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Research Article

Numerical Solution of Singularly Perturbed Boundary Value Problems with Twin Boundary Layers using Exponential Fitted Scheme

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Abstract. This paper deals with a numerical method with fitted operator difference method for twin (dual) boundary layers singularly perturbed boundary value problems. In this method, Numerov method is extended to the given second order problem having derivative of first order. Using the non standard differences and modified upwind difference for the first order derivatives, the discrete scheme is deduced. A fitting parameter is utilized in the difference scheme, which handles the rapid changes that occur in the boundary layers due to the small perturbation parameter. Tridiagonal solver is implemented to solve the system of the method. Convergence analysis of the deduced method is discussed. Maximum errors in the solution of the model numerical examples are tabulated and comparison is made, to illustrate and support the method. Solutions are depicted graphically to show the layer behaviour.

Keywords. Singular perturbation problem; Twin layers; Fitting factor; Tridiagonal system

MSC. 65L10; 65L11; 65L12

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1. Introduction

Singular perturbation theory is a area of mathematics with fair history and a assure for significant applications all over engineering and science. This problem is defined as, which has no single asymptotic series expansion is valid uniformly on entire domain, as the perturbation parameter $\varepsilon \to 0$. Such problems exist very often in fluid mechanics, aerodynamics, elasticity, magneto hydrodynamics and other area of the fluid motion. The numerical behaviour of these equations gives difficulties due to the occurrence of boundary and/or interior layers. If we use the standard computational methods for such problems, huge oscillations take place and corrupt the solution in the overall domain because of the layer behavior. Hence, more simpler and efficient techniques are essential to solve these boundary value problems.

For a comprehensive theory and analytical discussion on these problems, one can refer to Bender and Orszag [3], O'Malley [11], and Miller et al. [8]. The study papers by Reddy and Kadalbajoo [7], Kadalbajoo and Patidar [6] give an scholarly outline of the problems and their treatment on boundary layers. Abrahamsson [2] deduced a priori estimates with a turning point for the solutions of these problems. A set of conditions for a convergent scheme for turning point problem is derived by Farrell [5]. A computational method is derived by combining fitted difference schemes with classical numerical scheme by Natesan and Ramanujam [10] for turning point problem, Natesan et al. [9] proposed a uniform numerical scheme on Shishkin mesh for turning point problems.

2. Numerical Method

Consider the singular perturbation problem of the form:

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad -1 \le x \le 1, \tag{1}$$

with boundary conditions $y(-1) = \alpha$ and $y(1) = \beta$ (2)

where $0 < \varepsilon \ll 1$, α and β are finite constants. Here, assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions such that $a(0) = 0$, $|a(x)| \ge a_0 > 0$ for $0 < x \le 1$, $a'(0) \le 0$, $|a'(x)| \ge \frac{|a'(0)|}{2}$, $b(x) \ge b_0 > 0$, for all $x \in D = [-1,1]$.

With these assumption, the problem eq. (1)-eq.(2) has unique solution exhibit two boundary layers at both end points $x = -1, 1$. Partition the domain $[-1,1]$ into N equal sub domains with mesh length h, i.e., $h = \frac{2}{N}$ and $x_i = -1 + ih$ for $i = 0,1,\dots,N$. Denote $\frac{N}{2} = l$. Then, split the domain $[-1,1]$ into two sub regions $[x_{i-1}, x_i]$ for $i = 1,2,\dots,l-1$; and $[x_i, x_{i+1}]$ for $i = l+1, l+2,\dots,N-1$. For dual or twin layer problem, in $[x_{i-1}, x_i]$ for $i = 1,2,\dots,l-1$ layer exists at left end point and in $[x_i, x_{i+1}]$ for $i = l+1, l+2,\dots,N-1$ layer is at right end point. Hence, we derive the numerical scheme for both left end layer in $[-1,0]$ and right end layer in $[0,1]$ cases.

In $[-1,0]$, using the theory of singular perturbations, the zeroth order asymptotic solution of eq. (1) is (cf. O'Malley [11])

$$\lim_{h\to 0} y_i \approx y_0(-1) + (\alpha - y_0(-1)\exp\left(-\left(\frac{a^2(-1) - \varepsilon b(-1)}{a(-1)}\right)\left(\frac{-1}{\varepsilon} + i\rho\right)\right), \tag{3}$$

where $\rho = \frac{h}{\varepsilon}$. By the Numerov method, we have

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h^2}{12}(y''_{i-1} + 10y''_i + y''_{i+1}) + O(h^6). \tag{4}$$

Now, we extend the scheme eq. (4) for singular perturbation problem with first order derivative as follows: Using eq. (1), we have

$$\left.\begin{aligned}\varepsilon y''_{i+1} &= -a_{i+1}y'^{*}_{i+1} - b_{i+1}y_{i+1} + f_{i+1},\\ \varepsilon y''_{i} &= -a_{i}y'_{i} - b_{i}y_{i} + f_{i},\\ \varepsilon y''_{i-1} &= -a_{i-1}y'^{*}_{i-1} - b_{i-1}y_{i-1} + f_{i-1}.\end{aligned}\right\} \tag{5}$$

We approximate y'^{*}_{i+1}, y'^{*}_{i-1} using non standard finite differences and y'_i by modified upwind finite difference as

$$y'^{*}_{i+1} \approx \frac{y_{i-1}-4y_i+3y_{i+1}}{2h}, \quad y'_i \approx \frac{y_{i+1}-y_i}{h} - \frac{h}{2}y''_i, \quad y'^{*}_{i-1} \approx \frac{-3y_{i-1}+4y_i-y_{i+1}}{2h}. \tag{6}$$

Substituting eq. (5) and eq. (6) in eq. (4) and exercising, we get

$$\begin{aligned}&\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}(-3y_{i-1}+4y_i-y_{i+1}) + \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)}(y_{i+1}-y_i)\\ &+ \frac{a_{i+1}}{24h}(y_{i-1}-4y_i+3y_{i+1}) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i\varepsilon}{12\left(1-\frac{a_i\rho}{2}\right)}y_i + \frac{b_{i+1}}{12}y_{i+1}\\ &\quad = \frac{1}{12}\left(f_{i-1} + \frac{10\varepsilon}{12\left(1-\frac{a_i\rho}{2}\right)}f_i + f_{i+1}\right).\end{aligned} \tag{7}$$

Now, introducing the fitting factor $\sigma(\rho)$ in eq. (7), we get

$$\begin{aligned}&\sigma\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}(-3y_{i-1}+4y_i-y_{i+1}) + \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)}(y_{i+1}-y_i)\\ &+ \frac{a_{i+1}}{24h}(y_{i-1}-4y_i+3y_{i+1}) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i\varepsilon}{12\left(1-\frac{a_i\rho}{2}\right)}y_i + \frac{b_{i+1}}{12}y_{i+1}\\ &\quad = \frac{1}{12}\left(f_{i-1} + \frac{10\varepsilon}{12\left(1-\frac{a_i\rho}{2}\right)}f_i + f_{i+1}\right).\end{aligned} \tag{8}$$

The fitting factor $\sigma(\rho)$ is deduced in a manner that the solution of eq. (7) uniformly converges to the solution of eq. (1)-eq. (2).

Multiplying eq. (8) by h and take the limit as $h \to 0$ (cf. Doolan et al. [4]), and then using eq. (3), we get

$$\sigma = \rho\frac{a(-1)}{12}\left(\coth\left(\left(\frac{a^2(-1)-\varepsilon b(-1)}{a(-1)}\right)\frac{\rho}{2}\right) + 5\frac{e^{-b(-1)\frac{\rho}{2}}}{\sinh\left(\left(\frac{a^2(-1)-\varepsilon b(-1)}{a(-1)}\right)\frac{\rho}{2}\right)}\right) \tag{9}$$

is a fitting factor in the interval $[-1,0]$.

The tridiagonal system of the eq. (8) can be written as follows:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,2,\ldots,l-1, \tag{10}$$

where

$$\begin{aligned}E_j &= \frac{\varepsilon\sigma}{h^2} - \frac{3a_{i-1}}{24h} + \frac{b_{i-1}}{12} + \frac{a_{i+1}}{24h},\\ F_j &= \frac{2\varepsilon\sigma}{h^2} - \frac{4a_{i-1}}{24h} - \frac{10b_i}{12\left(1-\frac{a_i\rho}{2}\right)} + \frac{4a_{i+1}}{24h} + \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)},\end{aligned}$$

$$G_j = \frac{\varepsilon\sigma}{h^2} - \frac{a_{i-1}}{24h} + \frac{b_{i+1}}{12} + \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)} + \frac{3a_{i+1}}{24h},$$
$$H_j = \frac{1}{12}\left(f_{i-1} + \frac{10}{\left(1-\frac{a_i\rho}{2}\right)} f_i + f_{i+1}\right).$$

Finally, we discuss our method in $[0,1]$ for the problems with right end boundary layer of the underlying interval.

In the interval $[0,1]$, from the theory of singular perturbation the zeroth order asymptotic approximation to the solution of eq. (1) is

$$\lim_{h\to 0} y(ih) \approx y_0(-1) + (\beta - y_0(1)\exp\left\{a(1)\left(\frac{1}{\varepsilon} - i\rho\right)\right\} \tag{11}$$

where $\rho = \frac{h}{\varepsilon}$.

Now, for the right end boundary layer, we approximate y'^{*}_{i+1}, y'^{*}_{i-1} using non standard finite differences and y'_i by modified backward finite difference

$$y'^{*}_{i+1} \approx \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h}, \quad y'_i \approx \frac{y_i - y_{i-1}}{h} + \frac{h}{2}y''_i, \quad y'^{*}_{i-1} \approx \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h}. \tag{12}$$

Substituting eq. (5) and eq. (12) in eq. (4) and exercising, we get

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}(-3y_{i-1} + 4y_i - y_{i+1}) + \frac{10a_i}{12h\left(1+\frac{a_i\rho}{2}\right)}(y_i - y_{i-1})$$
$$+ \frac{a_{i+1}}{24h}(y_{i-1} - 4y_i + 3y_{i+1}) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i\varepsilon}{12\left(1+\frac{a_i\rho}{2}\right)}y_i + \frac{b_{i+1}}{12}y_{i+1}$$
$$= \frac{1}{12}\left(f_{i-1} + \frac{10\varepsilon}{12\left(1+\frac{a_i\rho}{2}\right)}f_i + f_{i+1}\right).$$

Now inserting the fitting factor $\sigma(\rho)$ in this scheme, we get

$$\sigma\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{a_{i-1}}{24h}(-3y_{i-1} + 4y_i - y_{i+1}) + \frac{10a_i}{12h\left(1+\frac{a_i\rho}{2}\right)}(y_i - y_{i-1})$$
$$+ \frac{a_{i+1}}{24h}(y_{i-1} - 4y_i + 3y_{i+1}) + \frac{b_{i-1}}{12}y_{i-1} + \frac{10b_i\varepsilon}{12\left(1+\frac{a_i\rho}{2}\right)}y_i + \frac{b_{i+1}}{12}y_{i+1}$$
$$= \frac{1}{12}\left(f_{i-1} + \frac{10\varepsilon}{12\left(1+\frac{a_i\rho}{2}\right)}f_i + f_{i+1}\right). \tag{13}$$

Proceeding as in the left end layer and using eq. (11), we get the fitting factor as

$$\sigma = \rho\frac{a(0)}{12}\left(\coth\left(\left(\frac{a^2(1) - \varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right) + 5\frac{e^{b(1)\frac{\rho}{2}}}{\sinh\left(\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right)}\right) \tag{14}$$

which is the required fitting factor.

The tridiagonal system of the eq. (13) is given as:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = l+1, l+2, \ldots, N-1 \tag{15}$$

where

$$E_j = \frac{\varepsilon\sigma}{h^2} + \frac{a_{i+1}}{24h} + \frac{b_{i-1}}{12} - \frac{10a_i}{12h\left(1+\frac{a_i\rho}{2}\right)} - \frac{3a_{i-1}}{24h},$$

$$F_j = \frac{2\varepsilon\sigma}{h^2} + \frac{4a_{i+1}}{24h} - \frac{10b_i}{12\left(1+\frac{a_i\rho}{2}\right)} - \frac{4a_{i-1}}{24h} - \frac{10a_i}{12h\left(1+\frac{a_i\rho}{2}\right)}$$
$$G_j = \frac{\varepsilon\sigma}{h^2} + \frac{3a_{i+1}}{24h} + \frac{b_{i+1}}{12} - \frac{a_{i-1}}{24h},$$
$$H_j = \frac{1}{12}\left(f_{i-1} + \frac{10}{\left(1+\frac{a_i\rho}{2}\right)} f_i + f_{i+1}\right)$$

and σ is given by eq. (14).

Using eq. (10) in $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, l-1$; and eq. (15) in $[x_i, x_{i+1}]$ for $i = l+1, l+2, \ldots, N-1$, we have a system of $(N-2)$ equations with $(N+1)$ unknowns. From the given conditions eq. (2), we have two more equations. One more equation is required to solve for the unknowns $y_0, y_1, \ldots, y_N$. To have this equation, we consider the original differential equation eq. (1) at $x = x_l = 0$. Since $a(x) = 0$ at $x = x_l = 0$, eq. (1) reduces to

$$\varepsilon y''(x_l) + b(x_l)y = f(x_l) \tag{16}$$

Making use of the central finite difference for $y''(x)$ in eq. (16) at x_l, we get

$$[\varepsilon]y_{l-1} - [2\varepsilon - h^2 b_l]y_l + [\varepsilon]y_{l+1} = h^2 f_l \tag{17}$$

with the eq. (17), we solve the tridiagonal algebraic system eq. (10), eq. (15) by using an efficient and stable discrete invariant imbedding algorithm [1].

3. Convergence Analysis

The system of eq. (10) in matrix-vector form is

$$AY = C \tag{18}$$

in which $A = (m_{ij})$, $1 \le i, j \le l-1$ is a tridiagonal matrix with order $l-1$, where

$$m_{i_{i-1}} = \frac{\varepsilon\sigma}{h^2} - \frac{3a_{i-1}}{24h} + \frac{b_{i-1}}{12} + \frac{a_{i+1}}{24h},$$
$$m_{i_i} = \frac{2\varepsilon\sigma}{h^2} - \frac{4a_{i-1}}{24h} - \frac{10b_i}{12\left(1-\frac{a_i\rho}{2}\right)} + \frac{4a_{i+1}}{24h} + \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)},$$
$$m_{i_{i+1}} = \frac{\varepsilon\sigma}{h^2} - \frac{a_{i-1}}{24h} + \frac{b_{i+1}}{12} + \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)} + \frac{3a_{i+1}}{24h}$$

and $C = (d_i)$ is a column vector with

$$d_i = \frac{1}{12}\left(f_{i-1} + \frac{10}{\left(1-\frac{a_i\rho}{2}\right)} f_i + f_{i+1}\right), \quad \text{for } i = 1, 2, \ldots, l-1$$

with local truncation error

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{\frac{\sigma h^2 \varepsilon^2}{12}|y^{(4)}(x)|\right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{\frac{ah^2}{36}|y^{(3)}(x)|\right\}$$
$$+ \max_{x_{i-1}\le x\le x_{i+1}} \left\{\frac{10ah^2}{72\left(1-\frac{a_i\rho}{2}\right)}|y^{(3)}(x)|\right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{\frac{ah^2}{36}|y^{(3)}(x)|\right\}$$

i.e.,

$$|\tau_i| \le O(h^2). \tag{19}$$

The system of eq. (15) in matrix-vector form is

$$AY = C \tag{20}$$

in which $A = (m_{ij})$, $l+1 \le i,\ j \le N-1$ is a tridiagonal matrix of order $N-1$, with

$$m_{i_{i-1}} = \frac{\varepsilon\sigma}{h^2} - \frac{3a_{i-1}}{24h} + \frac{b_{i-1}}{12} + \frac{a_{i+1}}{24h} - \frac{10a_i}{12h\left(1+\frac{a_i\rho}{2}\right)},$$

$$m_{i_{i+1}} = \frac{\varepsilon\sigma}{h^2} - \frac{a_{i-1}}{24h} + \frac{b_{i+1}}{12} + \frac{3a_{i+1}}{24h},$$

$$m_{i_i} = \frac{2\varepsilon\sigma}{h^2} - \frac{4a_{i-1}}{24h} - \frac{10b_i}{12\left(1+\frac{a_i\rho}{2}\right)} + \frac{4a_{i+1}}{24h} - \frac{10a_i}{12h\left(1+\frac{a_i\rho}{2}\right)}$$

and $C = (d_i)$ is a column vector with

$$d_i = \frac{1}{12}\left(f_{i-1} + \frac{10}{\left(1+\frac{a_i\rho}{2}\right)}f_i + f_{i+1}\right), \quad \text{for } i = l+1\ (1)\ N-1$$

with local truncation error

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{\sigma h^2\varepsilon^2}{12}|y^{(4)}(x)|\right\} + \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{ah^2}{36}|y^{(3)}(x)|\right\}$$
$$+ \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{10ah^2}{72\left(1+\frac{a_i\rho}{2}\right)}|y^{(3)}(x)|\right\} + \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{ah^2}{36}|y^{(3)}(x)|\right\}$$

i.e., $|\tau_i| \le O(h^2)$ and $Y = (y_0, y_1, y_2, \dots, _N)^t$.

We also have

$$A\overline{Y} - T(h) = C, \tag{21}$$

where $\overline{Y} = (\overline{y}_0, \overline{y}_1, \dots, \overline{y}_N)^t$ is the actual solution and $T(h) = (T_0(h), T_1(h), \dots, T_N(h))^t$ is the truncation error.

From eq. (18), eq. (20) and eq. (21), we get

$$A(\overline{Y} - Y) = T(h). \tag{22}$$

Thus the error equation is

$$AE = T(h) \tag{23}$$

where $E = \overline{Y} - Y = (e_0, e_1, e_2, \dots, e_N)^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{i_j} = \frac{-\varepsilon\sigma}{h^2} + \frac{3a_{i-1}}{24h} - \frac{a_{i+1}}{24h} + \frac{b_{i+1}}{12} + \frac{10b_i}{12\left(1-\frac{a_i\rho}{2}\right)}, \quad \text{for } i = 1;$$

$$S_i = \sum_{j=1}^{N-1} m_{i_j} = \frac{b_i}{12}\left(2 + \frac{10}{\left(1-\frac{a_i\rho}{2}\right)}\right) + O(h^2) = B_{i_0}, \quad \text{for } i = 2, 3, \dots, N-2;$$

$$S_i = \sum_{j=1}^{N-1} m_{i_j} = \frac{-\varepsilon\sigma}{h^2} + \frac{1}{24h}(a_{i-1} - 3a_{i+1}) - \frac{10a_i}{12h\left(1-\frac{a_i\rho}{2}\right)} + \frac{10b_i}{12\left(1-\frac{a_i\rho}{2}\right)} + \frac{b_{i-1}}{12}, \quad \text{for } i = N-1.$$

Since $0 < \varepsilon \ll 1$, the matrix A is monotone and irreducible. Then, it follows that A^{-1} exists and its entries are non negative. Hence from eq. (23), we get

$$E = A^{-1}T(h) \tag{24}$$

and

$$\|E\| \le \|A^{-1}\| \cdot \|T(h)\| \tag{25}$$

Let $\bar{m}_{ki}$ be the $(ki)^{th}$ element of A^{-1}. Since $\bar{m}_{ki} \ge 0$, from the theory of matrices, we have

$$\sum_{i=1}^{N-1} \bar{m}_{ki} S_i = 1, \quad k = 1,2,\dots,N-1. \tag{26}$$

Therefore,

$$\sum_{i=1}^{N-1} \bar{m}_{ki} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{B_{i_o}} \le \frac{1}{|B_{i_o}|} \tag{27}$$

for some i_0 between 1 and $N-1$ and $B_{i_o} = \frac{b_i}{12}\left(2 + \frac{10}{\left(1-\frac{a_i\rho}{2}\right)}\right)$ which is constant.

We define

$$\|A^{-1}\| = \max_{1\le k\le N-1} \sum_{i=1}^{N-1} |\bar{m}_{ki}| \quad \text{and} \quad \|T(h)\| = \max_{1\le i\le N-1} |T_i(h)|.$$

Using eq. (19), eq. (24) and eq. (27), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{ki} T_i(h), \quad j = 1,2,3,\dots,N-1$$

implies

$$e_j \le \frac{O(h^2)}{|B_{i_o}|}, \quad j = 1\ (1)\ N-1 \tag{28}$$

Therefore, using eq. (28), we have

$$\|E\| = O(h^2)$$

i.e., the method is second order convergent on uniform mesh.

4. Numerical Illustrations

To demonstrate the applicability of proposed scheme computationally, we consider three examples. These problems have been widely discussed in the literature.

Example 1. $\varepsilon y''(x) - 2(2x-1)y'(x) - 4y(x) = 0$; $x \in (0,1)$ with $y(0) = 1$ and $y(1) = 1$. The exact solution of this problem is $y(x) = e^{-2x(1-x)/\varepsilon}$. This problem possesses two layers at $x = 0$ and $x = 1$. The maximum errors E_ε^N in computed solution are presented in Table 1. The layer behaviour with and without fitting factor is depict in Figures 1 and 2.

Example 2. Consider the singular perturbation problem $\varepsilon y''(x) - 2(2x-1)y'(x) - 4y(x) = 4(4x-1)$; $x \in (0,1)$ with $y(0) = 1$ and $y(1) = 1$. This problem have two boundary layers at $x = 0$ and at $x = 1$. The exact solution of the problem is not known. To find the maximum errors, we use double mesh principle [4], $G_\varepsilon^N = \max\limits_{x_i \in D_\varepsilon^N} \left|Y^N(x_i) - Y^{2N}(x_i)\right|$ and $G^N = \max\limits_{\varepsilon} G_\varepsilon^N$ where $Y^N(x_i)$ and $Y^{2N}(x_i)$ represent the numerical solutions with N and $2N$ mesh intervals respectively. These errors are shown in Table 2. Numerical solution is shown graphically in Figure 3.

Table 1. Maximum absolute errors in Example 1

$\varepsilon \setminus N$	16	32	64	128	256	512	1024
Proposed method							
10^{-5}	0.15402	0.069095	0.032901	0.016107	0.0080100	0.0040341	0.0020663
10^{-6}	0.15386	0.068978	0.032798	0.016011	0.0079155	0.0039395	0.0019690
10^{-7}	0.15385	0.068967	0.032788	0.016001	0.0079062	0.0039303	0.0019599
10^{-8}	0.15385	0.068966	0.032787	0.016000	0.0079052	0.0039294	0.0019590
10^{-9}	0.15385	0.068966	0.032787	0.016000	0.0079051	0.0039293	0.0019589
Results in Natesan et al. [9]							
10^{-5}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-6}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-7}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-8}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097
10^{-9}	0.1796	0.1178	0.0800	0.0495	0.0298	0.0172	0.0097

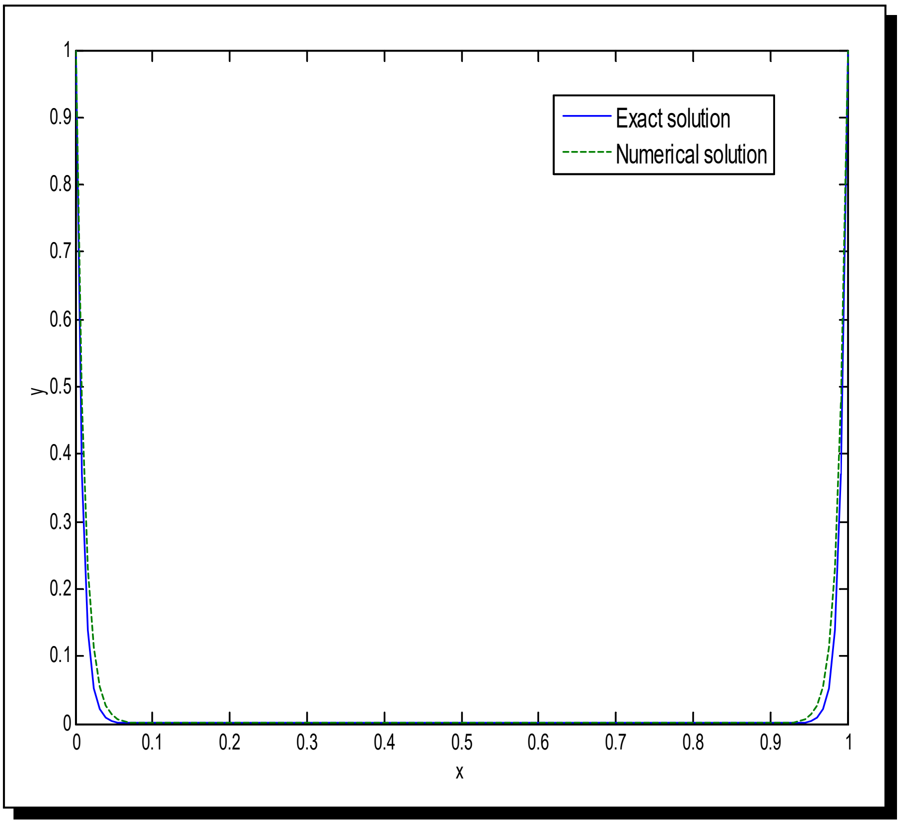

Figure 1. Exact and approximate solutions of Example 1 for $\varepsilon = 2^{-6}$ and $N = 64$ without fitting factor (artificial viscosity)

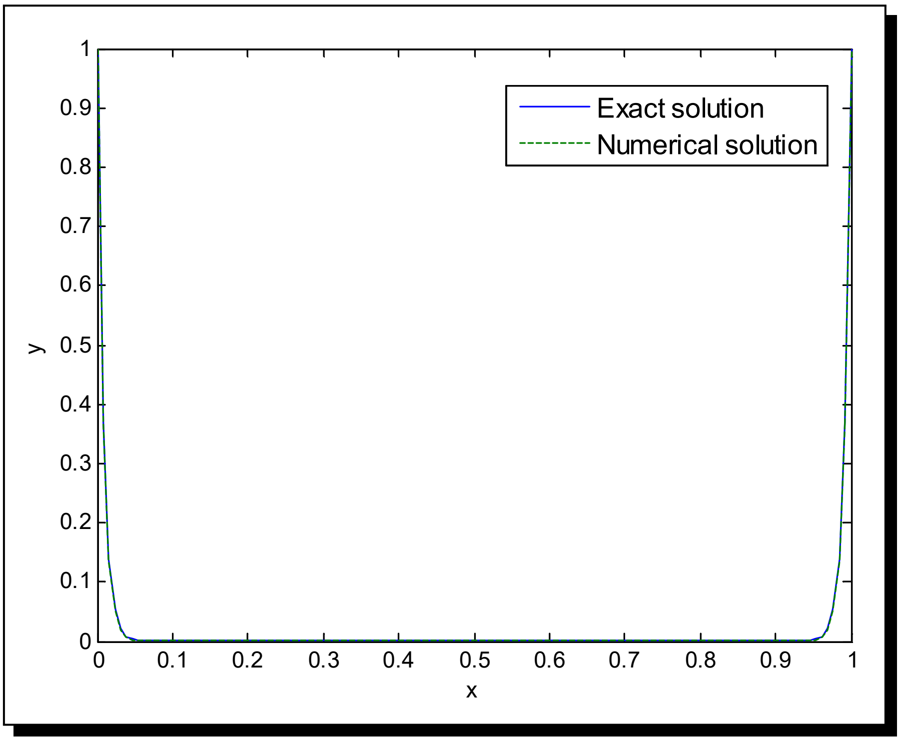

Figure 2. Exact solutions of Example 1 for $\varepsilon = 2^{-6}$ and $N = 64$ with fitting factor

Table 2. Maximum absolute errors in Example 2

$\varepsilon \setminus N$	16	32	64	128	256	512	1024
Proposed method							
10^{-5}	0.46947	0.20894	0.099095	0.048419	0.024054	0.012109	0.0062006
10^{-6}	0.46898	0.20858	0.098782	0.048126	0.023770	0.011824	0.0059085
10^{-7}	0.46893	0.20854	0.098751	0.048097	0.023742	0.011797	0.0058811
10^{-8}	0.46893	0.20854	0.098748	0.048097	0.023739	0.011794	0.0058783
10^{-9}	0.46893	0.20854	0.098748	0.048094	0.023739	0.011794	0.0058781
Results in Natesan et al. [9]							
10^{-5}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-6}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-7}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-8}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129
10^{-9}	0.1396	0.1129	0.0850	0.0601	0.0366	0.0225	0.0129

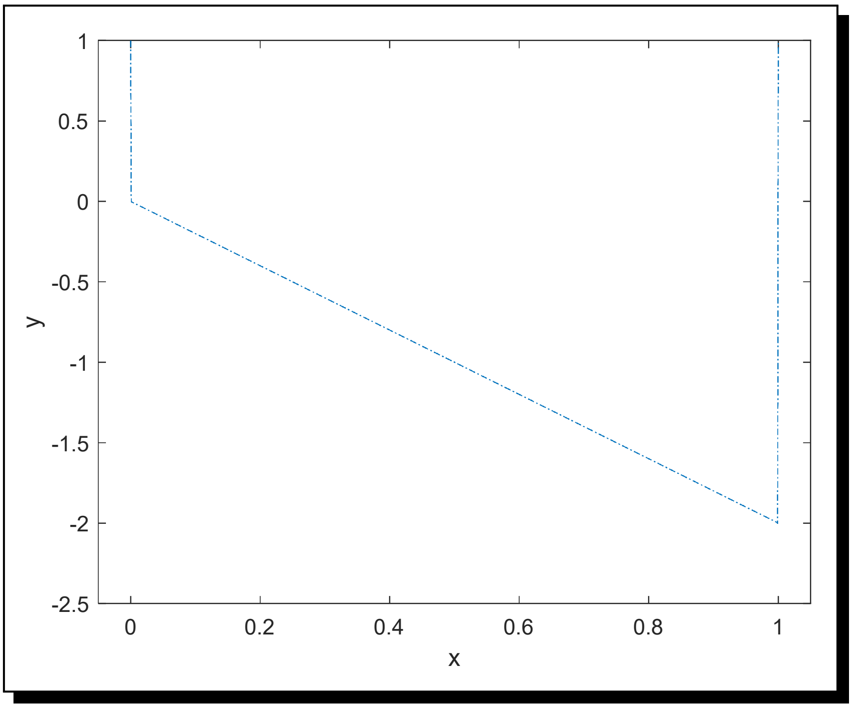

Figure 3. Numerical solution of Example 2 for $\varepsilon = 10^{-7}$ and $N = 2^{-9}$

5. Conclusions

We have proposed and demonstrated the implementation of an exponentially fitted operator finite difference method for singularly perturbed boundary value problems with twin layers. The method provides an alternative, supplementary technique to the conventional ways of solving singular perturbation problems. We have discussed convergence of the duduced method. To show the efficiency of the scheme, it is implemented on model examples having dual layer behaviour and compared the results with Natesan et al. [9]. The observation is that, the accuracy predicted is achieved with very modest computational effort with quadratic convergence. To show the importance of the fitting factor, we have presented the graphical solution of Example 1 with and without fitting factor.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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