(X)\} = \{(t,t) \mid t \in W_\tau(X)\} = \triangle_{W_\tau(X)}$ and T^{RG}_σ is reflexive.

(iii): Assume that T^{RG}_σ is symmetric. Then for all $t \in W_\tau(X)$ we have $(t,\hat{\sigma}[t]) \in T^{RG}_\sigma \Rightarrow (\hat{\sigma}[t],t) \in T^{RG}_\sigma$. Therefore $t = \hat{\sigma}[\hat{\sigma}[t]]$ and $\hat{\sigma}_{id}[t] = (\sigma\circ_G\sigma)\hat{}[t]$ for all $t \in W_\tau(X)$ and we have $\sigma\circ_G\sigma = \sigma_{id}$. Conversely, assume that $\sigma\circ_G\sigma = \sigma_{id}$. Then we have $T^{RG}_{\sigma\circ_G\sigma} = T_\sigma \circ T^{RG}_\sigma = T^{RG}_{\sigma_{id}}$. This means $T^{RG}_\sigma = (T^{RG}_\sigma)^{-1}$, thus T^{RG}_σ is symmetric. □

A tree transformation is called *injective* if σ is injective, i.e., if $\hat{\sigma}[t] = \hat{\sigma}[t']$ then $t = t'$, and T^{RG}_σ is called *surjective* if σ is surjective. Then we consider $\hat{\sigma}[W_\tau(X)] = \{t' \mid \exists\, t \in W_\tau(X), \hat{\sigma}[t] = t'\}$ is a subset of $W_\tau(X)$. Therefore, we consider T^{RG}_σ as a relation between $W_\tau(X)$ and $\hat{\sigma}[W_\tau(X)]$, so that $T^{RG}_\sigma \subseteq W_\tau(X) \times \hat{\sigma}[W_\tau(X)]$. We notice that $T^{RG}_\sigma \circ (T^{RG}_\sigma)^{-1} = \triangle_{W_\tau(X)}$ and $(T^{RG}_\sigma)^{-1} \circ T^{RG}_\sigma = \{(t,t') \mid \hat{\sigma}[t] = \hat{\sigma}[t']\} = ker\sigma$. Then, we have

Proposition 3.4. *Let $\sigma \in Hyp_G(\tau)$ be a regular generalized hypersubstitution of type τ and let $T^{RG}_\sigma = W_\tau \times \hat{\sigma}[W_\tau(X)]$ be the corresponding tree transformation. Then T^{RG}_σ is bijective iff $ker\sigma = \triangle_{W_\tau(X)} = T^{RG}_{\sigma_{id}}$.*

Proof. T^{RG}_σ is bijective iff $T^{RG}_\sigma \circ (T^{RG}_\sigma)^{-1} = (T^{RG}_\sigma)^{-1} \circ T^{RG}_\sigma = T^{RG}_{\sigma_{id}} = \triangle_{W_\tau(X)}$. □

As an example now we consider the variety $Rec := Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3\}$ is the nontrivial strongly solid variety of semigroup. By using Theorem 2.7 together with the identities of Rec, we can restrict our checking to the following regular generalized hypersubstitutions σ_t where $t \in \{x_1x_2\} \cup \{x_2x_1\} \cup \{x_1x_2x_j \mid j > 2\} \cup \{x_2x_1x_j \mid j > 2\} \cup \{x_1x_2x_1\} \cup \{x_2x_1x_2\} \cup \{x_jx_1x_2 \mid j > 2\} \cup \{x_jx_2x_1 \mid j > 2\} \cup \{x_jx_1x_2x_k \mid j,k > 2\} \cup \{x_jx_2x_1x_k \mid j,k > 2\}$. Here σ_t for a term $t \in W_{(2)}(X)$ denotes the hypersubstitution which maps the binary operation symbol to the term t. The multiplication of $\circ_G$ is described by the following tables.

$\circ_G$	$\sigma_{x_1x_2}$	$\sigma_{x_2x_1}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$
$\sigma_{x_1x_2}$	$\sigma_{x_1x_2}$	$\sigma_{x_2x_1}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$
$\sigma_{x_2x_1}$	$\sigma_{x_2x_1}$	$\sigma_{x_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$
$\sigma_{x_1x_2x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_2x_1x_j}$
$\sigma_{x_2x_1x_j}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_jx_2x_1x_j}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_2x_1x_j}$
$\sigma_{x_1x_2x_1}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$
$\sigma_{x_2x_1x_2}$	$\sigma_{x_2x_1x_2}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_2x_1x_2}$
$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_jx_2x_1x_j}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2}$
$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2}$
$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$
$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$

$\circ_G$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$
$\sigma_{x_1x_2}$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$
$\sigma_{x_2x_1}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$
$\sigma_{x_1x_2x_j}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$
$\sigma_{x_2x_1x_j}$	$\sigma_{x_2x_1x_j}$	$\sigma_{x_1x_2x_j}$	$\sigma_{x_kx_2x_1x_j}$	$\sigma_{x_kx_1x_2x_j}$
$\sigma_{x_1x_2x_1}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_jx_2x_1x_j}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_jx_2x_1x_j}$
$\sigma_{x_2x_1x_2}$	$\sigma_{x_2x_1x_2}$	$\sigma_{x_1x_2x_1}$	$\sigma_{x_kx_2x_1x_k}$	$\sigma_{x_kx_1x_2x_k}$
$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_1x_2}$	$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$
$\sigma_{x_jx_2x_1}$	$\sigma_{x_jx_2x_1x_j}$	$\sigma_{x_jx_1x_2x_j}$	$\sigma_{x_jx_2x_1x_j}$	$\sigma_{x_jx_1x_2x_j}$
$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$
$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$	$\sigma_{x_jx_2x_1x_k}$	$\sigma_{x_jx_1x_2x_k}$

Now, we want to describe the tree transformations corresponding to these regular generalized hypersubstitutions. By *leftmost*(t) and by *rightmost*(t) we denote the first and the last variable, respectively, of the term t.

$$
\begin{aligned}
T^{Rec}_{\sigma_{x_1x_2}} &= \{(t,t) \mid t \in W_{(2)}(X)\} = \Delta_{W_{(2)}(X)},\\
T^{Rec}_{\sigma_{x_2x_1}} &= \{(t,t) \mid t \in W_{(2)}(X)\} = \Delta_{W_{(2)}(X)},\\
T^{Rec}_{\sigma_{x_1x_2x_j}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = x_j\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = \mathit{leftmost}(t') = x_j\},\\
T^{Rec}_{\sigma_{x_2x_1x_j}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = x_j\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = \mathit{leftmost}(t') = x_j\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = x_j \text{ and } \mathit{leftmost}(t') = x_k \text{ where } j \neq k\},\\
T^{Rec}_{\sigma_{x_1x_2x_1}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } t' = x_1x_2x_1 \text{ or } t' = x_2x_1x_2\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = \mathit{leftmost}(t') = x_j\}\\
T^{Rec}_{\sigma_{x_2x_1x_2}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } t' = x_1x_2x_1 \text{ or } t' = x_2x_1x_2\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = \mathit{leftmost}(t') = x_j\},\\
T^{Rec}_{\sigma_{x_jx_1x_2}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{leftmost}(t') = x_j\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = \mathit{leftmost}(t') = x_j\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = x_j \text{ and } \mathit{leftmost}(t') = x_k \text{ where } j \neq k\},\\
T^{Rec}_{\sigma_{x_jx_2x_1}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{leftmost}(t') = x_j\}\\
&\quad \cup \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = \mathit{leftmost}(t') = x_j\},\\
T^{Rec}_{\sigma_{x_jx_1x_2x_k}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = x_j \text{ and } \mathit{leftmost}(t') = x_k \text{ where } j \neq k\}\\
T^{Rec}_{\sigma_{x_jx_2x_1x_k}} &= \{(t,t') \mid t' \in W_{(2)}(X) \text{ and } \mathit{rightmost}(t') = x_j \text{ and } \mathit{leftmost}(t') = x_k \text{ where } j \neq k\}.
\end{aligned}
$$

4. Properties of VRG-tree Transformations

Definition 4.1. Let V be a variety of algebras of type τ and $\sigma \in Reg_G(\tau)$. The set $T_\sigma^{VRG} := \{(t,\acute{t}) | t,\acute{t} \in W_\tau(X) \text{ and } \hat{\sigma}[t] \approx \acute{t} \in IdV\}$ is called the *VRG-tree transformation* defined by the regular generalized hypersubstitution σ.

Definition 4.2 ([9]). Let V be a variety of algebras of type τ. Two regular generalized hypersubstitutions σ_1, σ_2 of type τ are called *V-regular generalized equivalent* if and only if $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all $i \in I$. In this case we write $\sigma_1 \sim_{VRG} \sigma_2$.

Theorem 4.3 ([9]). *Let V be a variety of algebras of type τ, and let $\sigma_1, \sigma_2 \in Reg_G(\tau)$. Then the following are equivalent:*

(i) $\sigma_1 \sim_{VRG} \sigma_2$.

(ii) *For every $t \in W_\tau(X)$, the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in IdV$.*

(iii) *For every $\mathbf{A} \in V$, $\sigma_1[\mathbf{A}] = \sigma_2[\mathbf{A}]$ where $\sigma_k[\mathbf{A}] = (A; (\sigma_k(f_i)^{\mathbf{A}})_{i \in I}); k = 1,2$.*

Proposition 4.4 ([9]). *Let V be a variety of algebras of type τ. Then the following hold:*

(i) *For all $\sigma_1, \sigma_2 \in Reg_G(\tau)$, if $\sigma_1 \sim_{VRG} \sigma_2$ then σ_1 is a V-proper regular generalized hypersubstitution iff σ_2 is a V-proper regular generalized hypersubstitution.*

(ii) *For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in Reg_G(\tau)$, if $\sigma_1 \sim_{VRG} \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.*

Proposition 4.5. *Let $\sigma_1, \sigma_2 \in Reg_G(\tau)$ and let V be a variety of type τ. Then $\sigma_1 \sim_{VRG} \sigma_2$ iff $T_{\sigma_1}^{VRG} = T_{\sigma_2}^{VRG}$.*

Proof. ($\Rightarrow$) We have to show that $T_{\sigma_1}^{VRG} = T_{\sigma_2}^{VRG}$. Let $\sigma_1 \sim_{VRG} \sigma_2$ and $(t,\acute{t}) \in T_{\sigma_1}^{VRG}$. Then by Theorem 4.3(ii) we have $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in IdV$, for all $t \in W_\tau(X)$ and $\hat{\sigma}_1[t] \approx \acute{t} \in IdV$. Therefore $(t,\acute{t}) \in T_{\sigma_2}^{VRG}$ and thus $T_{\sigma_1}^{VRG} \subseteq T_{\sigma_2}^{VRG}$. Let $\sigma_1 \sim_{VRG} \sigma_2$ and $(t,\acute{t}) \in T_{\sigma_2}^{VRG}$. Then by Theorem 4.3(ii) we have $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in IdV$, for all $t \in W_\tau(X)$ and $\hat{\sigma}_2[t] \approx \acute{t} \in IdV$. Therefore $(t,\acute{t}) \in T_{\sigma_1}^{VRG}$ and thus $T_{\sigma_2}^{VRG} \subseteq T_{\sigma_1}^{VRG}$. Thus we conclude that $T_{\sigma_2}^{VRG} = T_{\sigma_1}^{VRG}$.

($\Leftarrow$) Assume that $T_{\sigma_1}^{VRG} = T_{\sigma_2}^{VRG}$. Let $t, t' \in W_\tau(X)$ and $(t,t') \in T_{\sigma_1}^{VRG} = T_{\sigma_2}^{VRG}$. Then $\sigma_1[t] \approx t'$ and $\sigma_2[t] \approx t'$ and so we get $\sigma_1[t] \approx \sigma_2[t]$ for all $t \in W_\tau(X)$. Then $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for all $i \in I$ and thus $\sigma_1 \sim_{VRG} \sigma_2$. □

Definition 4.6 ([9]). Let V be a variety of algebras of type τ. A regular generalized hypersubstitution $\sigma \in Reg_G(\tau)$ is called a *V-proper regular generalized hypersubstitution* if for every $s \approx t \in IdV$ one gets $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

We denote $P_{RG}(V)$ for the set of all V-proper regular generalized hypersubstitutions of type τ.

Proposition 4.7 ([9]). *The algebra $(P_{RG}(V); \circ_G, \sigma_{id})$ is a submonoid of $(Reg_G(\tau); \circ_G, \sigma_{id})$.*

Lemma 4.8. *If V is a variety of algebras of type τ, $\sigma_1 \in P_{RG}(V)$, and $\sigma_2 \in Reg_G(\tau)$ then $T_{\sigma_1}^{VRG} \circ T_{\sigma_2}^{VRG} = T_{\sigma_1 \circ_G \sigma_2}^{VRG}$.*

Proof. Let $(t,t') \in T^{VRG}_{\sigma_1} \circ T^{VRG}_{\sigma_2}$. Then there is a term t'' such that $(t,t'') \in T^{VRG}_{\sigma_2}$ and $(t'',t') \in T^{VRG}_{\sigma_1}$. Then $\hat{\sigma_2}[t] \approx t'' \in IdV$ and $\hat{\sigma_1}[t''] \approx t' \in IdV$. Since $\sigma_1 \in P_{RG}(V)$, so $\hat{\sigma_1}[\hat{\sigma_2}[t]] \approx \hat{\sigma_1}[t''] \approx t' \in IdV$. Thus $(\sigma_1 \circ_G \sigma_2)\hat{}[t] \approx t' \in IdV$ and then $(t,t') \in T^{VRG}_{\sigma_1 \circ_G \sigma_2}$. This shows that $T^{VRG}_{\sigma_1} \circ T^{VRG}_{\sigma_2} \subseteq T^{VRG}_{\sigma_1 \circ_G \sigma_2}$.

Let $(t,t') \in T^{VRG}_{\sigma_1 \circ_G \sigma_2}$. Thus $(\sigma_1 \circ_G \sigma_2)\hat{}[t] \approx t' \in IdV$ and so $\hat{\sigma_1}[\hat{\sigma_2}[t]] \approx t' \in IdV$ with $t'' \approx \hat{\sigma_2}[t] \in IdV$ we have $\hat{\sigma_1}[t''] \approx t' \in IdV$ because $\sigma_1 \in P_{RG}(V)$. Then $(t,t'') \in T^{VRG}_{\sigma_2}, (t'',t') \in T^{VRG}_{\sigma_2}$ and therefore $(t,t') \in T^{VRG}_{\sigma_1} \circ T^{VRG}_{\sigma_2}$. This shows that $T^{VRG}_{\sigma_1 \circ_G \sigma_2} \subseteq T^{VRG}_{\sigma_1} \circ T^{VRG}_{\sigma_2}$. □

We consider the set $\mathscr{T}_{P_{RG}}(V) := \{T^{VRG}_{\sigma} | \sigma \in P_{RG}(V)\}$ and we may take the relation product as a binary relation with $T^{VRG}_{\sigma_{id}} := \{(t,t) \mid t \approx t \in IdV\} = IdV$ as identity element. Then we get

Proposition 4.9. *The monoid $(\mathscr{T}_{P_{RG}(V)}; \circ, T^{VRG}_{\sigma_{id}})$ is a homomorphic image of $(P_{RG}(V); \circ_G, \sigma_{id})$.*

Proof. Let $\varphi : P_{RG}(V) \to \mathscr{T}_{P_{RG}(V)}$ be defined by $\varphi(\sigma) := T^{VRG}_{\sigma}$. It is clear that σ is well defined. Then by Lemma 4.8, we have $\varphi(\sigma_1 \circ_G \sigma_2) = T^{VRG}_{\sigma_1 \circ_G \sigma_2} = T^{VRG}_{\sigma_1} \circ T^{VRG}_{\sigma_2} = \varphi(\sigma_1) \circ \varphi(\sigma_2)$. So we get φ is a homomorphism and $\varphi(\sigma_{id}) = T^{VRG}_{\sigma_{id}}$. □

Definition 4.10 ([9])**.** Let V be a variety of algebras of type τ. A regular generalized hypersubstitution $\sigma \in Reg_G(\tau)$ is called an *inner regular generalized hypersubstitution* of a variety V if for every $i \in I$,

$$\hat{\sigma}[f_i(x_1,\dots,x_{n_i})] \approx f_i(x_1,\dots,x_{n_i}) \in IdV.$$

Let $P^0_{RG}(V)$ be the set of all inner regular generalized hypersubstitutions of V. By Definition 4.10, $P^0_{RG}(V)$ is the equivalence class $[\sigma_{id}]_{\sim_{VRG}}$.

Proposition 4.11 ([9])**.** *The algebra $(P^0_{RG}(V); \circ_G, \sigma_{id})$ is a submonoid of $(P_{RG}(V); \circ_G, \sigma_{id})$.*

Proposition 4.12. *Let V be a variety of algebras of type τ and let $\sigma \in Reg_G(\tau)$. Then T^{VRG}_{σ} is reflexive iff $\sigma \in P^0_{RG}(V)$.*

Proof. Assume that T^{VRG}_{σ} is reflexive. Then $\hat{\sigma}[t] \approx t \in IdV$ for all $t \in W_\tau(X)$. This is valid also for $t = f_i(x_1,\dots,x_{n_i}), i \in I$ and then $\hat{\sigma}[f_i(x_1,\dots,x_{n_i})] \approx f_i(x_1,\dots,x_{n_i}) \in IdV$, i.e. $\hat{\sigma}[f_i(x_1,\dots,x_{n_i})] \approx \hat{\sigma}_{id}[f_i(x_1,\dots,x_{n_i})] \in IdV$ and $\sigma \sim_{VRG} \sigma_{id}$. Therefore, $\sigma \in P^0_{RG}(V)$. Conversely, assume that $\sigma \in P^0_{RG}(V)$. Then $\sigma \sim_{VRG} \sigma_{id}$ and by Theorem 4.3 we have $\hat{\sigma}[t] \approx t \in IdV$ for all $t \in W_\tau(X)$. But this means $(t,t) \in T^{VRG}_{\sigma}$ and T^{VRG}_{σ} is reflexive. □

Definition 4.13. Let $\sigma \in Reg_G(\tau)$ and V be a variety of algebras of type τ. The set

$$ker^G_V \sigma := \{(t,t') | t,t' \in W_\tau(X) \text{ and } \hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV\}$$

will be called the *kernel of σ with respect to V* or *semantical kernel* of σ.

Proposition 4.14. *If $\sigma_1 \sim_{VRG} \sigma_2$, then $ker^G_V \sigma_1 = ker^G_V \sigma_2$.*

Proof. By Proposition 4.4. □

Proposition 4.15. *Let V be a variety of algebras of type τ and let $\sigma \in Reg_G(\tau)$. Then $ker_V^G \sigma \subseteq ker_V^G(\rho \circ_G \sigma)$ for all $\rho \in P_{RG}(V)$.*

Proof. For any $(t,t') \in ker_V^G \sigma$ we have $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$. Since ρ is a V-proper regular generalized hypersubstitution this implies that $\hat{\rho}[\hat{\sigma}[t]] \approx \hat{\rho}[\hat{\sigma}[t']] \in IdV$ and so $(t,t') \in ker_V^G(\rho \circ_G \sigma)$. □

Proposition 4.16. *Let V be a variety of algebras of type τ and let $\sigma \in Reg_G(\tau)$. Then $(T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = ker_V^G \sigma$.*

Proof. We have

$$\begin{aligned}(t,t'') \in (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} &\Leftrightarrow \exists\, t'((t,t') \in T_\sigma^{VRG} \text{ and } (t',t'') \in (T_\sigma^{VRG})^{-1}) \\ &\Leftrightarrow \exists\, t'((t,t') \in T_\sigma^{VRG} \text{ and } (t'',t') \in T_\sigma^{VRG}) \\ &\Leftrightarrow \exists\, t'(\hat{\sigma}[t] \approx t' \in IdV \text{ and } \hat{\sigma}[t''] \approx t' \in IdV) \\ &\Leftrightarrow \hat{\sigma}[t] \approx \hat{\sigma}[t''] \in IdV \\ &\Leftrightarrow (t,t'') \in ker_V^G \sigma.\end{aligned}$$

□

Theorem 4.17. *For any V-proper generalized hypersubstitution σ, the following are equivalent:*

(i) T_σ^{VRG} *is transitive.*

(ii) $\sigma \circ_G \sigma \sim_{VRG} \sigma$.

(iii) $T_\sigma^{VRG} \subseteq ker_V^G \sigma$.

Proof. (i)⇒(ii): Assume that T_σ^{VRG} is transitive. Then $T_\sigma^{VRG} \circ T_\sigma^{VRG} \subseteq T_\sigma^{VRG}$ and therefore $T_\sigma^{VRG} \circ T_\sigma^{VRG} = T_{\sigma \circ_G \sigma}^{VRG} \subseteq T_\sigma^{VRG}$. This means, if $(t,t') \in T_{\sigma \circ_G \sigma}^{VRG}$, i.e. $(\sigma \circ_G \sigma)\hat{}[t] \approx t' \in IdV$ then $(t,t') \in T_\sigma^{VRG}$ i.e. $\hat{\sigma}[t] \approx t' \in IdV$. But $(\sigma \circ_G \sigma)\hat{}[t] \approx \hat{\sigma}[t] \in IdV$ for all $t \in W_\tau(X)$ and so $\sigma \circ_G \sigma \sim_{VRG} \sigma$ by Theorem 4.3.

(ii)⇒(i): Assume that $\sigma \circ_G \sigma \sim_{VRG} \sigma$. By Lemma 4.8, we get $T_{\sigma \circ_G \sigma}^{VRG} = T_\sigma^{VRG}$ and $T_\sigma^{VRG} \circ T_\sigma^{VRG} = T_{\sigma \circ_G \sigma}^{VRG} = T_\sigma^{VRG}$ and so T_σ^{VRG} is transitive.

(ii)⇒(iii): Assume that $\sigma \circ_G \sigma \sim_{VRG} \sigma$ and let $(t,t') \in T_\sigma^{VRG}$, i.e. $\hat{\sigma}[t] \approx t' \in IdV$. Then $\hat{\sigma}[\hat{\sigma}[t]] \approx \hat{\sigma}[t'] \in IdV$ since σ is V-proper regular generalized hypersubstitution and we have $(\sigma \circ_G \sigma)\hat{}[t] \approx \hat{\sigma}[t'] \in IdV$. From $(\sigma \circ_G \sigma)\hat{}[t] \approx \hat{\sigma}[t'] \in IdV$ we obtain that $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$ and then $(t,t') \in ker_V^G \sigma$. This shows that $T_\sigma^{VRG} \subseteq ker_V^G \sigma$.

(iii)⇒(ii): Assume that $T_\sigma^{VRG} \subseteq ker_V^G \sigma$. Then $(t,t') \in T_\sigma^{VRG}$, i.e. $\hat{\sigma}[t] \approx t' \in IdV$ and since σ is V-proper regular generalized hypersubstitution we get $\hat{\sigma}[\hat{\sigma}[t]] \approx \hat{\sigma}[t'] \in IdV$. Since $(t,t') \in ker_V^G \sigma$ we have $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$ and then $\hat{\sigma}[\hat{\sigma}[t]] \approx \hat{\sigma}[t] \in IdV$, i.e. $\sigma \circ_G \sigma \sim_{VRG} \sigma$. □

Theorem 4.18. *Let V be a variety of algebras of type τ. Then:*

(i) T_σ^{VRG} *is surjective iff* $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = IdV$.

(ii) T_σ^{VRG} *is injective iff* $(T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = ker_V^G \sigma = \triangle_{W_\tau}(X)$.

(iii) T_σ^{VRG} *is bijective iff* $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = \triangle_{W_\tau}(X)$.

Proof. (i): Assume that T_σ^{VRG} is surjective. We will show that $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = IdV$. Let

$(t,t') \in IdV$. Assume that $t \approx t' \in IdV$. Since T_σ^{VRG} is surjective, for any t' there is a term t'' such that $(t'',t') \in T_\sigma^{VRG}$, i.e. $\hat{\sigma}[t''] \approx t' \in IdV$. Then we have also $\hat{\sigma}[t''] \approx t \in IdV$ and $(t'',t) \in T_\sigma^{VRG}$, i.e. $(t,t'') \in (T_\sigma^{VRG})^{-1}$. Let $(t,t') \in T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1}$. Thus $IdV \subseteq T_\sigma^{V} \circ (T_\sigma^{VRG})^{-1}$. Since $(t,t') \in T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1}$, there exists t'' such that $(t,t'') \in (T_\sigma^{VRG})^{-1}$ and $(t'',t') \in T_\sigma^{VRG}$. Then we have $(t'',t) \in T_\sigma^{VRG}$ and $(t'',t') \in T_\sigma^{VRG}$, i.e. $\hat{\sigma}[t''] \approx t \in IdV$ and $\hat{\sigma}[t''] \approx t' \in IdV$. So $t \approx t' \in IdV$. Conversely, we assume that $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = IdV$. Let $t \in W_\tau(X)$. We will show that there is a term $t' \in W_\tau(X)$ with $\hat{\sigma}[t'] \approx t \in IdV$. From $t \approx t \in IdV = T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1}$ we obtain existence of $t' \in W_\tau(X)$ such that $(t',t) \in T_\sigma^{VRG}$, but this means $\hat{\sigma}[t] \approx t \in IdV$ and this shows that T_σ^{VRG} surjective.

(ii): $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = ker_V^G \sigma$ is clear. Assume that T_σ^{VRG} is injective and let $(t,t') \in (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG}$. Then we have $(t,t'') \in T_\sigma^{VRG}$ and $(t',t'') \in T_\sigma^{VRG}$ so $t = t'$. We get $(t,t') \in \triangle_{W_\tau(X)}$ and thus $(T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} \subseteq \triangle_{W_\tau(X)}$. Assume that $(t,t') \in \triangle_{W_\tau(X)}$. Then $t = t'$ and $\hat{\sigma}[t] = \hat{\sigma}[t'] := t''$. Thus $\hat{\sigma}[t] \approx t'' \in IdV$ and $\hat{\sigma}[t'] \approx t'' \in IdV$ and $(t,t') \in T_\sigma^{V}, (t',t'') \in T_\sigma^{VRG}$. Then $(t,t') \in (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG}$, i.e. $\triangle_{W_\tau(X)} \subseteq (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG}$. This gives $\triangle_{W_\tau(X)} = (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG}$. Conversely, assume that $\triangle_{W_\tau(X)} = (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG}$ and that $(t,t''),(t',t'') \in T_\sigma^{VRG}$. Then $(t,t') \in (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = \triangle_{W_\tau(X)}$, i.e. $t = t'$ and therefore T_σ^{VRG} is injective.

(iii): Assume that T_σ^{VRG} is bijective. Since σ is V-proper regular generalized hypersubstitution, then by (i) and (ii): $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = IdV \subseteq ker_V^G \sigma = (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = \triangle_{W_\tau(X)}$ and therefore $IdV = \triangle_{W_\tau(X)}$ and $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = \triangle_{W_\tau(X)}$. Conversely, $T_\sigma^{VRG} \circ (T_\sigma^{VRG})^{-1} = (T_\sigma^{VRG})^{-1} \circ T_\sigma^{VRG} = \triangle_{W_\tau(X)}$, then by (ii) T_σ^{VRG} is injective. Therefore, $T_\sigma^{VRG} = \{(t,t') \mid \hat{\sigma} \approx t' \in IdV\} = IdV = \triangle_{W_\tau(X)}$. □

5. Conclusion

We use the extension of regular generalized hypersubstitutions to define tree transformations which is useful for abstract data type specifications in Theoretical Computer Science and this work we study some algebraic properties of tree transformations.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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