Communications in Mathematics and Applications
Vol. 10, No. 4, pp. 733–744, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i4.1250

Research Article

The Analysis of Bifurcation Solutions by Angular Singularities

Hussein K. Kadhim* and Mudhir A. Abdul Hussain

Department of Mathematics, Faculty of Education for Pure Sciences, University of Basrah, Basrah, Iraq
***Corresponding author:** khashanhussein@gmail.com

Abstract. This paper studies a nonlinear wave equation's bifurcation solutions of elastic beams situated on elastic bases with small perturbation by using the local method of Lyapunov-Schmidt. We have found the Key function corresponding to the functional related to this equation. The bifurcation analysis of this function has been investigated by the angular singularities. We have found the parametric equation of the bifurcation set (caustic) with the geometric description of this caustic. Also, the critical points' bifurcation spreading has been found.

Keywords. Bifurcation solutions; Angular Singularities; Caustic

MSC. 34C23; 34F10; 37G10

Received: May 22, 2019 **Accepted:** July 8, 2019

1. Introduction

The nonlinear problems which occur in mathematics and physics may be formed in the form of operator equation

$$f(x,\lambda)=b, \quad x\in O,\ b\in Y,\ \lambda\in\mathbb{R}^n. \tag{1}$$

In which f is a smooth Fredholm map whose index is zero and X, Y are *Banach's* spaces and $O\subseteq X$ is open. The method of reduction for these problems to the finite dimensional equation

$$\Theta(\xi,\lambda)=\beta, \quad \xi\in M,\ \beta\in N \tag{2}$$

may be used, where M and N are smooth finite dimensional manifolds. Lyapunov-Schmidt method can reduce equation (1) to equation (2), in which equation (2) has all the analytical and topological features of equation (1) (bifurcation diagram, multiplicity, etc.), as such information can be found in [7], [8], [9], [13]. Singularities of smooth maps play an important part in the investigation of bifurcation solutions of BVPs. One can find a good review of these studies in [6]. In initial years, the study of singularities of smooth maps and its applications to the BVPs took an important character in the works of Sapronov and his group. For example, in [12] Shvyreva studied the boundary singularities of the function

$$\tilde{W}(\eta,\gamma) = \eta_1^4 + (c\eta_1+\eta_2)^2 - 2\varepsilon_1\eta_1^2 + 2\varepsilon_2\eta_1^2\eta_2 + 2\varepsilon_3\eta_1\eta_2 + 2\varepsilon_4\eta_1 + 2\varepsilon_5\eta_2,$$

where $\eta = (\eta_1,\eta_2)$, $\gamma = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$, $\eta_1,\eta_2 \geq 0$ and considered the functional,

$$V(u,\lambda) = \int_0^\pi \left(\frac{(u')^2}{2} + \lambda(\cos(u(x))-1)\right)dx,$$

with the extra condition and in [1] Abdul Hussain has studied the following problem,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^2 = 0, \quad u(0) = u(1) = u''(0) = u''(1) = 0,$$

with the extra condition $u(x_1) \geq 0$, $u(x_2) \geq 0$, $x_1, x_2 \in [0,1]$. Our study differs from the previous studies that we have taken a new boundary value problem with a key function defined on 3-space domain. Lyapunov-Schmidt method supposes that $f : \Omega \subset E \to F$ is a smooth nonlinear Fredholm map of index zero. The map f has variational property, when there is a functional $V : \Omega \subset E \to \mathbb{R}$ such that $f = \mathrm{grad}_H V$ or equivalently, $\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda),\, h\rangle_H$, for all $x \in \Omega$, $h \in E$, where $\langle\cdot,\cdot\rangle_H$ is the scalar inner product in Hilbert space H. Also it assumes that $E \subset F \subset H$. The solutions of equation $f(x,\lambda) = 0$ are the own critical points of functional $V(x,\lambda)$. The finite dimensional reduction method (Lyapunov-Schmidt method) can reduce the problem, $V(x,\lambda) \to \mathrm{extr}\, x \in E$, $\lambda \in \mathbb{R}^n$ into equivalent problem $W(\xi,\lambda) \to \mathrm{extr}$, $\xi \in \mathbb{R}^n$, where $W(\xi,\lambda)$ is called key function. If we let $N = \mathrm{span}\{e_1,\ldots,e_n\}$ is a subspace of Banach space E, where $e_1,\ldots,e_n$ is an orthonormal set in H, then the key function $W(\xi,\lambda)$ may be defined by the form of $W(\xi,\lambda) = \inf\limits_{x:\langle x,e_i\rangle=\xi_i} V(x,\lambda)$, $\xi m = (\xi_1,\ldots,\xi_n)$. The function W possesses every the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, etc.) [8]. The study of bifurcating solutions of functional V is tantamount to the study of bifurcating solutions of key function. If f possesses a variational property, then the equation $\Theta(\xi,\lambda) = \mathrm{grad}\, W(\xi,\lambda) = 0$ is called bifurcating equation.

Definition 1.1 ([4])**.** The set of every λ for which the function $f(x,\lambda)$ possesses degenerate critical points is called bifurcation set (Caustic) and denoted by Σ.

2. Angular Singularities of Fredholm Functional [11]

To investigate the behavior of a Fredholm functional in a vicinity of an angular singular point, one uses the reduction to an analogous problem

$$W(x) \to \mathrm{extr}$$

where

$$x \in C,\ C = \left\{x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n : x_2 \geq 0, x_3 \geq 0\right\}.$$

We state that a point $a \in C$ is conditionally critical for a smooth function W in $\mathbb{R}^n$ if $\mathrm{grad}\, W(a)$ is perpendicular to the least face of C containing a.

The multiplicity of the conditionally critical point a (and is denoted by $\overline{\mu}$) is the dimension of the quotient algebra denotes the set

$$\widehat{Q} = \frac{\prod_a(\mathbb{R}^n)}{I},$$

where $\prod_a(\mathbb{R}^n)$ is smooth functions' the germs ring on $\mathbb{R}^n$ at the point a and

$$I := \left(\frac{\partial W}{\partial x_1}, x_2\frac{\partial W}{\partial x_2}, x_3\frac{\partial W}{\partial x_3}, \dots, \frac{\partial W}{\partial x_n}\right)$$

is the angular Jacobi ideal in $\prod_a(\mathbb{R}^n)$. The multiplicity $\overline{\mu}$ of a conditionally critical point a is the equal sum to multiplicities $\mu + \mu_0$, where μ is the (usual) multiplicity of W on $\mathbb{R}^n$, while μ_0 is the (usual) multiplicity of the restriction $W|\partial C$ (where ∂C is the boundary of the set C).

Then, we reduce the space of $W(x)$, $x \in \mathbb{R}^n$ to the space C by letting $\{e_1, \dots, e_n\}$ to be an orthonormal set in H. By Lyapunov-Schmidt method one can write any element $z \in E$ as the form, $z = u + v$ where $u = \sum_{i=1}^n x_i e_i$, $v \perp e_i$, $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.

If we consider there is a condition on E (domain of functional V) is as following, let z to be in E where z fulfills the following condition:

$$\langle z, e_3 \rangle. \tag{3}$$

Thus, we get $x_3 \geq 0$.

In addition, let $\widehat{\mathbb{R}}^n$ be a space with coordinates $y = (y_1, y_2, \dots, y_n)$ defined by map

$$\pi : \widehat{\mathbb{R}}^n \to \mathbb{R}^n, \quad \pi(y) = (y_1, y_2^2, y_3 \dots, y_n).$$

At that point the function $W(x)$, $x \in \mathbb{R}^n$ is defined on space $\widehat{\mathbb{R}}^n$ by the relation $\widehat{W}(\pi(y)) = \widehat{W}(y)$. The function $\widehat{W}$ is invariable with respect to the natural involution

$$J(y_1, y_2, \dots, y_n) = (y_1, -y_2, \dots, y_n).$$

From the definition of function π we obtain $x_2 = y_2^2$, so we have $x_2 \geq 0$. From above we conclude the domain of $W(x)$, $x \in \mathbb{R}^n$ may reduce to the space C.

If a critical point is "usual", then spreadings of bifurcating extremes (bif-spreadings) are represented by the row $(c_0, c_1, \dots, c_n)$, where c_i is the number of critical points of the Morse index i. If we are dealing with an angular critical point, then bif-spreadings are represented by the following matrix of order $3 \times (n+1)$:

$$\begin{pmatrix} c_0^1 & c_1^1 & \cdots & c_n^1 \\ c_0^2 & c_1^2 & \cdots & c_n^2 \\ c_0 & c_1 & \cdots & c_n \end{pmatrix}.$$

Here c_i^j is the numeral of the angular critical points of index i (for $j = 1, 2$), while c_i is the numeral of usual (situated inside C) critical points of index i.

3. The Nonlinear Wave Equation's Bifurcation Solutions

This section investigates a fourth order nonlinear differential equation's the bifurcation solutions. This equation describes the oscillations and motion of wave of the elastic beams on elastic foundations of periodic solutions that can be described by means of the coming ODE

$$az'''' + bz'' + cz + z^2 + z^3 = 0, \tag{4}$$

$$z(0) = z(1) = z''(0) = z''(1) = 0,$$

where a, b and c are the problem's the parameters, $z = z(x)$, $x \in [0,1]$, $' = \frac{d}{dx}$. Assume that $f : E \to M$ is a nonlinear Fredholm operator whose index equal zero from Banach space E to Banach space M, where $E = C^4([0,1],\mathbb{R})$ is the space of every continuous functions that have derivative of order at most four, $M = C^0([0,1],\mathbb{R})$ is the space of every continuous function and f are defined by the operator equation

$$f(u,\lambda) = az'''' + bz'' + cz + z^2 + z^3 = 0, \tag{5}$$

where $\lambda = (a,b,c)$. Every solution of the equation (4) (1-periodic solution) is a solution of the operator equation (5). Since, the operator f possesses variational property, then there exists functionals V such that,

$$f(z,\lambda) = \mathrm{grad}_H V(z,\lambda),$$

where

$$V(z,\lambda) = \int_0^1 \left(a\frac{(z'')^2}{2} - b\frac{(z')^2}{2} + c\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} \right) dx,$$

where z fulfills the condition (3) (when $n = 3$).

In this case every solution of equation (5) is functional V's a critical point.

The purpose of the paper is to find the solution areas of equation (4) where each bifurcating solution of equation (4) equals a critical point of functional V and each critical point of functional V coincides a critical point of the key function of functional V. Therefor, in subsections below, we shall investigate a function's the extremes bifurcation in which is its extremes bifurcation's study tantamount investigating the key function's the extremes bifurcation of functional V (i.e. the study of functional V's bifurcating solutions is tantamount to the study of bifurcating solutions of this function). Hence,the study of equation (4)'s bifurcating solutions is equivalent to the study of bifurcating solutions of this function.

3.1 Singularities of the Function of Codimension Twenty Six

In this section, we consider the function that have codimension twenty six at the origin [2] defined by

$$\begin{aligned} W(s_1,\rho) = {} & \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^4}{4} - x_1^3x_3 + x_1^2x_2^2 + x_1^2x_3^2 + x_1x_2^2x_3 + x_2^2x_3^2 + x_1^3 - x_1^2x_3 + x_1x_2^2 + x_1x_3^2 \\ & + x_2^2x_3 + x_3^3 + \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2 \end{aligned} \tag{6}$$

where $s_1 = (x_1,x_2,x_3)$, $x_3 \geq 0$ and $\rho = (\lambda_1,\lambda_2,\lambda_3)$.

Function (6) has multiplicity 27 and then it has codimension 26. The main purpose is to find geometrical description (bifurcation diagram) of the caustic of function (6) and then to determine the critical points' the spreading of this function. To avoid some difficulties in the study of function (6), we assume the following $x_1 = x$, $x_2^2 = y$ and $x_3 = z$. So the study of function (6) is tantamount to the study of the following function

$$\begin{aligned} W(s_2,\rho) = {} & \frac{x^4}{4} + \frac{y^2}{4} + \frac{z^4}{4} - x^3z + x^2y + x^2z^2 + xyz + yz^2 + x^3 - x^2z + xy + xz^2 + yz + z^3 \\ & + \lambda_1x^2 + \lambda_2y + \lambda_3z^2, \end{aligned} \tag{7}$$

where $s_2 = (x,y,z)$, $y \geq 0$, $z \geq 0$ and $\rho = (\lambda_1, \lambda_2, \lambda_3)$. Since, the germ of function (7) is

$$W_0 = \frac{x^4}{4} + \frac{y^2}{4} + \frac{z^4}{4}.$$

So, from Section 2 we have $I = \left(\frac{\partial W_0}{\partial x}, y\frac{\partial W_0}{\partial y}, z\frac{\partial W_0}{\partial z}\right) = \left(x^3, \frac{y^2}{2}, z^4\right) = (x^3, y^2, z^4)$. Accordingly, the multiplicity of function (7) is $\overline{\mu} = 24$ where $\mu = 6$ and $\mu_0 = 18$. Since multiplicity is equal to the number of critical points [3], so the number of critical points of function (7) is 24, twelve points lie on the boundary $y = 0$, six points lie on the boundary $z = 0$ and the last six points lie in the domain's the interior of this function. So the caustic of function (7) is the union of six sets,

$$\Sigma = \Sigma_{0,0,0} \cup \Sigma_{1,0,1}^{int} \cup \Sigma_{1,0,1}^{ext} \cup \Sigma_{1,1,0}^{int} \cup \Sigma_{1,1,0}^{ext} \cup \Sigma_{1,1,1}$$

where $\Sigma_{0,0,0}$ is the subset (component) of the caustic corresponding to degeneration at the vertex $(0,0,0)$, $\Sigma_{1,0,1}^{int}$ and $\Sigma_{1,0,1}^{ext}$ are the subsets (components) of the caustic corresponding to the degeneration of angular singularities along the boundary $y = 0$ and along the normal, respectively, $\Sigma_{1,1,0}^{int}$ and $\Sigma_{1,1,0}^{ext}$ are the subsets (components) of the caustic corresponding to the degeneration of angular singularities along the boundary $z = 0$ and along the normal, respectively, while $\Sigma_{1,1,1}$ is the component corresponding to the degeneration of interior (nonboundary) critical points.

3.2 Degeneration at the Vertex $(0,0,0)$

To determine the set $\Sigma_{0,0,0}$, we must find the following union

$$\left\{\lambda : \frac{\partial W(0,0,0,\lambda)}{\partial x} = 0\right\} \cup \left\{\lambda : \frac{\partial W(0,0,0,\lambda)}{\partial y} = 0\right\} \cup \left\{\lambda : \frac{\partial W(0,0,0,\lambda)}{\partial z} = 0\right\}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. This union yields the coming equation $\lambda_2 = 0$.

3.3 Degeneration Along the Boundary $y = 0$ (internal degeneration)

To determine the set $\Sigma_{1,0,1}^{int}$, we consider boundary critical points of function (7) such that the determinate of Hessian matrix of this function vanishes at these points, i.e the coming relations are valid:

$$\frac{\partial W(x,0,z,)}{\partial x} = \frac{\partial W(x,0,z,)}{\partial z} = \det(H(x,0,z,)) = 0, \quad z > 0,$$

where H is Hessian matrix at $(x,0,z,)$, det is its determinate and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. These relations yield the following relations,

$$\begin{aligned} x^3 - 3x^2z + 2xz^2 + 3x^2 - 2xz + 2x\lambda_1 + z^2 &= -x^3 + 2x^2z + z^3 - x^2 + 2xz + 3z^2 + 2z\lambda_3 \\ &= 2\lambda_1 - 12x - 14\lambda_3xz - 8\lambda_1xz + 11xz^3 - 5\lambda_1z^2 \\ &\quad - 5\lambda_3x^2 + 2\lambda_1\lambda_3 - 2\lambda_3x - 2\lambda_1z - 6z\lambda_3 - z^3 \\ &\quad - 8z^4 - \frac{49x^4}{2} - 14xz - 2x\lambda_1 - 22x^3z + \frac{45x^2z^2}{2} \\ &\quad - 49x^2z - 13xz^2 - 5z^2 - 51x^3 - 39x^2 - 2\lambda_3 = 0. \end{aligned}$$

Theoretically, it is difficult to solve the above relations, so we use MAPLE 2016 soft program to eliminate variables x, z to yield the parametric equation which represents the set $\Sigma_{1,0,1}^{int}$, but we wo not write it here because its length is very long (it may reach more than eight pages).

3.4 Degeneration Along the Boundary $y = 0$ (external degeneration)

To determine the set $\Sigma^{ext}_{1,0,1}$, we consider boundary critical points of function (7) such that its first order partial derivative with regard to y vanishes at these points, i.e, the coming relations are valid:

$$\frac{\partial W(x,0,z,\lambda)}{\partial x} = \frac{\partial W(x,0,z,\lambda)}{\partial z} = \frac{\partial W(x,0,z,\lambda)}{\partial y} = 0, \quad z > 0,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. These relations yield the following relations,

$$x^3 - 3x^2z + 2xz^2 + 3x^2 - 2xz + 2x\lambda_1 + z^2 = -x^3 + 2x^2z + z^3 - x^2 + 2xz + 3z^2 + 2z\lambda_3 = x^2 + xz + z^2 + x + z\lambda_2 = 0.$$

Theoretically, it is difficult to solve the above relations,so we use MAPLE 2016 soft program to eliminate variables x, z to yield the parametric equation which represents the set $\Sigma^{ext}_{1,0,1}$, but we won't write it here because its length is long.

3.5 Degeneration Along the Boundary $z = 0$ (internal degeneration)

The coming theorem gives the equation which represents the set $\Sigma^{int}_{1,1,0}$.

Theorem 3.1. *The parametric equation which represents the set $\Sigma^{int}_{1,1,0}$ is given by the form* (8) *below.*

Proof. To determine the set $\Sigma^{int}_{1,1,0}$, we consider boundary critical points of function (7) such that the determinate of Hessian matrix of this function vanishes at these points, i.e, the coming relations are valid:

$$\frac{\partial W(x,y,0,\lambda)}{\partial x} = \frac{\partial W(x,y,0,\lambda)}{\partial y} = \det(H(x,y,0,\lambda)) = 0, \quad y > 0,$$

where H is Hessian matrix at $(x,y,0,\lambda)$, det is its determinate and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. These relations yield the following relations,

$$\begin{aligned} x^3 + 3x^2 + 2xy + 2\lambda_1 x + y &= \frac{y}{2} + x^2 + x + \lambda_2 \\ &= -2\lambda_1 - 12x - 2y - 5\lambda_3 x^2 + 2\lambda_1\lambda_3 - 2\lambda_3 x + 2\lambda_3 y + 2\lambda_1 y + \frac{3}{2}y^2 \\ &\quad - \frac{49x^4}{2} - 2\lambda_1 x + 2x^2 y + 4xy - 51x^3 - 39x^2 - 2\lambda_3 \\ &= 0 \end{aligned}$$

For solving these relations simultaneously, we use MAPLE 2016 soft program for eliminating x, y to get the following equation which represents the set $\Sigma^{int}_{1,1,0}$,

$$\begin{aligned} &2888\lambda_1^5\lambda_2 - 1444\lambda_1^5\lambda_3 - 25612\lambda_1^4\lambda_2^2 + 15352\lambda_1^4\lambda_2\lambda_3 - 912\lambda_1^4\lambda_3^2 + 91488\lambda_1^3\lambda_2^3 \\ &\quad - 64168\lambda_1^3\lambda_2^2\lambda_3 + 7296\lambda_1^3\lambda_2\lambda_3^2 - 144\lambda_1^3\lambda_3^3 - 164392\lambda_1^2\lambda_2^4 + 132512\lambda_1^2\lambda_2^3\lambda_3 \\ &\quad - 21888\lambda_1^2\lambda_2^2\lambda_3^2 + 864\lambda_1^2\lambda_2\lambda_3^3 + 148472\lambda_1\lambda_2^5 - 135668\lambda_1\lambda_2^4\lambda_3 + 29184\lambda_1\lambda_2^3\lambda_3^2 \\ &\quad - 1728\lambda_1\lambda_2^2\lambda_3^3 - 53868\lambda_2^6 + 55208\lambda_2^5\lambda_3 - 14592\lambda_2^4\lambda_3^2 + 1152\lambda_2^3\lambda_3^3 + 1444\lambda_1^5 \\ &\quad - 17836\lambda_1^4\lambda_2 + 6676\lambda_1^4\lambda_3 + 80992\lambda_1^3\lambda_2^2 - 42632\lambda_1^3\lambda_2\lambda_3 + 2706\lambda_1^3\lambda_3^2 - 187976\lambda_1^2\lambda_2^3 \\ &\quad + 111984\lambda_1^2\lambda_2^2\lambda_3 - 10080\lambda_1^2\lambda_2\lambda_3^2 + 378\lambda_1^2\lambda_3^3 + 228332\lambda_1\lambda_2^4 - 152648\lambda_1\lambda_2^3\lambda_3 \\ &\quad + 17082\lambda_1\lambda_2^2\lambda_3^2 - 540\lambda_1\lambda_2\lambda_3^3 - 114428\lambda_2^5 + 90412\lambda_2^4\lambda_3 - 15492\lambda_2^3\lambda_3^2 + 1026\lambda_2^2\lambda_3^3 \\ &\quad - 4320\lambda_1^4 + 35652\lambda_1^3\lambda_2 - 10846\lambda_1^3\lambda_3 - 92764\lambda_1^2\lambda_2^2 + 44442\lambda_1^2\lambda_2\lambda_3 - 2871\lambda_1^2\lambda_3^2 \end{aligned}$$

$$+122860\lambda_1\lambda_2^3-62682\lambda_1\lambda_2^2\lambda_3+5274\lambda_1\lambda_2\lambda_3^2-324\lambda_1\lambda_3^3-81028\lambda_2^4+43886\lambda_2^3\lambda_3$$
$$-2223\lambda_2^2\lambda_3^2+4876\lambda_1^3-33236\lambda_1^2\lambda_2+8320\lambda_1^2\lambda_3+45196\lambda_1\lambda_2^2-20476\lambda_1\lambda_2\lambda_3$$
$$+1272\lambda_1\lambda_3^2-25700\lambda_2^3+11140\lambda_2^2\lambda_3-1140\lambda_2\lambda_3^2+90\lambda_3^3-2457\lambda_1^2+15202\lambda_1\lambda_2$$
$$-3118\lambda_1\lambda_3-7653\lambda_2^2+3494\lambda_2\lambda_3-195\lambda_3^2+466\lambda_1-2796\lambda_2+466\lambda_3=0 \tag{8}$$

3.6 Degeneration Along the Boundary $z=0$ (external degeneration)

The coming theorem gives the equation which represents the set $\Sigma_{1,1,0}^{ext}$.

Theorem 3.2. *The parametric equation which represents the set* $\Sigma_{1,1,0}^{ext}$ *is given by the form* (16) *below.*

Proof. To determine the set $\Sigma_{1,1,0}^{ext}$, we consider boundary critical points of function (7) such that function (7)'s partial derivative of first order with regard to z vanishes at these points, i.e, the coming relations are valid:

$$\frac{\partial W(x,y,0,\lambda)}{\partial x}=\frac{\partial W(x,y,0,\lambda)}{\partial y}=\frac{\partial W(x,y,0,\lambda)}{\partial z}=0,\quad y>0,$$

where $\lambda=(\lambda_1,\lambda_2,\lambda_3)$. These relations yield the coming relations,

$$x^3+3x^2+2xy+2x\lambda_1+y=\frac{y}{2}+x^2+x+\lambda_2=-x^3-x^2+xy+y=0.$$

We may represent the above relations as equations system,

$$x^3+3x^2+2xy+2x\lambda_1+y=0, \tag{9}$$
$$\frac{y}{2}+x^2+x+\lambda_2=0, \tag{10}$$
$$-x^3-x^2+xy+y=0., \tag{11}$$

From equation (10), we get, $y=-2x^2-2x-2\lambda_2$ and substituting y in equations (9), (11) respectively we obtain,

$$-3x^3-3x^2+2x\lambda_1-4x\lambda_2-2x-2\lambda_2=0, \tag{12}$$
$$3x^3+5x^2+2x\lambda_2+2x+2\lambda_2=0. \tag{13}$$

By adding equation (12) into (13), one get: $2x(x-\lambda_2+\lambda_1)=0$, which implies $x=0$ or $x=\lambda_2-\lambda_1$. By substituting $x=0$ and $x=\lambda_2-\lambda_1$ in equation (12) respectively, we get

$$\lambda_2=0, \tag{14}$$
$$3\lambda_1^3-9\lambda_1^2\lambda_2+9\lambda_1\lambda_2^2-3\lambda_2^3-5\lambda_1^2+12\lambda_1\lambda_2-7\lambda_2^2+2\lambda_1-4\lambda_2=0. \tag{15}$$

Hence, the product of multiplying two left sides of equations (14) and (15) with equal it to zero represents the set $\Sigma_{1,1,0}^{ext}$ such that it may be expressed as follows,

$$\lambda_2(3\lambda_1^3-9\lambda_1^2\lambda_2+9\lambda_1\lambda_2^2-3\lambda_2^3-5\lambda_1^2+12\lambda_1\lambda_2-7\lambda_2^2+2\lambda_1-4\lambda_2)=0$$

or

$$\lambda_2(\lambda_1-\lambda_2-1)(3\lambda_1^2-6\lambda_1\lambda_2+3\lambda_2^2-2\lambda_1+4\lambda_2)=0. \tag{16}$$

3.7 Degeneration of Interior (Nonboundary)

To determine the set $\Sigma_{1,1,1}$, we consider function (7)'s the critical points defined by the system,

$$\frac{\partial W(x,y,z,\lambda)}{\partial x}=\frac{\partial W(x,y,z,\lambda)}{\partial y}=\frac{\partial W(x,y,z,\lambda)}{\partial z}=0, \quad y>0,\ z>0,$$

where $\lambda=(\lambda_1,\lambda_2,\lambda_3)$, or

$$\begin{aligned}&x^3-3x^2z+2xz^2+3x^2+2xy-2xz+2x\lambda_1+yz+z^2+y=0,\\ &\frac{y}{2}+x^2+xz+z^2+x+z+\lambda_2=0,\\ &-x^3+2x^2z+z^3-x^2+xy+2xz+2yz+3z^2+2z\lambda_3+y=0.\end{aligned} \tag{17}$$

Then, make the Hessian matrix's the determinate of function (7)equal to zero to get the equation,

$$\begin{aligned}&-16xyz-2\lambda_3x-5\lambda_1z^2-5\lambda_3x^2+2\lambda_1\lambda_3+2\lambda_3y+2\lambda_1y-2\lambda_1z+11xz^3-2\lambda_1-2\lambda_3\\ &\quad+\frac{45x^2z^2}{2}-yz^2-22x^3z+2x^2y-4yz-\frac{49x^4}{2}-13xz^2+4xy-14xz-49x^2z-14\lambda_3xz\\ &\quad-2x\lambda_1+\frac{3}{2}y^2-8\lambda_1xz-8z^4-5z^2-39x^2-51x^3-2y-12x-z^3-6z\lambda_3=0.\end{aligned} \tag{18}$$

Theoretically, it is difficult to solve the system (17) with equation (18). So, we use MAPLE 2016 soft program in eliminating the variables x, y and z to get the the parametric equation of the set $\Sigma_{1,1,1}$, but we won't write it here because its length is very long (it may reach more than fifteen pages).

Theorem 3.3. *The matrices of bif-spreadings of the critical points of function* (7) *are as follows:*

1. *If* $\lambda_1=1$, *then we have*

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&3&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&3&0\\0&1&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&3&0\\0&0&0\\0&0&0\end{pmatrix}.$$

2. *If* $\lambda_1\neq 1$ *then we have*

$$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&2&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&2&0\\0&1&0\\0&1&0\end{pmatrix}.$$

Proof. Firstly, if we write every equation of the caustic components' the equations by making its left hand side=its right hand side= 0, then the parametric equation of caustic of function (7) will consist of the product of multiplying of the left hand sides of all the equations of caustic components with making it equaling to zero.

To find the caustic of the function (7) it is convenient to fix the value of λ_1 and then find all sections of caustic in $\lambda_2\lambda_3$-plane as λ_1 changes.

The caustic of function (7) as in Figure 1 decomposes the plane of parameters into seven regions W_i, $i=1,2,3,4,5,6,7$; every region contains a fixed number of critical points such that the critical points' the spreading is as follows: if the parameters λ_2, λ_3 belong to

1. W_1 or W_2, then have two critical points (one saddle point on boundary $y=0$ and one minimum point on boundary $z=0$), or

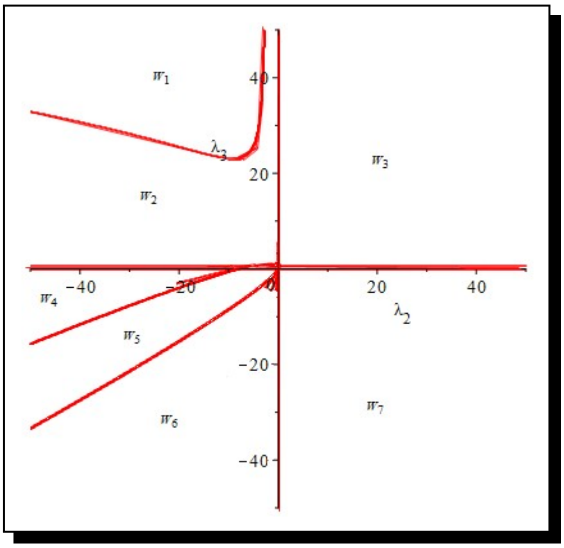

Figure 1. Portrays the caustic of function (7) in $\lambda_2\lambda_3$-plane when $\lambda_1 = 1$

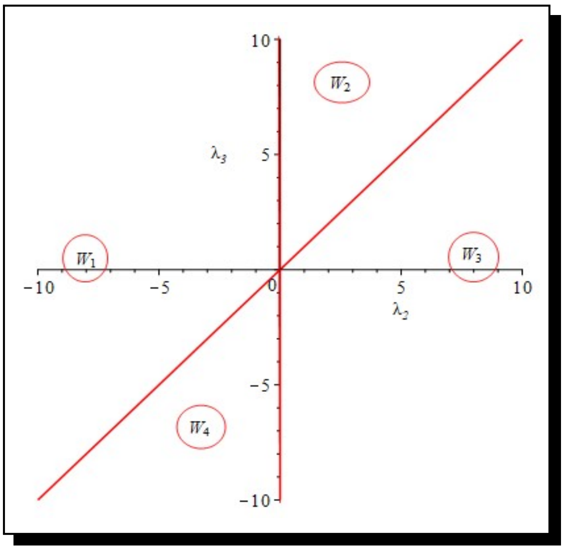

Figure 2. Portrays the caustic of function (7) in $\lambda_2\lambda_3$-plane when $\lambda_1 \neq 1$

2. W_3, then have one saddle critical point on boundary $y = 0$, or
3. W_4 or W_5, then have four critical points (three saddle points on boundary $y = 0$ and one minimum point on boundary $z = 0$), or
4. W_6, then have five critical points (three saddle points on boundary $y = 0$, one saddle point on boundary $z = 0$ and one saddle point in the interior), or
5. W_7, then have three saddle critical points on boundary $y = 0$.

Hence, the matrices of bif-spreadings are as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The caustic of function (7) as in Figure 2 decomposes the plane of parameters into four regions W_i, $i = 1,2,3,4$; every region contains a fixed number of critical points such that the critical points' the spreading is as follows: if the parameters λ_2, λ_3 belong to

1. W_1, then we have one minimum critical point on boundary $z = 0$, or
2. W_3, then we have two saddle critical points on boundary $y = 0$, or
3. W_4, then we have four critical points (two saddle points on boundary $y = 0$, one saddle point on boundary $z = 0$ and one saddle point in the interior), or
4. W_2, then we have not any real critical point in the domain $\{(x, y, z) : y \geq 0, z \geq 0\}$

Therefor, the matrices of bif-spreadings are as follows:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In addition, the values of the Morse index at a given vertex corresponds to one of the previous seven regions in figure 1 are defined as follows: $index = 0 \Leftrightarrow \lambda_2$, $\lambda_3 \in W_3 \cup W_7$; and $index = 1 \Leftrightarrow \lambda_2$, $\lambda_3 \in W_1 \cup W_2 \cup W_4 \cup W_5 \cup W_6$. And the values of the Morse index at a given vertex belongs to one of the previous four regions in Figure 2 are defined as follows: $index = 0 \Leftrightarrow \lambda_2$, $\lambda_3 \in W_2 \cup W_3$; and $index = 1 \Leftrightarrow \lambda_2$, $\lambda_3 \in W_1 \cup W_4$. □

In the following theorem, we prove that investigating of functional V's extremes bifurcation is reduced to investigating of function (4)'s extremes bifurcation.

Theorem 3.4. *The normal form of the key function W_1 corresponding to the functional V is given by,* $W_1(y,\rho) = \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^4}{4} - x_1^3 x_3 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_1 x_2^2 x_3 + x_2^2 x_3^2 + x_1^3 - x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_3^3 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$ *where* $y = (x_1, x_2, x_3)$, $x_3 \geq 0$ *and* $\rho = (\lambda_1, \lambda_2, \lambda_3)$.

Proof. By using the system of Lyapunov-Schmidt, the linearized equation which corresponds equation (5) at the point $(0,\lambda)$ possesses the form,

$$Ah = 0, \ h \in E$$
$$h(0) = h(1) = h''(0) = h''(1) = 0$$

where $A = a\frac{d^4}{dx^4} + b\frac{d^2}{dx^2} + c$.

The solution of the linearized equation which satisfies the initial conditions is given by $e_q(x) = c_q \sin(q\pi x)$, $q = 1,2,\ldots$ and the characteristic equation which corresponds this solution is

$$a(q\pi)^4 - b(q\pi)^2 + c = 0.$$

This equation gives in 3-*space* characteristic planes $\mathscr{P}_q$. The characteristic planes $\mathscr{P}_q$ consist of the points (a,b,c) for which the linearized equation has non-zero solutions [5]. The point of intersection of the characteristic planes in 3-*space* is bifurcation point, so the bifurcation point for the equation (5) is $(a,b,c) = (0,0,0)$. Localized parameters a, b and c as following: $a = 0 + \delta_1$, $b = 0 + \delta_2$, $c = 0 + \delta_3$, δ_1, δ_2 and δ_3 are small parameters, lead to bifurcation along the modes, $e_1(x) = c_1 \sin(\pi x)$, $e_2(x) = c_2 \sin(2\pi x)$ and $e_3(x) = c_3 \sin(3\pi x)$. Since, $\|e_1\| = \|e_2\| = \|e_3\| = 1$ then we have $c_1 = c_2 = c_3 = \sqrt{2}$.

Let $N = Ker(A) = span\{e_1, e_2, e_3\}$, then the space E is decomposed as direct sum of two subspaces, N and the orthogonal complement to N,

$$E = N \oplus N^{\perp},\ N^{\perp} = \left\{v \in E : \int_0^1 v e_k dx = 0,\ k = 1,2,3\right\}.$$

Similarly, the space M may be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$M = N \oplus \tilde{N}^{\perp},\ \tilde{N}^{\perp} = \left\{\omega \in M : \int_0^1 \omega e_k dx = 0,\ k = 1,2,3\right\}.$$

There exists two projections $P : E \to N$ and $I - P : E \to N^{\perp}$ such that $Pz = \omega$ and $(I - P)z = v$, (I is the identity operator). Hence every vector $z \in E$ can be formulated in the form,

$$z = \omega + v,\ \omega = x_1 e_1 + x_2 e_2 + x_3 e_3 \in N,\ v \in N^{\perp},\ x_i = \langle z, e_i \rangle.$$

Thus, by the implicit function's theorem , there exists a smooth map $\Theta : N \to N^{\perp}$, such that

$$\tilde{W}(u,\gamma) = V(\Theta(u,\gamma),\gamma),$$

$$u = (x_1, x_2, x_3),\quad \gamma = (\delta_1, \delta_2, \delta_3)$$

and then the key function $\tilde{W}$ can be formulated in the the form,

$$\begin{aligned}\tilde{W}(u,\gamma) &= V(x_1e_1 + x_2e_2 + x_3e_3 + \Theta(x_1e_1 + x_2e_2 + x_3e_3,\gamma),\gamma)\\ &= W_2(u,\gamma) + o(|u|^4) + O(|u|^4)O(\gamma),\end{aligned}$$

where

$$\begin{aligned}W_2(u,\gamma) = {}& \frac{3}{8}{x_1}^4 + \frac{3}{8}{x_2}^4 + \frac{3}{8}{x_3}^4 - \frac{1}{2}{x_1}^3 x_3 + \frac{3}{2}{x_1}^2{x_2}^2 + \frac{3}{2}{x_1}^2{x_3}^2 + \frac{3}{2}{x_2}^2 x_3 x_1 + \frac{3}{2}{x_2}^2{x_3}^2\\ &+ \frac{8\sqrt{2}{x_1}^3}{9\pi} - \frac{8\sqrt{2}{x_1}^2 x_3}{15\pi} + \frac{32\sqrt{2}{x_2}^2 x_1}{15\pi} + \frac{72\sqrt{2}x_1{x_3}^2}{35\pi} + \frac{32\sqrt{2}{x_2}^2 x_3}{21\pi} + \frac{8\sqrt{2}{x_3}^3}{27\pi}\\ &+ \left(\frac{1}{2}\pi^4 a - \frac{1}{2}\pi^2 b + \frac{c}{2}\right){x_1}^2 + \left(8\pi^4 a - 2\pi^2 b + \frac{c}{2}\right){x_2}^2 + \left(\frac{81\pi^4 a}{2} - \frac{9}{2}\pi^2 b + \frac{c}{2}\right){x_3}^2.\end{aligned}$$

Because z fulfills the condition (3) (when $n = 3$), we can get $x_3 \geq 0$.

The geometrical form of critical points' bifurcations and the first asymptotic of branches of bifurcating for the function $\tilde{W}$ are completely determined by its principal part W_2. If, we replace x_1, x_2 and x_3 by $\sqrt[4]{\frac{2}{3}}x_1$, $\sqrt[4]{\frac{2}{3}}x_2$ and $\sqrt[4]{\frac{2}{3}}x_3$ in function W_2 respectively, then W_1 and W_2 are contact equivalence,since in this case they have the same germ,

$$W_0(x_1,x_2,x_3) = \frac{3}{8}{x_1}^4 + \frac{3}{8}{x_2}^4 + \frac{3}{8}{x_3}^4$$

and deformation. Therefore the caustic of the function W_2 coincides with the caustic of the function W_1.

Thus,the function W_1 has all the topological and analytical properties of functional V, so the study of bifurcation analysis of the equation (5) is equivalent to the study of bifurcation analysis of the function W_1. This shows that the study of bifurcation of extremes of the functional V is reduced to the study of bifurcation of extremes of the function (7). □

4. Conclusion

In this paper the solution areas for the equation (4) are found in two cases, in the first case, seven regions are found, and in the second case, four regions are found. Each region contains a fixed number and quality of solutions. Each solution represents a critical point of functional, which in turn corresponds to a critical point of the key function of functional. Furthermore, the geometrical description of the branching diagram (caustic) was found with spreading of the branching of the critical points for both cases. Studying the branching solutions for the equation (4) is an application for studying singularities of the function (6).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. A. Abdul Hussain, *Corner Singularities of Smooth Functions in the Analysis of Bifurcations Balance of the Elastic Beams and Periodic Waves*, Ph.D. Thesis, Voronezh University, Russia (2005) (in Russian).

[2] V. I. Arnold, *Singularities of Differential Maps*, M. Science (1989).

[3] G. Berczi, *Lectures on Singularities of Maps*, Trinity Term, Oxford (2010).

[4] B. M. Darinskii, C. L. Tcarev and Yu. I. Sapronov, Bifurcations of extremals of Fredholm functionals, *Journal of Mathematical Sciences* **145**(6) (2007).

[5] E. V. Derunova and Yu. I. Sapronov, Application of normalized key functions in a problem of branching of periodic extremals, *Russian Mathematics (Iz. Vuz)* **59**(8) (2015).

[6] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*: Vol. I (Applied Mathematical Sciences 51), Springer, New York (1985), DOI: 10.1007/978-1-4612-5034-0.

[7] B. V. Loginov, *Theory of Branching Nonlinear Equations in The Conditions of Invariance Group*, Fan, Tashkent (1985).

[8] Yu. I. Sapronov, Regular perturbation of Fredholm maps and theorem about odd field, *Works Dept. of Math.*, Voronezh University, **10**, 82 – 88 (1973).

[9] Yu. I. Sapronov, Finite dimensional reduction in the smooth extremely problems, *Uspehi Math. Science* **51**(1) (1996), 101 – 132.

[10] Yu. I. Sapronov and V. R. Zachepa, *Local Analysis of Fredholm Equation*, Voronezh University (2002) (in Russian).

[11] Yu. I. Sapronov, B. M. Darinskii and C. L. Tcarev, *Bifurcation of Extremely of Fredholm Functionals*, Voronezh University (2004).

[12] O. V. Shveriova, Caustic and bif-spreadings for a boundary extremal with a triple degeneration along the boundary, *Tr. Mat. Fak. Voronezh. Gos. Univ.* (N.S.) **7** (2002), 149 – 160 (in Russian).

[13] M. M. Vainberg and V. A. Trenogin, *Theory of Branching Solutions of Non-linear Equations*, M. Science (1969), DOI: 10.1137/1018033.