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An Original Speech Around the Difficult Part of the Berge Conjecture (Solved by Chudnovsky, Robertson, Seymour and Thomas)

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Abstract. The *Berge conjecture* was proven by Chudnovsky, Robertson, Seymour and Thomas in a paper of 146 pages long (see [1]); in this manuscript, via an original speech and simple results, we rigorously simplify the understanding of this solved conjecture. It will appear that what Chudnovsky, Robertson, Seymour and Thomas were proved in their paper of 146 pages long, was an analytic conjecture stated in a very small class of graphs.

We say that a graph B is *berge* (see [4]) if every graph $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5 ($\bar{B}$ is the complementary graph of B). A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$, where $\chi(G')$ is the *chromatic number* of G' and $\omega(G')$ is the *clique number* of G'. The Berge conjecture states that a graph H is *perfect* if and only if H is *berge*. Indeed, the difficult part of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B (*Briefly, the difficult part of the Berge conjecture will be called the Berge problem* (see [4])). The *Hadwiger conjecture* (see [4]), states that every graph G satisfies $\chi(G) \leq \eta(G)$ (where $\eta(G)$ is the *hadwiger number* of G (*i.e.* the maximum of p such that G is *contractible* to the complete graph K_p)). In [4], it is presented an original investigation around the *Hadwiger conjecture* and the *Berge problem*. More precisely, in [4], via two simple Theorems, it is shown that the *Hadwiger conjecture* and the *Berge problem* are curiously resembling, so resembling that they seem identical (indeed, they can be restated in ways that resemble each other). In this paper, via only original speech and results, we rigorously simplify the understanding of the Berge problem. Moreover, it will appear that what Chudnovsky, Robertson, Seymour and Thomas were proved in their manuscript of 146 pages long, was an analytic conjecture stated in a very small class of graphs.

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1. Preliminary and Definitions

Recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \alpha(G), \bar{G}]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the *chromatic number* (i.e. the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color), $\omega(G)$ is the *clique number* of G (i.e. the size of a largest *clique* of G. Recall (see [2] for instance), that a graph F is a *subgraph* of G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph F is an *induced subgraph* of G by Z, if F is a subgraph of G such that $V(F) = Z$, $Z \subseteq V(G)$, and two vertices of F are adjacent in F, if and only if they are adjacent in G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G *induced* by $V(G) \setminus X$. A *clique* of G is a subgraph of G that is *complete*; such a subgraph is necessarily an induced subgraph (recall that a graph K is complete if every pair of vertices of K is an edge of K), $\alpha(G)$ is the *stability number* of G (i.e. the size of a largest *stable set* of G. Recall that a *stable set* of a graph G is a set of vertices of G that induces a subgraph with no edges), and $\bar{G}$ is the *complementary* graph of G (recall $\bar{G}$ is the complementary graph of G, if $V(G) = V(\bar{G})$ and two vertices are adjacent in G if and only if they are not adjacent in $\bar{G}$). We say that a graph B is *berge*, if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. The Berge conjecture states that a graph H is *perfect* if and only if H is *berge*. Indeed the *difficult part* of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B. Briefly, the *difficult part* of the Berge conjecture will be called the *Berge problem*. It is easy to see:

Assertion 1.1. *Let G be a graph and F be a subgraph of G. Then $\omega(G) \leq \chi(G)$ and $\chi(F) \leq \chi(G)$.*

That being so, this paper is divided into two simple sections. In Section 2, we introduce a graph parameter denoted by β (β is called the *berge index*), and, using this graph parameter, we recall the original version of the *Berge problem* known in [4]. After recalling the original version of the *Berge problem*, we introduce *uniform* graphs and *relative* subgraphs (uniform graphs and relative subgraphs are crucial for the proof of the result which rigorously simplify the understanding of the Berge problem, and the resulting analytic conjecture), and we give some elementary properties of these graphs. In Section 3, we define another graph parameter denoted by b (the graph parameter b is called the *berge caliber*, and is related to the berge index), and using the parameter b, we give the strong version of the *Berge problem*. This strong version is determinant and crucial for the simplification and the understanding of the *Berge problem*. Moreover, using this strong version of the *Berge problem*, it will immediately follow that what Chudnovsky, Robertson, Seymour and Thomas were proved in their manuscript of 146 pages long, was an analytic conjecture stated in a very small class of graphs. In this paper, every graph is finite, is simple and undirected.

2. The Berge Index of A Graph, The Original Version of The Berge Problem, Uniform Graphs and Relative Subgraphs

In this section, we introduce some important definitions that are not standard. In particular, we present a graph parameter called the *berge index* (and denoted by β), and we use it to recall the original version of the Berge problem known in [4].

Definition 2.1. We recall (see [4]) that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$. trpl(F) denotes the set of all true pals of F, and Ω denotes the set of all *complete multipartite graphs* ((*i.e.* the set of all graphs Q which are *complete* $\omega(Q)$*-partite* (Recall that a graph Q is a *complete* $\omega(Q)$*-partite* graph, if there exists a partition $\Xi(Q) = \{Y_1, \ldots, Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable sets, such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow x$ and y are adjacent in Q))). It is immediate that $\chi(Q) = \omega(Q)$ for all $Q \in \Omega$. It is also immediate that, for every $Q \in \Omega$, the partition $\Xi(Q) = \{Y_1, \ldots, Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable sets is canonical, and every $Y \in \Xi(Q)$ is a *maximal stable set* of Q.

So $G \in$ trpl(F) *means* G is a true pal of F, and $Q \in \Omega$ *means* Q is a complete $\omega(Q)$-partite graph (We *recall* (see above) that Ω is the set of all *complete multipartite graphs*. For example, if G is a multipartite graph with $\omega(G) = 1$, then $G \in \Omega$; if G is a multipartite graph with $\omega(G) = 2$, then $G \in \Omega$; if G is a multipartite graph with $\omega(G) = 3$, then $G \in \Omega$; $\ldots$ etc).

Now, using the previous definitions, then we have the following assertion.

Assertion 2.2. *Let F be a graph. Then, there exists a graph $P \in \Omega$ such that P is a true pal of F (i.e. there exists $P \in \Omega$ such that $P \in$ trpl(F)).*

Proof. Indeed, let F be a graph and let $\Xi(F) = \{Y_1, \ldots, Y_{\chi(F)}\}$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $V(Q) = V(F)$; (ii) $\Xi(Q) = \{Y_1, \ldots, Y_{\chi(F)}\}$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow x$ and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$). Now put $Q = P$; Assertion 2.2 follows. $\square$

Using Assertion 2.2, let us define

Definition 2.3. We say that a graph P is a *parent* of a graph F, if $P \in \Omega \bigcap$ trpl(F). In other words, a graph P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F (note that such a P clearly exists, via Assertion 2.2). *parent*(F) denotes the set of all parents of F.

So $P \in$ *parent*(F) *means* P is a parent of F. Using the definition of a parent, the definition of a true pal (see Definition 2.1), the definition of a *berge graph* (see

Preliminary and Definitions) and the definition of Ω (see Definition 2.1), then the following two Assertions are immediate.

Assertion 2.4. *Let F be a graph and let $P \in \text{parent}(F)$; then $\chi(F) = \chi(P) = \omega(P)$.*

Assertion 2.5. *Let $G \in \Omega$. Then, G is berge.*

Assertion 2.6. *Assertion* 2.5 *says that the set Ω is an obvious example of berge graphs. Curiously the set Ω is fundamental for the rigorous simplification of the Berge problem.*

Now, we redefine the *berge index* known in [4].

Definition 2.7. Let G be a graph; put $\mathscr{B}(G) = \{F; G \in parent(F)$ and F is $berge\}$ (i.e., $\mathscr{B}(G)$ is the set of graphs F such that G is a parent of F and F is berge), and let $parent(G)$ be the set of all parents of G. Then the *berge index* of G (see [4]) is denoted by $\beta(G)$, where $\beta(G)$ is defined as follows. $\beta(G) = \min_{F \in \mathscr{B}(G)} \omega(F)$ *if* $G \in \Omega$; and $\beta(G) = \min_{P \in parent(G)} \beta(P)$ *if* $G \notin \Omega$.

Using Assertion 2.3 and Assertion 2.5, then it becomes easy to see.

Assertion 2.8 ([4])**.** *Let G be a graph; then the berge index $\beta(G)$ exists and is well defined.*

Using the definition of the graph parameter β, then the following two assertions are immediate.

Assertion 2.9. *Let B be berge and let P be a parent of B; then $\beta(P) \leq \omega(B)$.*

Assertion 2.10. *Let K be a complete graph; then $\beta(K) = \omega(K) = \chi(K)$.*

Now the following Theorem is the original version of the *Berge problem*.

Theorem 2.11 ([4], The original version of the Berge problem)**.** *The following are equivalent*

(i) *The Berge problem holds* (*i.e., for every berge graph B, we have $\chi(B) = \omega(B)$*)*.*
(ii) *For every graph F, we have $\chi(F) = \beta(F)$.*
(iii) *For every $G \in \Omega$, we have $\omega(G) = \beta(G)$.*

In section 3, *the strong version of the Berge problem* which is crucial for the rigorous simplification of the Berge problem, will be implicitly based on the original version of the Berge problem given by Theorem 2.11. Now we are going to define a new class of graphs in Ω (called *uniform graphs*); we will also define *relative subgraphs*, and we will present some properties related to these graphs. These properties are elementary and curiously, are also crucial for the result which rigorously simplifies the Berge problem and the resulting analytic conjecture. Before, let us define.

Definition 2.12. An *optimal coloration* of a graph G is a partition $\Xi(G) = \{Y_1, \ldots, Y_{\chi(G)}\}$ of $V(G)$ into $\chi(G)$ stable sets; $\ominus(G)$ denotes the set of all optimal colorations of G.

So, $\Xi(G) \in \ominus(G)$ *means* $\Xi(G)$ is an optimal coloration of G.

Definition 2.13. Let G be a graph and let $\Xi(G) \in \ominus(G)$. We say that $\Xi(G)$ is *the canonical coloration* of G, if and only if $\ominus(G) = \{\Xi(G)\}$ (observe that such a canonical coloration does not always exists).

Using the definition of $\ominus(G)$ (see Definition 2.12), then the following Assertion is immediate.

Assertion 2.14. *Let* $G \in \Omega$ *and let* $\Xi(G) \in \ominus(G)$. *Then* $\ominus(G) = \{\Xi(G)\}$ (*i.e.,* $\Xi(G)$ *is* the *canonical coloration of G, via Definition* 2.13).

So, let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$; then Assertion 2.14 clearly says that $\Xi(G)$ is the canonical coloration of G (indeed, we have no choice, since $\ominus(G) = \{\Xi(G)\}$).

Definition 2.15. For a set X, $\text{card}(X)$ is the cardinality of X. Now let $G \in \Omega$, and let $\Xi(G) \in \ominus(G)$ (note $\Xi(G)$ is the canonical coloration of G, via Assertion 2.14); then we define $T(G)$, $t(G)$, $N_2(G)$ and $n(G)$ as follows: $T(G) = \{x \in V(G); \exists Y \in \Xi(G)$, and $Y = \{x\}\}$, $t(G) = \text{card}(T(G))$, $N_2(G) = \{Y \in \Xi(G); \text{card}(Y) \geq 2\}$, and $n(G) = \text{card}(N_2(G))$.

It is immediate that these definitions make sense, since $G \in \Omega$, and so $\Xi(G)$ is the canonical coloration of G, via Assertion 2.14. Using Definition 2.15, we clearly have:

Assertion 2.16. *Let* $G \in \Omega$ *and let* $T(G)$. *Now put* $G' = G \setminus T(G)$ (*note* G' *is the induced subgraph of* G *by* $V(G) \setminus T(G)$). *Then we have the following three properties*

(2.16.1) $G' \in \Omega$.

(2.16.2) $\omega(G') = n(G)$.

(2.16.3) $\omega(G) = n(G) + t(G) = \omega(G') + t(G)$.

Now we define *uniform* graphs and *relative* subgraphs.

Definition 2.17 (*Fundamental* 1)**.** Let G be a graph and let $\Xi(G) \in \ominus(G)$ (see Definition 2.12); we say that G is *uniform*, if $G \in \Omega$ and for all $Y \in \Xi(G)$, we have $\text{card}(Y) = \alpha(G)$.

The previous definition makes sense, since $G \in \Omega$ and so $\Xi(G)$ is canonical (by using Assertion 2.14). Using the definition of a uniform graph, then the following three assertions are immediate.

Assertion 2.18. *If G is a complete graph or if* $V(G) = \emptyset$ *or if* $\omega(G) \leq 1$, *then G is uniform.*

Assertion 2.19. *Let G be uniform. We have the following three properties*

(2.19.1) *If G is a complete graph or if $\alpha(G) \leq 1$ or if $n(G) = 0$, then $\omega(G) = t(G)$.*

(2.19.2) *If G is not a complete graph or if $\alpha(G) > 1$ or if $n(G) \neq 0$, then $\omega(G) = n(G)$.*

(2.19.3) *If $t(G) \neq 0$, then G is a complete graph.*

Assertion 2.20. *Let F be a graph (F is not necessarily in Ω); then there exist a uniform graph P such that P is a parent of F (see Definition 2.3 for the meaning of parent).*

Proof. *If* $\omega(F) \leq 1$, clearly F is uniform (use Assertion 2.18); now put $P = F$, clearly P is a uniform graph which is a parent of F. Now, *if* $\omega(F) \geq 2$, let $\Xi(F) = \{Y_1, \ldots, Y_{\chi(F)}\}$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $\Xi(Q) = \{Z_1, \ldots, Z_{\chi(F)}\}$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that, $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q; (ii) For every $j = 1, 2, \ldots, \chi(F)$ and for every $Z_j \in \Xi(Q) = \{Z_1, \ldots, Z_{\chi(F)}\}$, $\text{card}(Z_j) = \alpha(F)$. Clearly $Q \in \Omega$, $\text{card}(V(Q)) = \chi(F)\alpha(F)$, Q is uniform, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly “isomorphic" to a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is “isomorphic" to a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$) and Q is uniform. Using the previous and the definition of a parent, then we immediately deduce that Q is a uniform graph which is a parent of F. Now put $Q = P$; Assertion 2.20 follows. □

Definition 2.21 (*Fundamental* 1)**.** Let G and F be uniform. Let $\ominus(G) = \{\Xi(G)\}$ and let $\ominus(F) = \{\Xi(F)\}$. We say that F is a *relative subgraph* of G, *if* $\Xi(F) \subseteq \Xi(G)$.

It is immediate that this definition makes sense, since in particular $(G, F) \in \Omega \times \Omega$ [because G and F are uniform], and so $\Xi(G)$ and $\Xi(F)$ are canonical (i.e. $\ominus(G) = \{\Xi(G)\}$ and $\ominus(F) = \{\Xi(F)\}$, via Definition 2.13 and Assertion 2.14). *It is also immediate that relative subgraphs are defined for uniform graphs, and only for uniform graphs.* Using the definition of a relative subgraph, then the following three assertions are immediate.

Assertion 2.22. *Let R and P be uniform such that $\omega(P) \geq 1$ and $\omega(R) \geq 1$. If R is a relative subgraph of P, then $\alpha(R) = \alpha(P)$ and $\omega(R) \leq \omega(P)$.*

Assertion 2.23. *Let P be uniform and let R be a relative subgraph of P. We have the following three properties*

(2.23.1) *$\alpha(R) = 0$ or $\alpha(R) = \alpha(P)$ (the law of all or nothing).*

(2.23.2) if *$\alpha(R) \geq 1$ or if $\omega(R) \geq 1$, then $\alpha(R) = \alpha(P)$ and $\omega(R) \leq \omega(P)$.*

(2.23.3) *R is uniform.*

Assertion 2.24. *Let G be uniform and let R be a relative subgraph of G. Now let $\Xi(G)$ be the canonical coloration of G, and let $\Xi(R)$ be the canonical coloration of R. Then we have the following two properties*

(2.24.1) $\omega(R) = \omega(G) \Leftrightarrow R = G \Leftrightarrow \Xi(R) = \Xi(G) \Leftrightarrow V(R) = V(G)$.

(2.24.2) $\omega(R) < \omega(G) \Leftrightarrow$ *there exists* $Y \in \Xi(G)$ *such that* R *is a relative subgraph of* $G \setminus Y$.

We will see in Section 3 that uniform graphs play a major role in the proof of Theorem which rigorously simplifies the Berge problem and the resulting analytic conjecture. That being so, uniform graphs have also nice properties related to *isomorphisms* (recall that two graphs are *isomorphic* if there exists a one to one correspondence between their vertex set that preserves adjacency). Using the definition of a uniform graph, the definition of a relative subgraph and the definition of isomorphism of graphs, then the following two assertions are immediate.

Assertion 2.25 (*Natural Isomorphism* 1)**.** *Let* P *and* Q *be uniform; then the following are equivalent*

(i) P *is isomorphic to* Q.

(ii) $\omega(P) = \omega(Q)$ *and* $\alpha(P) = \alpha(Q)$.

Assertion 2.26 (*Natural Isomorphism* 2)**.** *Let* R *and* P *be uniform such that* $\omega(P) \geq 1$ *and* $\omega(R) \geq 1$. *Then the following are equivalent*

(i) $\omega(P) \geq \omega(R)$ *and* $\alpha(P) = \alpha(R)$.

(ii) R *is isomorphic to a relative subgraph of* P.

Some of the previous elementary assertions will help us to give the strong version of the Berge problem, and to deduce the analytic conjecture that was solved implicitly by Chudnovsky, Robertson, Seymour and Thomas.

3. The Strong Version of The Berge Problem and The Statement of The Analytic Conjecture that Was Solved Implicitly by Chudnovsky, Robertson, Seymour and Thomas

Here, we use again definitions that are not standard; in particular, we introduce a graph parameter denoted by b and called the *berge caliber* (the *berge caliber* b is related to the *berge index* β introduced in Section 2), and using the parameter b, we give the strong version of the Berge problem. This strong version, immediately implies that, what was solved Chudnovsky, Robertson, Seymour and Thomas, was an analytic conjecture stated on uniform graphs (see Definition 2.17 for the meaning of uniform graphs). Before, let us define

Definition 3.1 (*Fundamental* 3)**.** We say that a graph G is *bergerian*, *if* G is uniform and *if* $\omega(G) = \beta(G)$.

Using Assertion 2.10, then it becomes immediate to see that every complete graph is bergerian. So the set of all complete graphs is an obvious example of bergerian graphs.

Definition 3.2 (*Fundamental* 4)**.** Let G be uniform. We say that a graph F is a *bergerian subgraph* of G, if F is *bergerian* and is a relative subgraph of G. We say that F is a *maximal bergerian* subgraph of G (we recall that G is uniform), *if* F is a *bergerian subgraph* of G and $\omega(F)$ is *maximum* for this property (it is immediate that such a F exists and is well defined).

Now we define the *berge caliber*.

Definition 3.3 (*Fundamental* 5)**.** Let G be uniform, and let F be a *maximal bergerian* subgraph of G (see Definition 2.3), then the *berge caliber* of G is denoted by $b(G)$, where $b(G) = \omega(F)$.

It is immediate that $b(G)$ exists and is well defined. *It is also immediate that the berge caliber* (*i.e. the graph parameter* b) *is defined for uniform graphs and only for uniform graphs*. Now, using the definition of a uniform graph, the definition of a relative subgraph and the definition of the berge caliber, then the following two assertions are immediate.

Assertion 3.4. *Let G be uniform and let $b(G)$ be the berge caliber of G. Consider $\beta(G)$ ($\beta(G)$ is berge index of G). We have the following five properties*

(3.4.1) $\omega(G) \geq b(G)$.

(3.4.2) *G is bergerian* $\Longleftrightarrow \beta(G) = \omega(G) = b(G)$.

(3.4.3) *G is not bergerian* $\Longleftrightarrow \omega(G) > b(G) \Longleftrightarrow \omega(G) \neq \beta(G) \Longleftrightarrow \omega(G) \neq b(G)$.

(3.4.4) *If $\omega(G) \leq 1$ or if $\alpha(G) \leq 1$ or if G is a complete graph or if $b(G) = 0$, then G is bergerian.*

(3.4.5) *If $\omega(G) \geq 1$, then $b(G) \geq 1$.*

Property (3.4.4) of Assertion 3.4 gives classical example of bergerian graphs.

Assertion 3.5. *Let G be uniform and let R be a relative subgraph of G. Now let $b(G)$ be the berge caliber of G, and let $b(R)$ be the berge caliber of R. Then we have the following two simple properties*

(3.5.1) $b(R) \leq b(G)$.

(3.5.2) *If G is bergerian, then R is also bergerian (in other words, if $\omega(G) = b(G)$, then $\omega(R) = b(R)$ (see property* (3.4.2) *of Assertion* 3.4).

Definition 3.6 (*Fundamental* 6)**.** Let G be uniform, and let $b(G)$ be the *berge caliber* of G (see Definition 3.3); then we define f_G as follows:

$$f_G = 48b(G)^{b(G)} + 1000\,.$$

It is immediate that f_G exists and is well defined. Now the following Theorem is the strong version of the Berge problem, this strong version is implicitly based on the original version of the Berge problem given by Theorem 2.11.

Theorem 3.7 (*The strong version of the Berge problem*)**.** *The following are equivalent*

(i) *For every uniform graph U, we have $\omega(U) = b(U)$.*
(ii) *Every uniform graph Q is bergerian.*
(iii) *For every uniform graph G, we have $f_G^4 \geq \omega(G)$.*

Using Assertion 2.25 or Assertion 2.26, then the following proposition becomes easy to prove

Proposition 3.8. *Let Q be uniform and let R be a relative subgraph of Q. Now let $b(Q)$ be the berge caliber of Q, and let $b(R)$ be the berge caliber of R.* if $\omega(R) > b(R)$ (i.e. if *R is not bergerian*), *then $b(Q) = b(R)$.*

Proof. Otherwise (we reason by reduction to absurd), observing that $Q \in \Omega$, and since R is a relative subgraph of Q, then property (3.5.1) of Assertion 3.5 implies that $b(Q) > b(R)$. Now, let R' be a relative subgraph of R such that $\omega(R') = b(R)+1$ (*It is immediate that such a R' exists, since $\omega(R) > b(R)$*); clearly R' is a relative subgraph of the uniform graph Q (because R' is a relative subgraph of R and R is a relative subgraph of the uniform graph Q) and $\omega(R') \geq 1$. So R' is a relative subgraph of the uniform graph Q and $\omega(R') \geq 1$; properties (2.23.2) and (2.23.3) of Assertion 2.23 imply that $\alpha(R') = \alpha(Q)$ and R' is uniform. Now, let F be a maximal bergerian subgraph of Q; observing that $\omega(F) = b(Q)$ (because F is a maximal bergerian subgraph of Q), clearly $\omega(F) \geq \omega(R')$ (since we have seen above that $b(Q) > b(R)$ and $\omega(R') = b(R) + 1$). In particular F is a relative subgraph of Q (because F is a maximal bergerian subgraph of Q), now, observing that $\omega(F) \geq 1$ (since we have seen above that $\omega(F) \geq \omega(R')$ and $\omega(R') \geq 1$), clearly F is a relative subgraph of the uniform graph Q and $\omega(F) \geq 1$; properties (2.23.2) and (2.23.3) of Assertion 2.23 imply that $\alpha(F) = \alpha(Q)$ and F is uniform. Now, consider R' and F, by using the previous, we easily deduce that R' and F are uniform such that $\alpha(F) = \alpha(R') = \alpha(Q)$ and $\omega(F) \geq \omega(R') \geq 1$; clearly R' and F are uniform, and $\alpha(F) = \alpha(R')$ and $\omega(F) \geq \omega(R') \geq 1$, then, by using Assertion 2.26 (Natural Isomorphism 2), it follows that R' is isomorphic to a relative subgraph of F. Since in particular F is bergerian (because F is a maximal bergerian subgraph of Q), clearly R' is isomorphic to a relatif subgraph of F and F is bergerian; then by applying property (3.5.2) of Assertion 3.5, it follows that R' is also bergerian. Therefore $b(R') = \omega(R')$, and clearly $b(R') = b(R) + 1$ (since we he seen above that $\omega(R') = b(R) + 1$). Now, recalling that R' is a relative subgraph of R, then property (3.5.1) of Assertion 3.5 implies that $b(R') \leq b(R)$. This last inequality is impossible, since $b(R') = b(R) + 1$. Proposition 3.8 follows. □

***Proof of Theorem* 3.7.** It is immediate that (i)⇒(ii) and it also immediate that (ii)⇒(iii).

(iii)⇒(i). Assume otherwise (we reason by reduction to absurd). Let U be uniform such that, $\omega(U) \neq b(U)$, clearly (by using property (3.4.1) of Assertion 3.4), we have

$$\omega(U) > b(U) \tag{3.1}$$

fix once and for all U (U is fixed once and for all; so U does not move anymore); then we have these simple remarks. □

Remark 3.9. U is uniform such that $\omega(U) > 1$ and $\alpha(U) > 1$ and $b(U) \geq 1$.

Otherwise, clearly U is uniform such that $\omega(U) \leq 1$ or $\alpha(U) \leq 1$ or $b(U) = 0$, and using property (3.4.4) of Assertion 3.4, we easily deduce that U is bergerian; therefore $\omega(U) = b(U)$ and this contradicts (3.1).

Remark 3.10. Look at the uniform graph U and consider f_U^4 (see Definition 3.6 for the meaning of f_U). Then there exists a uniform graph U' such that $\omega(U') = 3000[\omega(U) + 1000f_U^4]$ and $\alpha(U') = \alpha(U)$.

Indeed, observing (by using Remark 3.9) that U is uniform such that $\alpha(U) > 1$, then it becomes immediate to deduce that there exists a uniform graph U' such that $\omega(U') = 3000[\omega(U) + 1000f_U^4]$ and $\alpha(U') = \alpha(U)$.

Remark 3.11. Look at the uniform graph U and let the uniform graph U' defined in Remark 3.10. Then U is isomorphic to a relative subgraph of U'.

Indeed, noticing (by using Remark 3.9) that $\omega(U) > 1$, then, using the previous inequality and the definition of U', we immediately deduce that $\omega(U') > \omega(U) > 1$ and $\alpha(U') = \alpha(U)$; clearly U and U' are uniform such that $\alpha(U) = \alpha(U')$ and $\omega(U') > \omega(U) > 1$; now using the previous and Assertion 2.26, we easily deduce that U is isomorphic to a relative subgraph of U'. Remark 3.11 follows

Remark 3.12. Look at the uniform graph U and let the uniform graph U' defined in Remark 3.10. Then $b(U') = b(U)$.

Indeed, observing (by (3.1)) that $\omega(U) > b(U)$ and remarking (by Remark 3.11) that U is isomorphic to a relative subgraph of U', then, using Proposition 3.8, we immediately deduce that $b(U') = b(U)$. Remark 3.12 follows

Remark 3.13. Let the uniform graph U' defined in Remark 3.10. Then $\omega(U') \leq f_{U'}^4$.

Indeed, observing that U' is uniform, then U' satisfies property (iii) of Theorem 3.7 and therefore $\omega(U') \leq f_{U'}^4$. Remark 3.13 follows

Remark 3.14. Look at the uniform graph U and let the uniform graph U' defined in Remark 3.10. Then $\omega(U') \leq f_U^4$.

Indeed observing (by Remark 3.12) that $b(U) = b(U')$, then the previous equality immediately implies that $f_U = f_{U'}$ and therefore

$$f_U^4 = f_{U'}^4. \tag{3.2}$$

Now using equality (3.2) and Remark 3.13, we easily deduce that $\omega(U') \leq f_U^4$. Remark 3.14 follows

These simple remarks made, let U' be the uniform graph introduced in Remark 3.10, then (by Remark 3.14) $\omega(U') \leq f_U^4$; observing (by the definition of U') that $\omega(U') = 3000[\omega(U) + 1000 f_U^4]$, then the previous inequality immediately becomes $3000[\omega(U) + 1000 f_U^4] \leq f_U^4$; and this is clearly impossible, since $\omega(U) > 1$ (by using Remark 3.9) and $f_U^4 > 1$ (by using the definition of f_U). So, assuming that property (iii) of Theorem 3.7 is true and property (i) of Theorem 3.7 is false gives rise to a serious contradiction. So (iii)⇒(i)). Theorem 3.7 clearly follows

Corollary 3.9. *If for every uniform graph U, we have $f_U^4 \geq \omega(U)$, then for every berge graph B, we have $\chi(B) = \omega(B)$.*

Proof. Indeed, let B be berge and let Q be a uniform graph such that Q is a parent of B (it is immediate that such a Q exists, by Assertion 2.20); then using Assertion 2.9, we immediately deduce that

$$\beta(Q) \leq \omega(B). \tag{3.3}$$

Now, observing (by the hypotheses) that for every uniform graph U, we have $f_U^4 \geq \omega(U)$, then, using Theorem 3.7, we immediately deduce that

$$\text{every uniform graph } U \text{ is bergerian.} \tag{3.4}$$

Using property (3.4.2) of Assertion 3.4, then we immediately deduce that (3.4) implies that

$$\text{for every uniform graph } U, \text{ we have } \omega(U) = b(U) = \beta(U). \tag{3.5}$$

Recalling that Q is uniform, then, using (3.5), we immediately deduce that inequality (3.3) is of the form

$$\omega(Q) \leq \omega(B). \tag{3.6}$$

Now, observing that $\omega(Q) = \chi(B)$ (note that Q is a parent of B and use Assertion 2.4), then, inequality (3.6) immediately becomes $\chi(B) \leq \omega(B)$, and the last inequality implies that $\chi(B) = \omega(B)$. □

Corollary 3.10. *If for every uniform graph U, we have $f_U^4 \geq \omega(U)$, then the Berge problem holds.*

Proof. Use Corollary 3.9 and definition of Berge problem. □

Corollary 3.10 *immediately gives the statement of an analytic conjecture on uniform graphs. Indeed, Corollary* 3.10 *clearly says that, if for every uniform U, we have $f_U^4 \geq \omega(U)$, then, the Berge problem follows*. Using Corollary 3.10 and Theorem 3.7, then the following corollary becomes immediate.

Corollary 3.11. *The following are equivalent*

(i) *For every uniform graph U, we have $\omega(U) = b(U)$.*
(ii) *Every uniform graph Q is bergerian.*
(iii) *For every uniform graph G, we have $f_G^4 \geq \omega(G)$*
(iv) *For every uniform graph U, we have $f_U^4 \geq \omega(U)$*
(v) *The Berge problem holds.*

From Corollary 3.11, it immediately follows that Chudnovsky, Robertson, Seymour and Thomas were implicitly proved in their paper of 146 pages long (see [1]) the following analytic conjecture stated on uniform graphs.

Conjecture 3.12. *For every uniform graph U, we have $f_U^4 \geq \omega(U)$, where $f_U = 48b(U)^{b(U)} + 1000$, and where $b(U)$ is the berge caliber of U.*

It is easy to show that Conjecture 3.12 is equivalent to the following:

Conjecture 3.13. *There exists a fixed positive integer t such that for every uniform graph U, we have $\omega(U) \leq b(U) + t$, where $b(U)$ is the berge caliber of U.*

So, to give an elementary proof of the Berge problem (and at the same time an elementary proof of the Berge conjecture), it suffices to show that: *for every uniform graph U, we have $f_U^4 \geq \omega(U)$* or *there exists a fixed positive integer t such that for every uniform graph U, we have $\omega(U) \leq b(U) + t$.*

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