Communications in Mathematics and Applications
Vol. 10, No. 4, pp. 783–796, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i4.1252

Research Article

Enumeration of Glued Graphs of Paths

Monthiya Ruangnai[1] and Sayan Panma[2,*]

[1] *PhD Program in Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[2] *Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
*__Corresponding author:__ panmayan@yahoo.com

Abstract. Let G_1 and G_2 be two vertex-disjoint graphs with H_1 a subgraph of G_1 and H_2 a subgraph of G_2. Let $f: H_1 \to H_2$ be an isomorphism between these subgraphs. The glued graph of G_1 and G_2 at H_1 and H_2 with respect to f is the graph that results from combining $G_1 \cup G_2$ by identifying the subgraphs H_1 and H_2 according to the isomorphism f between H_1 and H_2. We refer G_1 and G_2 as its original graphs and refer H as its clone where H is a copy of H_1 and H_2. In this paper, we enumerate all non-isomorphic resulting glued graphs between two paths at connected clones. Moreover, we also give the characterization of the glued graph at a connected clone.

Keywords. A glued graph; The glue operator; Glued graph of paths; Graphs enumeration; Graph isomorphisms

MSC. 05C30; 05C38; 05C60; 05C76

Received: May 23, 2019 **Accepted:** July 18, 2019

1. Introduction

Graphs have proven to be an extremely useful tool for analyzing situations involving a set of elements which are related by some property. The most obvious examples of graphs are sets with physical links such as subway systems, telephone communication systems, oil pipelines, and electrical networks. There are numerous ways to do operations on graphs so as to obtain new graphs and study their properties for more advantages. Consequently, in this paper, we are interested in the glue operator which has been introduced in Uiyyasathian's doctoral thesis [8] and Promsakon's master's degree thesis [6].

Let G_1 and G_2 be two vertex-disjoint graphs with H_1 a subgraph of G_1 and H_2 a subgraph of G_2. Let $f : H_1 \to H_2$ be an isomorphism between these subgraphs. The glued graph of G_1 and G_2 at H_1 and H_2 with respect to f is the graph that results from combining $G_1 \cup G_2$ by identifying the subgraphs H_1 and H_2 according to the isomorphism f between H_1 and H_2. The glued graph is denoted by $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$. We refer G_1 and G_2 as its original graphs and refer H as its clone where H is a copy of H_1 and H_2.

Some more recent works on glued graphs are investigated. These works vary on their original graphs and clones. In 2006, Promsakon [6] examined some coloring properties of the glued graph. The original graphs such as forests, trees, bipartite graphs, k-partite graphs, chordal graphs, and interval graphs are studied. Mekwian [4] applied this new binary operation of glued graphs for solving E-logistics network problems in the next year.

In 2009, Uiyyasathian and Saduakdee [10] were interested in the perfection of graphs. They studied chromatic numbers of glued graphs at complete clones together with their clique numbers. In 2010, Uiyyasathian and Jongthawonwuth [9] investigated bounds of clique partition numbers of glued graphs at K_2-clones and K_3-clones. Uiyyasathian and Pimpasalee [5] also obtained bounds of clique covering numbers of glued graphs at complete clones in the same year.

Moreover, in 2010, Boonthong et al. [1] characterized another interesting research on glued graphs. They obtained the conditions of being Eulerian glued graphs. Malila [3] observed some values such as the domination number (in 2011) and the upper and the lower independence number (in 2014) on glued graphs of cycles having paths as the clones.

Furthermore, Seyyedi and Rahmati [7] were inspired by the work on clique coverings of Pimpasalee [5], so they studied some properties of glued graphs at some certain clones in the view of algebraic combinatorics in 2014.

We see that all mentioned topics have been focusing in the term of glued graphs. In addition, Gross and Yellen [2] also introduced the term "amalgamation" as a graph operation in their textbook in 2006. The definitions of both terms are equivalent. According to those research works, we see that there are various forms of original graphs and clones.

The original graphs we shall consider in this paper is a pair of non-trivial paths at any lengths. When gluing on a pair of isomorphic subgraphs of two original paths, the structure of resulting graphs may depend on exactly how the vertices and edges of the two subgraphs are matched via an isomorphism. Therefore, different types of resulting graphs such as a longer path, a cycle, a tree, or a multigraph could be exist. Our considering clone is a connected subgraph of indicated paths. The objective is to enumerate all non-isomorphic resulting graphs obtained from the gluing of two paths together. We make a systematic analysis to describe the number and the relation of all distinct resulting glued graphs of paths. We also investigate the characterization of the glued graph at a connected clone. We hope that our results can be applied to solve real-life problems. For example, a route map can be represented by a graph, such as to find the number of possible transit routes, to plan an itinerary, or to settle a location as the hub of the transportation.

2. Preliminaries

This section provides definitions and notations which are useful in the sequel of this research work (see for further details concerning graph theory and glued graphs in [12] and [6], respectively).

A graph G consists of a non-empty finite set $V(G)$ of elements called *vertices*, and a finite family $E(G)$ of unordered pairs of these elements of $V(G)$ called *edges*. We call $V(G)$ the *vertex set*, and $E(G)$ the *edge family* of G. A *subgraph* H of G is a graph which all vertices of H belong to $V(G)$ and all edges of H belong to $E(G)$. That is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *number of vertices* of a graph G is the cardinality of the vertex set $V(G)$, denoted by $|V(G)|$. Similarly, the *number of edges* of a graph G is the cardinality of the edge family $E(G)$, denoted by $|E(G)|$.

The *degree* of a vertex v of G is the number of edges incident with v, and is written $\deg(v)$. The *degree sequence* of a graph consists of the degrees of all vertices of the graph written in an increasing order, with repeats where necessary.

A *walk* in G is a finite sequence of vertices in G such that two consecutive vertices in the sequence are adjacent. A *path* is a walk in which no vertex is repeated. A path with n distinct vertices are denoted by P_n. A graph G is said to be *connected* if there is a path connecting between any two vertices of G.

A graph $G_1 = (V(G_1), E(G_1))$ is isomorphic to a graph $G_2 = (V(G_2), E(G_2))$ if there is a one-to-one correspondence f from $V(G_1)$ onto $V(G_2)$ such that $\{u,v\} \in E(G_1)$ if and only if $\{f(u), f(v)\} \in E(G_2)$. If such a function exists, it is called *an isomorphism from G_1 to G_2*, and written by $G_1 \cong G_2$.

Let G_1 and G_2 be disjoint graphs with H_1 a subgraph of G_1 and H_2 a subgraph of G_2. Let $f: H_1 \to H_2$ be an isomorphism between these subgraphs. The *glued graph of G_1 and G_2 at H_1 and H_2 with respect to f* (or the *amalgamation of G_1 and G_2 modulo the isomorphism $f: H_1 \to H_2$*) is the graph that results from combining $G_1 \cup G_2$ by identifying the subgraphs H_1 and H_2 according to the isomorphism f between H_1 and H_2. The glued graph is denoted by $\underset{H_1 \cong_f H_2}{G_1 \lhd\!\rhd G_2}$.

Let H be the copy of H_1 and H_2 in the glued graph, i.e., H is a subgraph of the glued graph and $H \cong H_1 \cong H_2$. We refer H as its *clone* and refer G_1 and G_2 as its *original graphs*.

Example 1. Consider two subgraphs H_1 and H_2 of G_1 and G_2, respectively, where $H_1 \cong H_2$ as in Figure 1. We obtain $H_1 = \triangle(1,2,3) \subseteq G_1$ and $H_2 = \triangle(a,b,c) \subseteq G_2$.

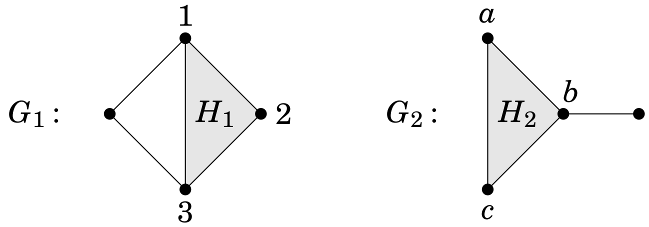

Figure 1. $H_1 = \Delta(1,2,3) \subseteq G_1$ and $H_2 = \Delta(a,b,c) \subseteq G_2$

Consider the following three isomorphisms f, g, and h, between H_1 and H_2:

$$f(1)=a,\ f(2)=b,\ f(3)=c;\ g(1)=b,\ g(2)=c,\ g(3)=a;\ h(1)=c,\ h(2)=b,\ h(3)=a.$$

Then we have the following three glued graphs as shown in Figure 2.

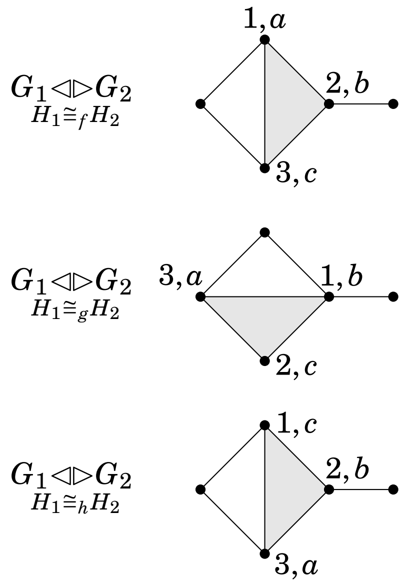

Figure 2. The results of graph gluing in different isomorphisms

This example shows that different isomorphisms could give different results or the same one. However, in some cases, it is possible that all isomorphisms give the same resulting graph. The next example will illustrate this fact.

Example 2. Consider the following $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ showing in Figure 3. We obtain $H_1 = \triangle(1,2,3) \subseteq G_1$ and $H_2 = \triangle(a,b,c) \subseteq G_2$. There are six isomorphisms between H_1 and H_2, but all of them give the same glued graph as shown in Figure 4.

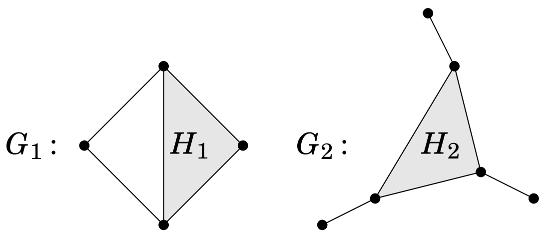

Figure 3. $H_1 = \Delta(1,2,3) \subseteq G_1$ and $H_2 = \Delta(a,b,c) \subseteq G_2$

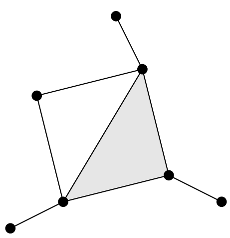

Figure 4. The unique resulting glued graph for any isomorphisms

In this particular case, we do not need to specify an isomorphism in the notation representing the glued graph, i.e, f can be omitted from the notation without ambiguity. Hence the glued graph notation can be written as $G_1 \underset{H_1 \cong H_2}{\lhd\!\rhd} G_2$.

Next, we are going to introduce an example for the gluing of P_3 and P_4 in which a clone H is a trivial subgraph.

Example 3. Figure 5 shows the outcome of the gluing between P_3 and P_4 where H, a trivial subgraph of these two paths, is the clone. There are four non-isomorphic resulting graphs, namely A, B, C, and D.

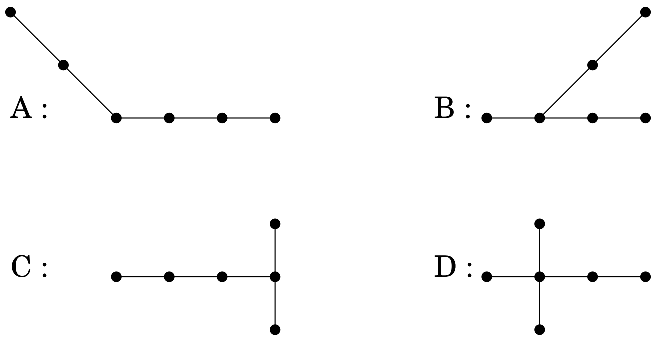

Figure 5. Four non-isomorphic glued graphs of P_3 and P_4 at the clone H

In this paper, we shall enumerate all non-isomorphic resulting graphs obtained by the glue operator on paths. The following proposition about the number of vertices and edges of a glued graph is very useful for our study.

Proposition 1 ([6]). *Let G_1 and G_2 be any two disjoint graphs. Let H be a clone of a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$. Then*

1. $\left|V(G_1 \underset{H}{\lhd\!\rhd} G_2)\right| = |V(G_1)| + |V(G_2)| - |V(H)|$, *and*
2. $\left|E(G_1 \underset{H}{\lhd\!\rhd} G_2)\right| = |E(G_1)| + |E(G_2)| - |E(H)|$.

3. Enumeration of Glued Graphs of Paths at Connected Clones

First of all, we should recall about some types of subgraphs. There are many kinds of them to be considered for an arbitrary graph. Table 1 as shown below illustrates the number of some specific subgraphs of a non-trivial path with $n = 2,3,4,5,6$ vertices.

As previously stated, our original graphs for this paper are two non-trivial paths at any lengths which the considering clone is a connected subgraph of these paths. Let's recall that we use the following symbols for convenience. Let $m,n \in \mathbb{N}$. Let X and Y be subgraphs of P_m and P_n, respectively, such that $X \cong Y$. Let H be a clone of a glued graph.

Table 1. Numbers of some particular subgraphs of a path P_n

P_n	Number of non-isomorphic subgraphs			
	Connected subgraphs	Null graphs	Spanning subgraphs	All subgraphs
P_2	2	2	2	3
P_3	3	3	3	6
P_4	4	4	5	11
P_5	5	5	7	18
P_6	6	6	11	29
P_7	7	7	15	44
P_8	8	8	22	66

We will denote by

$\underset{X\cong_f Y}{P_m \lhd\!\rhd P_n}$: a glued graph of P_m and P_n at X and Y with respect to f;

$\underset{X\cong Y}{P_m \lhd\!\rhd P_n}$: a glued graph of P_m and P_n at X and Y;

$\underset{H}{P_m \lhd\!\rhd P_n}$: a glued graph of P_m and P_n at a clone H;

$(P_m \lhd\!\rhd P_n)_H$: a set of resulting glued graph of P_m and P_n at a clone H;

$\left|(P_m \lhd\!\rhd P_n)_H\right|$: the number of elements in a set $(P_m \lhd\!\rhd P_n)_H$.

Since P_2 is a subgraph of P_n for all $n \in \mathbb{N}$ where $n \geq 2$, there are two common connected subgraphs of P_2 and P_n which can be considered as a clone in the glued graph. The subgraphs are P_1 and P_2. We see that P_1 and P_2 are subgraphs of both P_2 and P_n. If we choose a graph having more than 2 vertices or more than an edge as a clone of a glued graph between P_2 and P_n, then it might be a subgraph of P_n; but it is certainly **not** a subgraph of P_2. Such a case should not be considered.

We start with the problems of gluing P_2 and P_n. We shall conclude the number of all distinct resulting glued graphs of P_2 and P_n at connected clones.

Lemma 1. *Let $2 \leq n \in \mathbb{N}$. If a graph P_1 is a clone of a glued graph between P_2 and P_n, then* $\left|(P_2 \lhd\!\rhd P_n)_{P_1}\right| = \lceil \frac{n}{2} \rceil$.

Proof. Let X and Y be subgraphs of P_2 and P_n, respectively, such that $X \cong Y \cong P_1$. Then there exist $|X||Y| = 2n$ possible isomorphisms between X and Y. Clearly, no matter which vertices are chosen in P_2, the isomorphism type of the glued graph is the same. Moreover, we obtain $\lceil \frac{n}{2} \rceil$ distinct resulting graphs since the symmetry of P_n. Hence the number of all non-isomorphic glued graphs between P_2 and P_n with a clone P_1 is equal to $\lceil \frac{n}{2} \rceil$. □

Example 4. All resulting graphs obtained from the gluing between P_2 and P_n at a clone P_1 are shown in Figure 6. The glued vertex is indicated by the white circle. Due to the symmetry of a path graph, some of these results are not distinct.

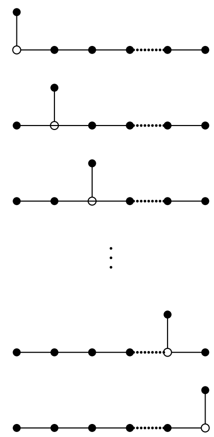

Figure 6. Resulting glued graphs between P_2 and P_n at a clone P_1

Lemma 2. *Let* $2 \le n \in \mathbb{N}$. *If a graph* P_2 *is a clone of a glued graph between* P_2 *and* P_n*, then* $\left|(P_2 \lhd\rhd P_n)_{P_2}\right| = 1$.

Proof. Let X and Y be subgraphs of P_2 and P_n, respectively, such that $X \cong Y \cong P_2$. Then we have $X = P_2$ and $Y = P_2 \subseteq P_n$. Thus the gluing of P_2 and P_n at a clone P_2 gives a unique resulting graph for any isomorphisms f between X and Y since $P_2 \subseteq P_n$. So f can be embedded in a graph P_n for all $n \ge 2$. That is, the only result of the gluing is a graph P_n. □

Similarly, we study the gluing of P_3 and P_n at a common connected subgraph between them via an isomorphism. Since P_3 is a subgraph of P_n for all $n \in \mathbb{N}$ where $n \ge 3$, we obtain three mutual connected subgraphs of P_3 and P_n to be considered as a clone in the glued graph. The subgraphs are P_1, P_2 and P_3. We will give the number of all distinct resulting glued graphs of P_3 and P_n using the same method as in the study of P_2 and P_n. We obtain the following results.

Lemma 3. *Let* $3 \le n \in \mathbb{N}$. *If a graph* P_1 *is a clone of a glued graph between* P_3 *and* P_n*, then*

$$\left|(P_3 \lhd\rhd P_n)_{P_1}\right| = \begin{cases} 2\left\lceil \frac{n}{2} \right\rceil - 1 & \text{if } n = 3, \\ 2\left\lceil \frac{n}{2} \right\rceil & \text{if } n > 3. \end{cases}$$

Proof. Let P_3 and P_n be paths with 3 and n vertices, respectively, where $n \ge 3$. Assume that $u_1, u_2, u_3 \in V(P_n)$ and $v_1, v_2, \ldots, v_n \in V(P_n)$. Let X be a subgraph of P_3 such that $X \cong N_1$, i.e., $|V(X)| = 1$ and $|E(X)| = 0$. Similarly, let Y be a subgraph of P_n such that $Y \cong N_1$, i.e., $|V(Y)| = 1$ and $|E(Y)| = 0$. We see that $X \cong Y \cong N_1$. Suppose that $V(X) = \{u_i | u_i \in V(P_3), \exists i \in \{1,2,3\}\}$ and $V(Y) = \{v_j | v_j \in V(P_n), \exists j \in \{1,2,\ldots,n\}\}$. Define $f_{ij} : V(X) \to V(Y)$ by $f_{ij}(u_i) = v_j$. We see that f_{ij} is an isomorphism from X to Y. Since $|V(P_3)| = 3$ and $|V(P_n)| = n$, there exist $3n$ distinct isomorphisms between any subgraphs X and Y. We will denote by G_{ij} a glued graph between P_3 and P_n at X and Y with respect to f_{ij}, i.e.,

$$G_{ij} := P_3 \underset{X \cong_{f_{ij}} Y}{\lhd\rhd} P_n.$$

Firstly, we obtain $G_{1j} \cong G_{3j}$ for all $j \in \{1,2,\ldots,n\}$ due to the symmetry of P_3. Thus, without loss of generality, we can only consider G_{ij} where $i \in \{1,2\}$. We also obtain that for each $i \in \{1,2\}$,

$G_{ij} \cong G_{i(n-j+1)}$ for all $j \in \{1,2,\dots,n\}$. Thus, without loss of generality, we can only consider G_{ij} where $i \in \{1,2\}$ and $j \in \{1,2,\dots,\lceil \frac{n}{2} \rceil\}$.

Case 1 (if $n > 3$): By the construction of glued graphs between P_3 and P_n, we see that G_{ij} are all distinct. Hence we can conclude that the number of non-isomorphic glued graphs between P_3 and P_n, where $n > 3$, at X and Y with respect to any isomorphisms f is $2\lceil \frac{n}{2} \rceil$.

Case 2 (if $n = 3$): The proof of this case is similar to the proof of Case 1, but there is a condition left to consider. From Case 1, we obtain the upper bound for the number of non-isomorphic glued graphs between P_3 and P_3 at X and Y with respect to any isomorphisms f is $2\lceil \frac{3}{2} \rceil = 4$. It is easily checked that $G_{ij} \cong G_{ji}$ where $i,j \in \{1,2\}$ since $n = 3$. That is, $G_{12} \cong G_{21}$. Thus the number of non-isomorphic glued graphs between P_3 and P_3 at a clone N_1 with respect to any isomorphisms f is $2\lceil \frac{3}{2} \rceil - 1 = 4 - 1 = 3$. □

Lemma 4. *Let* $3 \le n \in \mathbb{N}$. *If a graph* P_2 *is a clone of a glued graph between* P_3 *and* P_n, *then* $\left|(P_3 \lhd\!\rhd P_n)_{P_2}\right| = \lceil \frac{n}{2} \rceil$.

Proof. We see that the method of constructing glued graphs of P_3 and P_n at a clone P_2 is as same as in Lemma 1 of P_2 and P_n at a clone P_1. Then we get the number of all non-isomorphic glued graphs. □

Lemma 5. *Let* $3 \le n \in \mathbb{N}$. *If a graph* P_3 *is a clone of a glued graph between* P_3 *and* P_n, *then* $\left|(P_3 \lhd\!\rhd P_n)_{P_3}\right| = 1$.

Proof. It is similar to the proof of Lemma 2 since $P_3 \subseteq P_n$ for all $n \ge 3$. So any isomorphisms f between P_3 and Y such that $Y \cong P_3$ can be embedded in a graph P_n. Thus the only resulting graph is P_n. □

Example 5. Table 2 here determines the numbers of distinct resulting glued graphs of P_m and P_n, where $m = 2,3$, according to any isomorphisms between their common subgraphs.

By all the lemmas and example, the following results about the number of glued graphs of paths at connected clones are examined. We shall begin with some examples.

Notice that we study glued graphs that having P_m and P_n as original graphs, and P_r as their connected clone where $m,n,r \in \mathbb{N}$ and $r \le m \le n$.

Example 6. We consider if $r = 1$. Then the subgraph is a null graph with one vertex, $P_1\dot{}$We have $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ ways to create a glued graph according to the isomorphism. We also obtain that resulting graphs are all distinct since each one is a graph with $m+n-1$ vertices and $m+n-2$ edges. Furthermore, a glued graph satisfies either one of the following conditions:

(1) degree of a glued vertex is equal to 2; or

(2) degree of a glued vertex is equal to 3; or

(3) degree of a glued vertex is equal to 4.

Table 2. Numbers of all non-isomorphic glued graphs of P_m and P_n at connected clones

P_n	The number of non-isomorphic glued graphs				
	$\lvert(P_2 \lhd\rhd P_n)_{P_1}\rvert$	$\lvert(P_2 \lhd\rhd P_n)_{P_2}\rvert$	$\lvert(P_3 \lhd\rhd P_n)_{P_1}\rvert$	$\lvert(P_3 \lhd\rhd P_n)_{P_2}\rvert$	$\lvert(P_3 \lhd\rhd P_n)_{P_3}\rvert$
P_2	1	1	2	1	1
P_3	2	1	3	2	1
P_4	2	1	4	2	1
P_5	3	1	6	3	1
P_6	3	1	6	3	1
P_7	4	1	8	4	1
P_8	4	1	8	4	1
P_9	5	1	10	5	1
P_{10}	5	1	10	5	1
Conclusion	$\left\lceil \frac{n}{2} \right\rceil$	1	$2\left\lceil \frac{n}{2} \right\rceil$ if $n > 3$	$\left\lceil \frac{n}{2} \right\rceil$	1

Example 7. We consider if $r = 1$ and $m = n$. Similarly, we usually obtain $\lceil \frac{m}{2} \rceil \lceil \frac{m}{2} \rceil$ ways to construct resulting graphs, but some isomorphisms give the same result. Therefore, the number of resulting graphs in this case is $\lceil \frac{m}{2} \rceil \lceil \frac{m}{2} \rceil - \sum_{q=1}^{\lceil \frac{m}{2} \rceil - 1} q$.

Lemma 6. *Let $m,n \in \mathbb{N}$ be such that $m < n$. Let X and Y be subgraphs of P_m and P_n, respectively, such that $X \cong Y \cong P_1$. Then there exist $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ non-isomorphic glued graphs between P_m and P_n at X and Y with respect to the isomorphism f.*

Proof. Let $m,n \in \mathbb{N}$ be such that $m < n$. Let $1 \le i \le m$ and $1 \le j \le n$. Let u_i be a vertex of P_m such that the distance from u_i to an endpoint of P_m is $i-1$. Then the distance from u_i to the other endpoint is $m-i$.

Without loss of generality, we can consider $1 \le i \le \lceil \frac{m}{2} \rceil$ due to the symmetry of P_m.

Similarly, let v_j be a vertex of P_n such that the distance from v_j to an endpoint of P_n is $j-1$. Then the distance from v_j to the other endpoint is $n-j$. Also, without loss of generality, we can consider $1 \le j \le \lceil \frac{n}{2} \rceil$ due to the symmetry of P_n.

Let X and Y be subgraphs of P_m and P_n, respectively, such that $X \cong Y \cong P_1$. Define $f_{ij} : V(X) \to V(Y)$ by $f_{ij}(u_i) = v_j$. Thus there exist $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ isomorphisms f_{ij} between X and Y.

Let G_{ij} denote a glued graph between P_m and P_n at X and Y with respect to f_{ij}, i.e.,

$$G_{ij} := P_m \underset{\cong f_{ij}}{\lhd\rhd} P_n .$$

To show that $G_{ij} \not\cong G_{kl}$ for $i,k \in \{1,2,\dots,\lceil \frac{m}{2} \rceil\}$ and $j,l \in \{1,2,\dots,\lceil \frac{n}{2} \rceil\}$ so that we can conclude resulting graphs from the gluing between P_m and P_n can possibly be $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ non-isomorphic forms according to the isomorphisms.

Let $i,k \in \{1,2,\dots,\lceil \frac{m}{2} \rceil\}$ and $j,l \in \{1,2,\dots,\lceil \frac{n}{2} \rceil\}$. We consider the graph isomorphism between G_{ij} and G_{kl} for three cases as follows:

Case 1: $i = k$ and $j \neq l$

Case 1.1: $i = k = 1$ and $j \neq l$

Case 1.2: $i,k \neq 1$, $i = k$ and $j \neq l$;

Case 2: $i \neq k$ and $j = l$

Case 2.1: $i \neq k$ and $j = l = 1$

Case 2.2: $i \neq k$, $j = l$ and $j,l \neq 1$;

Case 3: $i \neq k$ and $j \neq l$

Case 3.1: ($i = 1$ or $k = 1$) and ($j = 1$ or $l = 1$)

Case 3.2: ($i = 1$ or $k = 1$) and $j,l \neq 1$

Case 3.3: $i,k \neq 1$ and ($j = 1$ or $l = 1$)

Case 3.4: $i,j,k,l \neq 1$.

Next, we shall verify these following cases:

Case 1.1: Assume that $i = k = 1$ and $j \neq l$.

1.1.1: Suppose that $j = 1$. Then a glued graph $G_{1j} \cong P_{m+n-1}$, i.e., G_{1j} has 2 endpoints. Also we get $l \neq 1$ and G_{1l} has 3 endpoints. Thus $G_{1j} \ncong G_{1l}$.

1.1.2: Suppose that $j \neq 1$ and $l \neq 1$. Then G_{1j} and G_{1l} are both glued graphs having 3 endpoints, say α, β, and γ. Suppose that w is the glued vertex of G_{1j} and z is the glued vertex of G_{1l}. Then we have

$d(w,\alpha) = j-1$, $d(w,\beta) = m-1$, and $d(w,\gamma) = n-j$;

$d(z,\alpha) = l-1$, $d(z,\beta) = m-1$, and $d(z,\gamma) = n-l$.

It is easily seen that $G_{1j} \ncong G_{1l}$ since $j \neq l$.

Case 1.2: Assume that $i,k \neq 1$, $i = k$ and $j \neq l$.

1.2.1: Suppose that $j = 1$. Thus we get $l \neq 1$. Then G_{ij} is a glued graph having 3 endpoints and G_{il} is a glued graph having 4 endpoints. So $G_{ij} \ncong G_{il}$.

1.2.2: Suppose that $j \neq 1$ and $l \neq 1$. Then G_{ij} and G_{il} are both glued graphs having 4 endpoints, say α, β, γ, and δ. Suppose that w is the glued vertex of G_{ij} and z is the glued vertex of G_{il}. Then we have

$d(w,\alpha) = i-1$, $d(w,\beta) = m-i$, $d(w,\gamma) = j-1$, and $d(w,\delta) = n-j$;

$d(z,\alpha) = k-1$, $d(z,\beta) = m-k$, $d(z,\gamma) = l-1$, and $d(z,\delta) = n-l$.

It is easily seen that $G_{ij} \ncong G_{il}$ since $j \neq l$.

Case 2.1: Assume that $i \neq k$ and $j = l = 1$.

2.1.1: Suppose that $i = 1$. Then a glued graph $G_{i1} \cong P_{m+n-1}$, i.e., G_{i1} has 2 endpoints. Also we get $k \neq 1$ and G_{k1} has 3 endpoints. Thus $G_{i1} \ncong G_{k1}$.

2.1.2: Suppose that $i \neq 1$ and $k \neq 1$. Then G_{i1} and G_{k1} are both glued graphs having 3 endpoints, say α, β, and γ. Suppose that w is the glued vertex of G_{i1} and z is the glued vertex of G_{k1}. Then we have

$d(w,\alpha) = i-1$, $d(w,\beta) = m-i$, and $d(w,\gamma) = n-1$;

$d(z,\alpha) = k-1$, $d(z,\beta) = m-k$, and $d(z,\gamma) = n-1$.

It is easily seen that $G_{i1} \ncong G_{k1}$ since $i \neq k$.

Case 2.2: Assume that $i \neq k$, $j = l$ and $j,l \neq 1$.

2.2.1: Suppose that $i = 1$. Thus we get $k \neq 1$. Then G_{ij} is a glued graph having 3 endpoints and G_{kj} is a glued graph having 4 endpoints. So $G_{ij} \ncong G_{kj}$.

2.2.2: Suppose that $i \neq 1$ and $k \neq 1$. Then G_{ij} and G_{kj} are both glued graphs having 4 endpoints, say α, β, γ, and δ. Suppose that w is the glued vertex of G_{ij} and z is the glued vertex of G_{kj}. Then we have

$d(w,\alpha) = i-1$, $d(w,\beta) = m-i$, $d(w,\gamma) = j-1$, and $d(w,\delta) = n-j$;

$d(z,\alpha) = k-1$, $d(z,\beta) = m-k$, $d(z,\gamma) = l-1$, and $d(z,\delta) = n-l$.

It is easily seen that $G_{ij} \ncong G_{kj}$ since $i \neq k$.

Case 3.1: Assume that ($i = 1$ or $k = 1$) and ($j = 1$ or $l = 1$).

Suppose that $i = 1$. Thus we get $k \neq 1$. If $j = 1$, then $G_{ij} \cong P_{m+n-1}$ and G_{kl} is a glued graph having 4 endpoints. So $G_{ij} \ncong G_{kl}$. If $l = 1$, then G_{ij} and G_{kl} are both glued graphs having 3 endpoints, say α, β, and γ. Suppose that w is the glued vertex of G_{ij} and z is the glued vertex of G_{kl}. Then we have

$d(w,\alpha) = m-1$, $d(w,\beta) = j-1$, and $d(w,\gamma) = n-j$;

$d(z,\alpha) = k-1$, $d(z,\beta) = m-k$, and $d(z,\gamma) = n-1$.

It is easily seen that $G_{ij} \ncong G_{kl}$ since $m < n$.

Case 3.2: Assume that ($i = 1$ or $k = 1$) and $j,l \neq 1$.

Suppose that $i = 1$. Thus we get $k \neq 1$. Then G_{ij} is a glued graph having 3 endpoints and G_{kl} is a glued graph having 4 endpoints. So $G_{ij} \ncong G_{kl}$.

Case 3.3: Assume that $i,k \neq 1$ and ($j = 1$ or $l = 1$).

Suppose that $j = 1$. Thus we get $l \neq 1$. Then G_{ij} is a glued graph having 3 endpoints and G_{kl} is a glued graph having 4 endpoints. So $G_{ij} \ncong G_{kl}$.

Case 3.4: Assume that $i,j,k,l \neq 1$.

Thus we have G_{ij} and G_{kl} are both glued graphs having 4 endpoints, say α, β, γ, and δ. Suppose that w is the glued vertex of G_{ij} and z is the glued vertex of G_{kl}. Then we have

$d(w,\alpha) = i-1$, $d(w,\beta) = m-i$, $d(w,\gamma) = j-1$, and $d(w,\delta) = n-j$;

$d(z,\alpha) = k-1$, $d(z,\beta) = m-k$, $d(z,\gamma) = l-1$, and $d(z,\delta) = n-l$.

It is easily seen that $G_{ij} \ncong G_{kl}$ since $i \neq k$ and $j \neq l$.

Hence $G_{ij} \ncong G_{kl}$ for $i,k \in \{1,2,\dots,\lceil \frac{m}{2} \rceil\}$ and $j,l \in \{1,2,\dots,\lceil \frac{n}{2} \rceil\}$, and thus we obtain $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ distinct resulting graphs from the gluing between P_m and P_n at a clone P_1 with respect to the isomorphisms. □

Therefore, we can conclude one of our main theorems.

Theorem 1. *Let $m,n \in \mathbb{N}$ and $2 \le m \le n$. If a graph P_1 is a clone of a glued graph between P_m and P_n, then*

$$\left|(P_m \lhd\rhd P_n)_{P_1}\right| \le \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } m = n = 2, \\ \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil & \text{if } 2 < m < n, \\ \lceil \frac{m}{2} \rceil^2 - \sum_{q=1}^{\lceil \frac{m}{2} \rceil - 1} q & \text{if } 2 < m = n. \end{cases}$$

Proof. We can conclude this theorem as follows:
Case 1 (if $m = n = 2$): The result is clear by Lemma 1.
Case 2 (if $2 < m < n$): The proof implies from Lemma 6.
Case 3 (if $2 < m = n$): It follows from Example 7. □

Example 8. For $4 \le n \in \mathbb{N}$. We obtain the following results:
If $n = 4$, then $\left|(P_4 \lhd\rhd P_n)_{P_1}\right| = 2\lceil \frac{n}{2} \rceil - 1$.
Also, if $n > 4$, then $\left|(P_4 \lhd\rhd P_n)_{P_1}\right| = 2\lceil \frac{n}{2} \rceil$.

We see that these two statements satisfy Theorem 1 if we consider P_1 as a clone of a glued graph between P_4 and P_n for $n \ge 4$.

Furthermore, we see that $\left|(P_2 \lhd\rhd P_n)_{P_2}\right| = 1$ where $n \ge 2$ and $\left|(P_3 \lhd\rhd P_n)_{P_3}\right| = 1$ where $n \ge 3$ from Lemma 2 and Lemma 5, respectively. Thus we give the following conclusion.

Theorem 2. *Let $m,n \in \mathbb{N}$ and $2 \le m \le n$. If a graph P_m is a clone of a glued graph between P_m and P_n, then* $\left|(P_m \lhd\rhd P_n)_{P_m}\right| = 1$.

Proof. The result is clear due to the embedding of P_m in P_n. There exists unique resulting glued graph between P_m in P_n at a clone P_m, which is always the graph P_n. □

Next, we recommend a solution for being the useful tool in finding out the number of all non-isomorphic glued graphs of P_m and P_n at a non-trivial clone P_r where $1 < r < m \le n$.

Remark 1. A path P_n usually has $n - r + 1$ subgraphs P_r.

Proof. Let $r,n \in \mathbb{N}$ where $2 \le n$. The number of choosing a possible endpoint of P_r in P_n is $n-r+1$. Thus there exists another vertex of P_n which will be the other endpoint of P_r. Since P_r is an unlabelled undirected path, the symmetry of P_r gives $n - r + 1$ distinct ways to choose P_r. □

4. Characterization of Glued Graphs of Paths with Connected Clones

In this section, we obtain the structure of resulting glued graphs between P_m and P_n having a connected path P_r as a clone where $m,n,r \in \mathbb{N}$ and $r \le m \le n$. The result is an observation due to the construction of glued graphs.

Theorem 3. *Let P_m and P_n be paths of m and n vertices, respectively. Let P_r be a clone of a glued graph between P_m and P_n where P_r is a non-trivial connected path of r vertices where*

$2 \le r \le m \le n$. *Then a resulting graph is satisfying the following properties:*

1. *a glued graph between* P_m *and* P_n *is a simple graph with* $m+n-r$ *vertices and* $m+n-r-1$ *edges.*
2. *a glued graph between* P_m *and* P_n *is a tree with either one of the following conditions:*
   - 2.1. *two endpoints with the degree sequence* $(1,1,2,2,2,\dots,2)$; *or*
   - 2.2. *three endpoints with the degree sequence* $(1,1,1,2,2,2,\dots,2,3)$; *or*
   - 2.3. *four endpoints with the degree sequence* $(1,1,1,1,2,2,2,\dots,2,3,3)$.

Proof. It is clear that all resulting graph obtaining from the gluing of two paths at any non-trivial connected clone is a tree. Thus it is a simple graph with $m+n-r$ vertices and $m+n-r-1$ edges by Proposition 1. The degree sequence of a resulting graph depends on the glued vertices and its endpoints. □

Corollary 1. *Let* P_m *and* P_n *be paths with* m *and* n *vertices, respectively. Let* P_1 *be a clone of a glued graph between* P_m *and* P_n. *Then a resulting graph is a tree with* $m+n-1$ *vertices and* $m+n-2$ *edges, satisfying either one of the followings:*

1. *the degree of the glued vertex is* 2, *and the degree sequence of a glued graph is* $(1,1,2,2,\dots,2)$;
2. *the degree of the glued vertex is* 3, *and the degree sequence of a glued graph is* $(1,1,1,2,2,\dots,2,3)$;
3. *the degree of the glued vertex is* 4, *and the degree sequence of a glued graph is* $(1,1,1,1,2,2,\dots,2,4)$.

Proof. Similarly, it can be implied by Theorem 3. □

5. Conclusion

In this paper, we study the resulting graphs obtaining from the gluing on a pair of isomorphic subgraphs of two original paths. Due to varieties of the occurring forms, we hope that our results will be applicable to solve real-life problems such as finding the number of possible transit routes, planning an itinerary, or sorting out a location as the hub of activities.

Acknowledgment

This research was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] V. Boonthong, P. Putthapiban, P. Chaisuriya and P. Pacheenburawana, Eulerian glued graphs, *Thai J. Math.* **8**(1) (2010), 103 – 109, URL http://thaijmath.in.cmu.ac.th/index.php/thaijmath/article/view/307.

[2] J. L. Gross and J. Yellen, *Graph Theory and Its Applications*, 2nd ed., Chapman & Hall, United States of America (2006), URL http://cds.cern.ch/record/902845.

[3] S. Malila, *The Upper and Lower Independence Numbers of Glued Graphs of Cycles*, Master Thesis, Mahidol University, Thailand (2014), URL: https://graduate.mahidol.ac.th/engine/currentstudents/detail/abstract_view.php?id=5436612.

[4] J. Mekwian, *Glued Graphs for Solving E-logistics Network Problems*, Master Thesis, King Mongkut's Institute of Technology Ladkrabang, Thailand (2007), URL: http://newtdc.thailis.or.th/docview.aspx?tdcid=330282.

[5] W. Pimpasalee and C. Uiyyasathian, Clique coverings of glued graphs at complete clone, *Int. Math. Forum* **5**(24) (2010), 1155 – 1166, URL: https://pdfs.semanticscholar.org/e9a7/9935e33fda38cc4b168594f6f7e1f90a2d09.pdf.

[6] C. Promsakon, *Colorability of Glued Graphs*, master thesis, Chulalongkorn University, Thailand (2006), URL https://cuir.car.chula.ac.th/handle/123456789/15010.

[7] S. M. Seyyedi and F. Rahmati, Some properties of glued graphs at complete clone in the view of algebraic combinatorics, *Sci. Int. (Lahore)* **27**(1) (2014), 39 – 47, URL: http://www.sci-int.com/pdf/112019674939-47%20-Seyyede%20Masoome%20Seyyedi%20%20COMPOSED%20MATH.--IRAN-GP.pdf.

[8] C. Uiyyasathian, *Maximal-Clique Partitions*, Doctoral Thesis, University of Colorado at Denver, USA (2003), 39 – 42, URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.93.4100&rep=rep1&type=pdf.

[9] C. Uiyyasathian and U. Jongthawonwuth, Clique partitions of glued graphs, *J. Math. Research* **2**(2) (2010), 104 – 111, DOI: 10.5539/jmr.v2n2p104.

[10] C. Uiyyasathian and S. Saduakdee, Perfect glued graph at complete clones, *J. Math. Research* **1**(1) (2009), 25 – 30, DOI: 10.5539/jmr.v1n1p25.

[11] R. Wilson and J. J. Watkins, *Graphs: An Introductory Approach A First course in Discrete Mathematics*, John Wiley & Sons, United States of America (1990), URL: https://www.wiley.com/enus/Graphs%3A+An+Introductory+Approach+A+First+Course+in+Discrete+Mathematics-p-9780471615545.

[12] R. Wilson, *Introduction to Graph Theory*, 4th edition, Addison Wesley Longman Limited, England (1996), URL: https://pdfslide.net/documents/robin-j-wilson-introduction-tograph-theory-4th-edition.html.