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Research Article

Composite Generalized Variational Inequalities With Wiener-Hopf Equations

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Abstract. An introduction and study of a composite generalized variational inequality problem with a composite Wiener-Hopf equation in separable real Hilbert space is performed. Projection operator technique has been employed, to establish the equivalence between the composite generalized variational inequality problem with a composite Wiener-Hopf equation. Equivalent formulation discuss the existence of solution of the problem. Under some specific conditions, the convergence analysis of the suggested iterative algorithm has been discussed. The paper also specks that the problem and results obtained are more general than the papers that are already available in the literature.

Keywords. Algorithms; Composite variational inequalities; Composite Wiener-Hopf equation; Convergence analysis; Monotone operators

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1. Introduction

Variational inequality is an inequality that includes a functional, that has to be solved for all possible values of a given variable pertaining to a convex set. Initial development of theory of variational inequalities lies in dealing of equilibrium problems, especially the Signorini problem. Subsequently, it was diversified in general sense to study a vast range of problems that appear in mechanics, optimization and control, non-linear programming, economics, finance, game theory and applied sciences etc, see e.g., [2–5, 7–11, 16, 17] and references therein.

Introduction of new and powerful methods in variational inequality theory made it to work authoritatively in several directions, which include a study of a wide class of unrelated problems in a unified and general framework as well. Development of an efficient and executable iterative algorithm for solving variational inequalities is one of toughest tasks in this theory. There exist a considerable number of iterative methods for solving variational inequalities. Among the various efficacious methods, the projection technique and its variant forms is the most successful one.

In this article we introduce and study a composite generalized variational inequality problem with a composite Wiener-Hopf equation in separable real Hilbert space. By using projection operator technique we establish the equivalence between the composite generalized variational inequality problem with a composite Wiener-Hopf equation. Under some specific conditions, the convergence analysis of the suggested iterative algorithm has been discussed.

2. Preliminaries

Let $\mathcal{H}$ be a separable real Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and K be a nonempty closed convex set in $\mathcal{H}$ and P_K is the projection mapping of $\mathcal{H}$ into K.

Definition 2.1. An operator $F : K \to \mathcal{H}$ is called

(i) monotone if,
$$\langle Fp - Fq, p - q\rangle \geq 0, \ \forall\ p, q \in K;$$

(ii) strongly monotone if, there exists a constant $\xi > 0$ such that
$$\langle Fp - Fq, p - q\rangle \geq \xi\|p - q\|^2, \ \forall\ p, q \in K;$$

(iii) η-expansive if, there exists $\eta > 0$ such that
$$\|Fp - Fq\| \geq \eta\|p - q\|, \ \forall\ p, q \in K;$$

(iv) μ-cocoercive if, there exists $\mu > 0$ such that
$$\langle Fp - Fq, p - q\rangle \geq \mu\|Fp - Fq\|^2, \ \forall\ p, q \in K;$$

(v) relaxed γ-cocoercive if, there exists $\gamma \geq 0$ such that
$$\langle Fp - Fq, p - q\rangle \geq (-\gamma)\|Fp - Fq\|^2, \ \forall\ p, q \in K;$$

(vi) relaxed (γ, r)-cocoercive if, there exist $\gamma, r > 0$ such that
$$\langle Fp - Fq, p - q\rangle \geq (-\gamma)\|Fp - Fq\|^2 + r\|p - q\|^2, \ \forall\ p, q \in K.$$

Remark 2.1. (i) If $\eta = 1$, then A is called expansive;

(ii) Every μ-cocoercive mapping is $\frac{1}{\mu}$-Lipschitz continuous.

Definition 2.2. The set valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is called

(i) relaxed monotone operator if, there exists a constant $\nu > 0$ such that
$$\langle w_1 - w_2, p - q\rangle \geq (-\nu)\|p - q\|^2, \ \forall\ w_1 \in T(p) \text{ and } w_2 \in T(q).$$

(ii) The set-valued mapping $T:\mathcal{H}\to 2^{\mathcal{H}}$ is λ-Lipschitz continuous if, there exists $\lambda>0$ such that

$$\|w_1-w_2\|\le\lambda\|p-q\|,\ \forall\ w_1\in T(p) \text{ and } w_2\in T(q).$$

Definition 2.3. A single-valued mapping $S:K\to K$ is called

(i) nonexpansive if,

$$\|Sp-Sq\|\le\|p-q\|,\ \forall\ p,q\in K.$$

(ii) strictly pseudo-contractive if, there exists $k\in[0,1)$ such that

$$\|Sp-Sq\|^2\le\|p-q\|^2+k\|(I-S)p-(I-S)q\|^2,\ \forall\ p,q\in K.$$

3. Formulation of the Problem and Iterative Algorithm

Let $\mathcal{H}$ be a separable real Hilbert space and K be a nonempty closed convex set in $\mathcal{H}$. Let $A,F:K\to\mathcal{H}$ and $g:K\to K$ be the single-valued continuous nonlinear mappings. Suppose $T:\mathcal{H}\to 2^{\mathcal{H}}$ is a set valued mapping. We consider the following composite generalized variational inequality problem (CGVIP) of finding $p\in\mathcal{H}$ such that $g(p)\in K$ and

$$\langle Aog(p)+F(w),g(q)-g(p)\rangle\ge 0,\ \forall\ g(q)\in K \text{ and } w\in T(p). \tag{3.1}$$

The solution set of CGVIP (3.1) is denoted by $VI(K,A,F,T,g)$.

Special cases:

(i) If $F,g=I$ (identity mapping), then CGVIP (3.1) is equivalent to finding $p\in K$ such that

$$\langle Ap+w,q-p\rangle\ge 0,\ \forall\ q\in K \text{ and } w\in Tp. \tag{3.2}$$

Problem (3.2) has been introduced and studied by Wu [15].

(ii) If $A=0$, $F=I$ and T is single-valued mapping, then CGVIP (3.1) is equivalent to finding $p\in\mathcal{H}$ such that $g(p)\in K$.

$$\langle Tp,g(p)-g(q)\rangle\ge 0,\ \forall\ g(q)\in K. \tag{3.3}$$

Problem (3.3) was studied by Noor [6].

(iii) If $F,T=0$ and $g=I$ (identity mapping), then CGVIP (3.1) is equivalent to finding $p\in K$ such that

$$\langle Ap,q-p\rangle\geqq 0,\ \forall\ q\in K. \tag{3.4}$$

Problem (3.4) has been introduced and studied by Stampachia [12] .

It is clear that for suitable choices of mappings involved in the formulation of CGVIP (3.1), one can obtain many variational inclusion problems studied in recent past.

Lemma 3.1 ([1]). *Given $z\in\mathcal{H}$ $p\in K$ satisfies the inequality,*

$$\langle p-z,q-p\rangle\ge 0,\ \forall\ q\in K,$$

if and only if $p=P_Kz$, where P_K is the projection of $\mathcal{H}$ into K. Furthermore, the projection P_K is a nonexpansive mapping.

Let P_K be the projection of $\mathcal{H}$ into K, and let $Q_K = I - SP_K$, where I is the identity mapping and S is a non-expansive mapping. If g^{-1} exists, then we consider the problem to finding $z \in \mathcal{H}$ such that

$$ASP_K z + F(w) + \rho^{-1} Q_K z = 0, \ \forall \ w \in TSP_K z, \tag{3.5}$$

where $\rho > 0$ is a constant.

Equation of the type (3.5) is called composite generalized Wiener-Hopf equation. The solution set of the problem (3.5) is denoted by $CC_1WE(H, A, S, F, g)$.

Remark 3.1. (i) If $g, F = I$, the identity mappings, then problem (3.5) is equivalent to finding $z \in \mathcal{H}$ such that

$$ASP_K z + w + \rho^{-1} Q_K z = 0, \ \forall \ w \in TSP_K z. \tag{3.6}$$

(ii) If $F = 0$, $Aog(p) = T(p)$, and $S = I$, identity mapping, then problem (3.5) is equivalent to finding $z \in \mathcal{H}$ such that

$$Tg^{-1} P_K z + \rho^{-1} Q_K z = 0, \text{ where } Q_K = I - P_K. \tag{3.7}$$

(iii) If $g = I$, identity mapping, then problem (3.7) is equivalent to finding $z \in \mathcal{H}$ such that

$$TP_K z + \rho^{-1} Q_K z = 0. \tag{3.8}$$

For the general treatment and applications of Wiener-Hopf equation (see [7, 14, 15]).

Lemma 3.2. *The element of $p \in K$ is a common element of $VI(K, A, F, T, g) \cap F(Sog)$ if and only if the composite Wiener-Hopf equation* (3.5) *has a solution $z \in \mathcal{H}$, where*

$$z = g(p) - \rho[Aog(p) + F(w)], \tag{3.9}$$

$$g(p) = SP_K(z), \tag{3.10}$$

where P_K is the projection of $\mathcal{H}$ into K and $\rho > 0$ is a constant.

Proof. Let $p \in \mathcal{H}$ be such that $g(p) \in K$ is a solution of problem (3.1). Then by Lemma 3.1, it follows that

$$g(p) = S(g(p)) = SP_K[g(p) - \rho(Aog(p) + F(w))]. \tag{3.11}$$

Using $Q_K = I - SP_K$ and applying (3.8) repeatedly, we obtain

$$\begin{aligned} Q_K[g(p) - \rho(Aog(p)) + F(w)] &= g(p) - \rho[Aog(p) + F(w)] - SP_K[g(p) - \rho(Aog(p) + F(w))] \\ &= -\rho(Aog(p) + F(w)). \end{aligned}$$

This implies that,

$$\begin{aligned} \rho^{-1} Q_K[z] + F(w) &= -Aog(p) \\ &= -Aog[g^{-1}(SP_K(g(p) - \rho(Aog(p) + F(w))))]. \end{aligned}$$

From which it follows that

$$ASP_K(z) + F(w) + \rho^{-1} Q_K(z) = 0,$$

where $z = g(p) - \rho(Aog(p) + F(w))$ and g^{-1} is the inverse of the mapping g.

Conversely, let $z \in \mathcal{H}$ be a solution of the problem (3.5). Then, we have

$$\rho(ASP_K(z)+F(w)) = -Q_K(z) = SP_K(z)-z. \tag{3.12}$$

Now, from (3.12) and Lemma 3.1, for all $g(q)\in K$, we obtain

$$0 \le \langle SP_K(z)-z, g(q)-SP_K(z)\rangle = \langle \rho(Aog(p)+F(w)), g(q)-SP_K(z)\rangle,$$

it follows that

$$\langle Aog(p)+F(w), g(q)-SP_K(z)\rangle \ge 0 \ \forall \ g(q)\in K.$$

Thus $g(p) = SP_K z$ is a solution of CGVIP (3.1) and from (3.12), we have

$$z = g(p)-\rho(Aog(p)+F(w)).$$

This completes the proof. □

Lemma 3.3 ([14]). *Let z_n be a sequence of non-negative real numbers such that*

$$z_{n+1} \leqq (1-\lambda_n)z_n + y_n, \ \forall \ n \le n_0,$$

where n_0 is some non-negative integer, $\lambda_n \in [0,1]$ with

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \ y_n = o(\lambda_n), \text{ then } \lim_{n\to\infty} a_n = 0.$$

Remark 3.2. It is obvious that composite generalized variational inequalities and composite generalized Wiener Hopf equations are equivalent.

4. Convergence Analysis

First we establish an iterative algorithm based on Lemma 3.2 for finding the solution of CGVIP (3.1) and then we prove a convergence result.

Algorithm 4.1. *For any $s_0 \in \mathcal{H}$, compute the sequence $\{s_n\}$ by the iterative processes*

$$g(p_n) = (\alpha I + (1-\alpha)S)P_K s_n,$$

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n[g(p_n)-\rho(Aog(p_n)+F(w_n))], \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and S is a strictly contractive mapping.

(I) If $F = g = I$, the identity mapping in Algorithm 4.1, then we have the following algorithm:

Algorithm 4.2. *For any $s_0 \in \mathcal{H}$, compute the sequence $\{s_n\}$ by the iterative processes*

$$p_n = (\alpha I + (1-\alpha)S)P_K s_n, \tag{4.2}$$

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n[p_n - \rho(Aog(p_n)+w_n)], \tag{4.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, for all $n \ge 0$, which was studied by Wu [15].

(II) If $g, F, S = I$, the identity mapping in Algorithm 4.1, then we have the following algorithm:

Algorithm 4.3. *For any $s_0 \in \mathcal{H}$, compute the sequence $\{s_n\}$ by the iterative processes*

$$p_n = P_K s_n, \tag{4.4}$$

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n[p_n - \rho(A(p_n)+w_n)], \tag{4.5}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$*, for all* $n \geq 0$*, which was studied by Noor* [7].

(III) If $g,F,S = I$, the identity mapping and $\alpha_n = 1$, $\forall\ n$ in Algorithm 4.1, then we have the following algorithm:

Algorithm 4.4. *For any* $s_0 \in \mathcal{H}$*, compute the sequence* $\{s_n\}$ *by the iterative processes*

$$p_n = P_K s_n, \tag{4.6}$$

$$s_{n+1} = p_n - \rho(A(p_n)+w_n), \quad n \geq 0, \tag{4.7}$$

which was studied by Verma [13].

Theorem 4.1. *Let* K *be a closed convex subset of a separable real Hilbert space* $\mathcal{H}$*. Let* $A,F : K \to \mathcal{H}$ *and* $S,g : K \to K$ *be the single-valued continuous nonlinear mappings such that* A *is relaxed* (γ,r)*-cocoerceive mapping and* μ*-Lipschitz continuous,* F *is* ν*-Lipschitz continuous and* S *is* C*-strictly psuedocontractive mapping such that* $F(Sog) \cap VI(K,A,F,T,g) \neq \phi$*, respectively. Let* $T : \mathcal{H} \to 2^{\mathcal{H}}$ *be a set-valued Lipschitz continuous operator and relaxed monotone with corresponding constants* $m > 0$ *and* $k > 0$*, respectively. Let* $\{s_n\}$ *and* $\{p_n\}$ *be sequences generated by Algorithm 4.1 and let* α_n *be a sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\alpha \in [k,1)$,

(iii) $0 < \rho < \frac{2(r-\gamma\mu-k)}{(\mu+\nu m)^2}$, $r > \gamma\mu + k$,

then the sequences $\{p_n\}$ *and* $\{s_n\}$ *converge strongly to* $p^* \in F(Sog) \cap VI(K,A,F,T,g)$ *and* $s^* \in CC_1WE(H,A,S,F,g)$*, respectively.*

Proof. Let $R = \alpha I + (I-\alpha)S$. In view of the restriction (ii), we find that R is nonexpansive with $F(R) = F(S)$. Let $g(p) \in K$ be the common elements of $F(S) \cap VI(K,A,F,T,g)$, we have $g(p^*) = RP_K s^*$,

$$s^* = (1-\alpha_n)s^* + \alpha_n[g(p^*) - \rho(Aog(p^*) + F(w^*))],$$

where $w^* \in Tg(p^*)$ and $s^* \in CC_1WE(H,A,S,F,g)$. Observing Algorithm 4.1, we have

$$\begin{aligned}
\|s_{n+1} - s^*\| &= \|(1-\alpha_n)s_n + \alpha_n[g(p_n) - \rho(Aog(p_n)+F(w_n))] - s^*\| \\
&= \|\big[(1-\alpha_n)s_n + \alpha_n[g(p_n) - \rho(Aog(p_n)+F(w_n))]\big] \\
&\quad - \big[(1-\alpha_n)s^* + \alpha_n[g(p^*) - \rho(Aog(p^*)+F(w^*))]\big]\| \\
&\leq (1-\alpha_n)\|s_n - s^*\| + \alpha_n\|g(p_n) - g(p^*) - \rho[(Aog(p_n)+F(w_n)) \\
&\quad - (Aog(p^*)+F(w^*))]\|.
\end{aligned} \tag{4.8}$$

On considering the second term of right side of (4.8). Assume T is relaxed monotone, A is μ-Lipschitz continuous, relaxed (γ,r)-cocoercive and m-Lipschitz continuous and F is ν-Lipschitz continuous, we have

$$\|g(p_n) - g(p^*) - \rho[(Aog(p_n)+F(w_n)) - (Aog(p^*)+F(w^*))]\|^2$$

$$= \|g(p_n)-g(p^*)\|^2 - 2\rho\langle(Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*),(g(p_n)-g(p^*))\rangle$$
$$+\rho^2\|((Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*))\|^2$$
$$= \|g(p_n)-g(p^*)\|^2 - 2\rho\langle Aog(p_n)-Aog(p^*)),g(p_n)-g(p^*)\rangle$$
$$-2\rho\langle F(w_n)-F(w^*),g(p_n)-g(p^*)\rangle + \rho^2\|(Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*))\|^2$$
$$\leq \|g(p_n)-g(p^*)\|^2 - 2\rho(-\gamma\|Aog(p_n)-Aog(p^*)\|+r\|g(p_n)-g(p^*)\|)$$
$$+2\rho k\|g(p_n)-g(p^*)\|+\rho^2\|(Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*))\|^2$$
$$\leq \|g(p_n)-g(p^*)\|^2 + 2\rho(\gamma\mu-r+k)\|g(p_n)-g(p^*)\|$$
$$+\rho^2\|(Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*))\|^2. \tag{4.9}$$

Now consider the third term of right-side of (4.9), we have

$$\|(Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*))\| = \|(Aog(p_n)-Aog(p^*))+(w_n-w^*)\|$$
$$\leq \|Aog(p_n)-Aog(p^*)\|+\|F(w_n)-F(w^*)\|$$
$$\leq (\mu+\nu m)\|g(p_n)-g(p^*)\| \tag{4.10}$$

Put (4.9) into (4.8), we have

$$\|g(p_n)-g(p^*)-\rho[(Aog(p_n)+F(w_n))-(Aog(p^*)+F(w^*))]\|^2$$
$$\leq \|g(p_n)-g(p^*)\|^2+2\rho(\gamma\mu-r+k)\|g(p_n)-g(p^*)\|+\rho^2(\mu+\nu m)^2\|g(p_n)-g(p^*)\|^2$$
$$=[1+2\rho(\gamma\mu-r+k)+\rho^2(\mu+\nu m)^2]\|g(p_n)-g(p^*)\|^2$$
$$=\theta^2\|g(p_n)-g(p^*)\|^2. \tag{4.11}$$

where $\chi=\sqrt{1+2\rho(\gamma\mu-r+k)+\rho^2(\mu+\nu m)^2}$. From (ii) condition, we get $\chi<1$. Putting equation (4.11) into (4.8), we get

$$\|s_{n+1}-s^*\| \leq (1-\alpha_n)\|s_n-s^*\|+\alpha_n\theta\|g(p_n)-g(p^*)\|. \tag{4.12}$$

Since R is non-expansive, we find that

$$\|g(p_n)-g(p^*)\| = \|RP_Ks_n-RP_Ks^*\| \leq \|s_n-s^*\|. \tag{4.13}$$

Put (4.13) into (4.12), we get

$$\|s_{n+1}-s^*\| \leq (1-\alpha_n)\|s_n-s^*\|+\alpha_n\theta\|s_n-s^*\|$$
$$\leq [1-\alpha_n(1-\theta)]\|s_n-s^*\|. \tag{4.14}$$

From condition (i) and using Lemma 3.1 into equation (4.14), we get

$$\lim_{n\to\infty}\|s_n-s^*\|=0.$$

On the other hand, observing (4.13), we get

$$\lim_{n\to\infty}\|p_n-p^*\|=0.$$

Therefore, the sequences $\{p_n\}$ and $\{s_n\}$ converges strongly to $p^*\in F(Sog)\cap VI(K,A,F,T,g)$ and $s^*\in CC_1WE(H,A,S,F,g)$, respectively. □

From Theorem 4.1, the following results are easy to derive.

Corollary 4.1. *Let K be a closed convex subset of a separable real Hilbert space $\mathcal{H}$. Let $A,F:K\to\mathcal{H}$ and $S,g:K\to K$ be the single-valued continuous nonlinear mappings such that A is μ-Lipschitz continuous and relaxed (γ,r)-cocoerceive mapping, F is ν-Lipschitz continuous and S is S is a nonexpansive mapping such that $F(Sog)\cap VI(K,A,F,T,g)\neq\phi$, respectively. Let $T:H\to 2^{\mathcal{H}}$ be a multi-valued Lipschitz continuous and relaxed monotone operator with corresponding constants $m>0$ and $k>0$, respectively. Let $\{s_n\}$ and $\{p_n\}$ be sequences generated by Algorithm 4.1 and let α_n be a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty}\alpha_n=\infty$,

(ii) $0<\rho<\frac{2(r-\gamma\mu-k)}{(\mu+\nu m)^2}$, $r>\gamma\mu+k$,

then the sequences $\{p_n\}$ and $\{s_n\}$ converge strongly to $p^*\in F(Sog)\cap VI(K,A,F,T,g)$ and $s^*\in CC_1WE(H,A,S,F,g)$, respectively.

5. Conclusion

In this article, we introduced and studied a composite generalized variational inequality problem with a composite Wiener-Hopf equation in separable real Hilbert space. Some preliminary results proved to obtain the main result. By using the projection operator technique we established the equivalence between the composite generalized variational inequality problem with a composite Wiener-Hopf equation. The existence and convergence analysis of the suggested iterative algorithm has been discussed under some specific conditions. Our obtained results extend and generalize most of the results for different systems existing in the literature. We remark that our results may be further considered in higher dimensional spaces.

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