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Research Article

Hyers-Ulam-Rassias Stability of the C^*-ternary Bi-homomorphisms and C^*-ternary Bi-derivations in C^*-ternary Algebras

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Abstract. In this paper, we prove Hyers-Ulam-Rassias stability of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras by using alternative fixed point theorem.

Keywords. Hyers-Ulam-Rassias stability; C^*-ternary bi-homomorphisms; C^*-ternary bi-derivations; C^*-ternary algebras

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1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, the set of real numbers, the set of complex numbers, respectively.

The stability problem of functional equations was initiated by Ulam in 1940 [23] arising from concerning the stability of group homomorphisms. These question form is the object of the stability theory. If the answers is affirmative, we say that the functional equation for homomorphism is *stable*. In 1941, Hyers [11] provided a first affirmative partial answer to Ulam's problem for the case of approximately additive mapping in Banach spaces. In 1978, Rassias [22] provided a generalization of Hyers's theorem for linear mapping by considering an unbounded Cauchy difference. In 1994, a generalization of Rassias's results was developed by

Găvruţa [9] by replacing the unbounded Cauchy difference by a general control function. For more information on that subject and further references we refer to a survey paper [4] and to a recent monograph on Ulam stability [5].

A C^*-ternary algebras is a complex Banach space $\mathbb{A}$, equipped with a ternary product $(x,y,z) \mapsto [x,y,z]$ of $\mathbb{A}^3$ into $\mathbb{A}$, which is $\mathbb{C}$-linear in the outer variables and conjugate $\mathbb{C}$-linear in the middle variable:

(i) $[\lambda x + y, v, w] = \lambda[x,v,w] + [y,v,w]$,

(ii) $[v,w,\lambda x + y] = \lambda[v,w,x] + [v,w,y]$,

(iii) $[v,\lambda x + y, w] = \bar{\lambda}[v,x,w] + [v,y,w]$,

and associative in the sense that

$$[[v,w,x],y,z] = [v,[y,x,w],z] = [v,w,[x,y,z]]$$

and satiesfies

$$\|[x,y,z]\| \le \|x\|\|y\|\|z\|, \ \ \|[x,x,x]\| = \|x\|^3$$

for all $v,w,x,y,z \in \mathbb{A}$.

If a C^*-ternary algebra $(\mathbb{A},[\cdot,\cdot,\cdot])$ has an identity, i.e. an element $e \in \mathbb{A}$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in \mathbb{A}$, then it is routine to verify that $\mathbb{A}$, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is unital C^*-algebra. Conversely, if $(\mathbb{A},\circ)$ is a unital C^*-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes $\mathbb{A}$ into C^*-ternary algebra. A C^*-ternary algebras have many applications in fractional quatum Hall effect, the nonstandard statistics, hypothetical, supersymmetric theory, and Yang-Baxter equation (see [1,13,24])

In 2010, Bae and Park [2] proved the Hyers-Ulam stability of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras.

Definition 1.1 ([2]). Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-ternary algebras. A $\mathbb{C}$-linear mapping $H : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ is called a C^*-ternary bi-homomorphisms if

$$\begin{aligned} &H([x,y,z],w) = [H(x,w),H(y,w),H(z,w)], \\ &H(x,[y,z,w]) = [H(x,y),H(x,z),H(x,w)] \end{aligned} \tag{1.1}$$

for all $x,y,z,w \in \mathbb{A}$. A $\mathbb{C}$-bilinear mapping $\delta : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is called a C^*-ternary bi-derivations if

$$\begin{aligned} &\delta([x,y,z],w) = [\delta(x,w),y,z] + [x,\delta(y,w),z] + [x,y,\delta(z,w)], \\ &\delta(x,[y,z,w]) = [\delta(x,y),z,w] + [y,\delta(x,y),w] + [y,z,\delta(x,w)] \end{aligned} \tag{1.2}$$

for all $x,y,z,w \in \mathbb{A}$.

It is easy to show that the definitions of bi-homomorphisms and bi-derivations are meaningless. Indeed, replacing w in $2w$ in (1.1), we get that

$$\begin{aligned} 2H([x,y,z],w) = H([x,y,z],2w) & \\ &= [H(x,2w),H(y,2w),H(z,2w)] \\ &= [2H(x,w),2H(y,w),2H(z,w)] \\ &= 8[H(x,w),H(y,w),H(z,w)] \\ &= 8H([x,y,z],w) \end{aligned}$$

and so $H([x,y,z],w)=0$ for all $x,y,w,z\in\mathbb{A}$. Similary, $H(x,[y,z,w])=0$ for all $x,y,z,w\in\mathbb{A}$. This implies that (1.1) is meaningless. Replacing w by iw in (1.2), we have

$$\begin{aligned}
i\delta([x,y,z],w) &= \delta([x,y,z],iw)\\
&= [\delta(x,iw),y,z]+[x,\delta(y,iw),z]+[x,y,\delta(z,iw)]\\
&= [i\delta(x,w),y,z]+[x,i\delta(y,w),z]+[x,y,i\delta(z,w)]\\
&= i[\delta(x,w),y,z]-i[x,\delta(y,w),z]+i[x,y,\delta(z,w)]\\
&= i\,([\delta(x,w),y,z]-[x,\delta(y,w),z]+i[x,y,\delta(z,w)])\\
&\neq i\delta([x,y,z],w)
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. Similary, $\delta(x,[y,z,w])=0$ for all $x,y,z,w\in\mathbb{A}$. This implies that (1.2) is also meaningless. Next, Park [16] corrected the above definition as follows

Definition 1.2. Let $\mathbb{A}$ and $\mathbb{B}$ be C^*-ternary algebras. A $\mathbb{C}$-bilinear mapping $H:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ is called a C^*-ternary bi-homomorphisms if

$$\begin{aligned}
H([x,y,z],[w,w,w]) &= [H(x,w),H(y,w),H(z,w)],\\
H([x,x,x],[y,z,w]) &= [H(x,y),H(x,z),H(x,w)]
\end{aligned}\tag{1.3}$$

for all $x,y,z,w\in\mathbb{A}$. A $\mathbb{C}$-linear mapping $\delta:\mathbb{A}\times\mathbb{A}\to\mathbb{A}$ is called a C^*-ternary bi-derivation if

$$\begin{aligned}
\delta([x,y,z],w) &= [\delta(x,w),y,z]+[x,\delta(y,w^*),z]+[x,y,\delta(z,w)],\\
\delta(w,[y,z,w]) &= [\delta(x,y),z,w]+[y,\delta(x^*,y),w]+[y,z,\delta(x,w)]2
\end{aligned}\tag{1.4}$$

for all $x,y,z,w\in\mathbb{A}$.

Moreover, Park[16] proved the Hyers-Ulam stability of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras and investigated the stability results of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras associated with the following bi-additive s-functional inequalities

$$\begin{aligned}
&\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\|\\
&\quad\le\left\|s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|
\end{aligned}\tag{1.5}$$

and

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right\|\\
&\quad\le\|s(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w))\|
\end{aligned}\tag{1.6}$$

where s is a fixed nonzero complex number with $|s|<1$ by direct method in the sense of Rassias.

The fixed point method was applied to study the stability of functional equations by Baker in 1991 [3] by using the Banach contraction principle. Next, Radu [20] proved a stability of functional equation by the alternative of fixed point which was introduced by Diaz and Margolis [8]. The fixed point method has provided a lot of influence in the development of stability. We refer to many works of stability of functional equations using the fixed point method in books on stability of functional equations (see[6, 7, 10, 12, 21]).

Throughout this paper, assume that $\mathbb{A}$ is a unital C^*-ternary algebra with norm $\|\cdot\|_{\mathbb{A}}$ and unit e, and that $\mathbb{B}$ is a C^*-ternary algebra with norm $\|\cdot\|_{\mathbb{B}}$ and unit e'. The purpose of the present paper is to investigate the stability of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras and prove the stability results of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras associated with the following bi-additive functional inequalities (1.5) and (1.6) by using the alternative fixed point theorem in sense of Găvruţ.

We recall a fundamental result in fixed point theory. The following is the definition of generalized metric space which was introduced by Luxemburg in 1958 [14].

Definition 1.3 ([14])**.** Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(iii) $d(x,z) \le d(x,y)+d(y,z)$, for all $x,y,z \in X$.

The following fixed point theorem will play important roles in proving our main results.

Theorem 1.4 ([8])**.** *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a strictly contractive mapping, that is,*

$$d(Tx,Ty) \le kd(x,y)$$

for all $x,y \in X$ and for some Lipshitz $k < 1$. Then for each given element $x \in X$, either

$$d(T^n x, T^{n+1} x) = \infty$$

for all nonnegative integer n or there exists a positive ineteger n_0 such that

(i) *$d(T^n x, T^{n+1} x) < \infty$ for all $n \ge n_0$,*

(ii) *the sequence $\{T^n x\}$ converges to a fixed point y^* of T,*

(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X \mid d(T^{n_0} x, y) < \infty\}$,*

(iv) *$d(y,y^*) \le \frac{1}{1-k} d(y,Ty)$, for all $y \in Y$.*

From now on, let Ω be the set of all the mappings $g : X \times X \to Y$ satisfying $g(0,0) = 0$.

Lemma 1.5 ([17])**.** *Let $\psi : X \times X \to [0,\infty)$ be a function. Consider the generalized metric d on Ω given by*

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h),$$

where $S_\psi(g,h) := \{K \in [0,\infty] \mid \|g(x,y) - h(x,y)\| \le K\psi(x,y)$ for all $x,y,\in X\}$ for all $g,h \in \Omega$. Then (Ω,d) is complete.

Lemma 1.6 ([2])**.** *Let X and Y be $\mathbb{C}$-linear spaces and let $f : X \times X \to Y$ be a bi-additive mapping such that $f(\lambda x, \mu y) = \lambda\mu f(x,y)$ for all $\lambda,\mu\mathbb{S} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and for all $x,y \in X$, then f is $\mathbb{C}$-bilinear.*

Lemma 1.7 ([2])**.** *Let X and Y be $\mathbb{C}$-linear spaces and let $f : X \times X \to Y$ be a mapping such that*

$$f(\lambda x+\lambda y, \mu z-\mu w)+f(\lambda x-\lambda y, \mu z+\mu w) = 2\lambda\mu f(x,z)-2\lambda\mu f(y,w)$$

for all $\lambda,\mu \in \mathbb{S}$ and all $x,y,z,w \in X$. Then f is $\mathbb{C}$-bilinear.

2. Stability of C^*-ternary Bi-homomorphisms in C^*-ternary Algebras

Let f be a mapping of $\mathbb{A}\times\mathbb{A}$ into $\mathbb{B}$. We define

$$E_{\lambda,\mu}f(x,y,z,w) := f\left((\lambda(x+y),\mu(z-w)\right)+f\left(\lambda(x-y),\mu(z+w)\right)-2\lambda\mu f(x,z)+2\lambda\mu f(y,w)$$

for all $\lambda,\mu\in\mathbb{S}$, for all $x,y,z,w\in\mathbb{A}$.

We prove the Hyers-Ulam-Rassias stability of C^*-ternary bi-homomorphisms for the functional equation $E_{\lambda,\mu}f(x,y,z,w)=0$ in C^*-ternary algebras.

Theorem 2.1. *Let $f:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ be a mapping with $f(0,0)=0$ such that $\varphi:\mathbb{A}^4\to[0,\infty)$ satisfying*

$$\left\|E_{\lambda,\mu}f(x,y,z,w)\right\|\le\varphi(x,y,z,w) \tag{2.1}$$

and

$$\|f([x,y,z],[w,w,w])-[f(x,w),f(y,w),f(z,w)]\|+\|f([x,x,x],[y,z,w])-[f(x,y),f(x,z),f(x,w)]\|$$
$$\le\varphi(x,y,z,w) \tag{2.2}$$

for all $\lambda,\mu\in\mathbb{S}$ and all $x,y,z,w\in\mathbb{A}$. If there exists a Lipschitz constant $L<1$ such that

$$\varphi(x,y,z,w)\le 4L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right) \tag{2.3}$$

for all $x,y,z,w\in\mathbb{A}$. Then there exists a unique C^*-ternary bi-homomorphism $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ such that

$$\|f(x,y)-F(x,y)\|\le\frac{L}{1-L}\psi(x,y) \tag{2.4}$$

where $\psi:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ is a function given by

$$\begin{aligned}\psi(x,y) &:= \varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right)+\frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right)+\frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right)\\ &\quad+\frac{1}{2}\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right)+\frac{1}{2}\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)\end{aligned} \tag{2.5}$$

for all $x,y\in\mathbb{A}$.

Proof. Consider the set

$$\Omega := \{g\mid\mathbb{A}\times\mathbb{A}\to\mathbb{B}, g(0,0)=0\}$$

Consider the generalized metric d on Ω given by

$$d(g,h)=d_\psi(g,h):=\inf S_\psi(g,h)$$

for all $g,h\in\Omega$. By Lemma 1.5, the generalized metric space (Ω,d) is complete. Now, we define a linear mapping $T:\Omega\to\Omega$ by

$$Tg(x,y):=\frac{1}{4}g(2x,2y)$$

for all $g\in\Omega$ and all $x,y\in\mathbb{A}$. By the proof of Theorem 4 in [18], letting $\lambda=\mu=1$, we have the inequality

$$\left\|f(x,y)-\frac{1}{4}f(2x,2y)\right\|$$
$$\le\frac{1}{4}\varphi(x,x,y,-y)+\frac{1}{8}\left[\varphi(x,x,y,y)+\varphi(x,x,-y,-y)+\varphi(0,0,y,-y)+\varphi(0,0,-y,y)\right]$$

for all $x, y \in \mathbb{A}$. By (2.3) and (2.5), we get

$$\begin{aligned}
&\left\| f(x,y) - \frac{1}{4} f(2x,2y) \right\| \\
&\le \frac{1}{4}\varphi(x,x,y,-y) + \frac{1}{8}\left[\varphi(x,x,y,y) + \varphi(x,x,-y,-y) + \varphi(0,0,y,-y) + \varphi(0,0,-y,y)\right] \\
&\le \frac{1}{4} 4L\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right) + \frac{1}{8}\left[4L\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right) + 4L\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right)\right. \\
&\quad \left. + 4L\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right) + 4L\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)\right] \\
&= L\psi(x,y)
\end{aligned} \tag{2.6}$$

for all $x, y \in \mathbb{A}$. It follows from (2.6) that we obtain

$$d(f, Tf) \le L.$$

By the proof Theorem 2.3 in [17], we obtain that

$$d(Tg, Th) \le L d(g,h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with Lipschitz constant L. By Theorem 1.4, there exists a unique mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ such that

$$F(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in \mathbb{A}$. Moreover, we obtain that $d(f,F) \le \frac{1}{1-L} d(f,Tf)$, which implies that

$$d(f,F) \le \frac{1}{1-L} d(f,Tf) \le \frac{L}{(1-L)}.$$

Therefore, the inequality (2.4) holds. Then we obtain that

$$\begin{aligned}
&\|F(\lambda x + \lambda y, \mu z - \mu w) + F(\lambda x - \lambda y, \mu z + \mu w) - 2\lambda\mu F(x,z) + 2\lambda\mu F(y,w)\| \\
&= \lim_{n\to\infty} \frac{1}{4^n} \left\| f\left(\lambda 2^n x + \lambda 2^n y, \mu 2^n z - \mu 2^n w\right) + f\left(\lambda 2^n x - \lambda 2^n y, \mu 2^n z + \mu 2^n w\right) \right. \\
&\qquad\qquad \left. -2\lambda\mu f\left(2^n x, 2^n z\right) + 2\lambda\mu f\left(2^n y, 2^n w\right) \right\| \\
&\le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z, 2^n w) \\
&\le \lim_{n\to\infty} \frac{4L}{4^n} \varphi(2^{n-1} x, 2^{n-1} y, 2^{n-1} z, 2^{n-1} w) \\
&= \lim_{n\to\infty} \frac{L}{4^{n-1}} \varphi(2^{n-1} x, 2^{n-1} y, 2^{n-1} z, 2^{n-1} w) \\
&\ \vdots \\
&\le \lim_{n\to\infty} L^n \varphi(x,y,z,w) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathbb{A}$. This implies that

$$F(\lambda x + \lambda y, \mu z - \mu w) + F(\lambda x - \lambda y, \mu z + \mu w) = 2\lambda\mu F(x,z) - 2\lambda\mu F(y,w) \tag{2.7}$$

for all $x, y, z, w \in \mathbb{A}$. By Lemma 1.7, the mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ is $\mathbb{C}$-bilinear. It follows that from (2.2) that

$$\begin{aligned}
&\|F([x,y,z],[w,w,w]) - [F(x,w),F(y,w),F(z,w)]\| \\
&+ \|F([x,x,x],[y,z,w]) - [F(x,y),F(x,z),F(x,w)]\|
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\frac{1}{64^n}\left\|f([2^n x,2^n y,2^n z],[2^n w,2^n w,2^n w])-\left[f(2^n x,2^n w),f(2^n y,2^n w),f(2^n z,2^n w)\right]\right\| \\
&\quad + \lim_{n\to\infty}\frac{1}{64^n}\left\|f([2^n x,2^n x,2^n x],[2^n y,2^n z,2^n w])-\left[f(2^n x,2^n y),f(2^n x,2^n z),f(2^n x,2^n w)\right]\right\| \\
&\le \lim_{n\to\infty}\frac{1}{64^n}\varphi(2^n x,2^n y,2^n z,2^n w) \\
&\ \vdots \\
&\le \lim_{n\to\infty}\left(\frac{L}{16}\right)^n\varphi(x,y,z,w)=0
\end{aligned}
$$

for all $x,y,z,w\in\mathbb{A}$. Therefore we get that

$$
\begin{aligned}
F([x,y,z],[w,w,w]) &= [F(x,w),F(y,w),F(z,w)] \\
F([x,x,x],[y,z,w]) &= [F(x,y),F(x,z),F(x,w)]
\end{aligned}
$$

for all $x,y,z,w\in\mathbb{A}$. This completes of proof. □

Theorem 2.2. *Let $f:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ be a mapping with $f(0,0)=0$ such that $\varphi:\mathbb{A}^4\to[0,\infty)$ satisfying* (2.1) *and* (2.2) *for all $\lambda,\mu\in\mathbb{S}$ and all $x,y,z,w\in\mathbb{A}$. If there exists a Lipschitz constant $L<1$ such that*

$$\varphi(x,y,z,w)\le\frac{L}{64}\varphi(2x,2y,2z,2w) \tag{2.8}$$

for all $x,y,z,w\in\mathbb{A}$. Then there exists a unique C^*-ternary bi-homomorphism $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ such that

$$\|f(x,y)-F(x,y)\|\le\frac{1}{1-L}\psi(x,y) \tag{2.9}$$

where $\psi:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ is a function given by

$$
\begin{aligned}
\psi(x,y):=&\,\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right)+\frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right)+\frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right) \\
&+\frac{1}{2}\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right)+\frac{1}{2}\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)
\end{aligned}
$$

for all $x,y\in\mathbb{A}$.

Proof. Now we define a linear mapping $T:\Omega\to\Omega$ by

$$Tg(x,y):=4g\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $g\in\Omega$ and all $x,y\in\mathbb{A}$. By the proof of Theorem 4 in [18], we have the inequality

$$
\begin{aligned}
&\left\|f(x,y)-\frac{1}{4}f(2x,2y)\right\| \\
&\quad\le\frac{1}{4}\varphi(x,x,y,-y)+\frac{1}{8}\left[\varphi(x,x,y,y)+\varphi(x,x,-y,-y)+\varphi(0,0,y,-y)+\varphi(0,0,-y,y)\right]
\end{aligned}
$$

for all $x,y\in\mathbb{A}$. Replacing x and y by $\frac{x}{2}$ and $\frac{y}{2}$ in the above inequality, respectively, we have

$$
\begin{aligned}
&\left\|f\left(\frac{x}{2},\frac{y}{2}\right)-\frac{1}{4}f(x,y)\right\| \\
&\le\frac{1}{4}\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right)+\frac{1}{8}\left[\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right)+\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right)+\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right)+\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)\right]
\end{aligned}
$$

and so we have

$$\begin{aligned}
&\left\|4f\left(\frac{x}{2},\frac{y}{2}\right)-f(x,y)\right\| \\
&\le \varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right)+\frac{1}{2}\left[\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right)+\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right)+\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right)+\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)\right] \\
&= \psi(x,y) \qquad (2.10)
\end{aligned}$$

for all $x,y\in\mathbb{A}$, that is, it follows from (2.10) that we obtain

$$d(f,Tf)\le 1.$$

By the proof Theorem 2.4 in [17], we obtain that

$$d(Tg,Th)\le Ld(g,h)$$

for all $g,h\in\Omega$, that is, T is a strictly contractive mapping of Ω with Lipschitz constant L. By Theorem 1.4, there exists a unique mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ such that

$$F(x,y)=\lim_{n\to\infty}4^n f\left(\frac{x}{2^n},\frac{y}{2^n}\right)$$

for all $x,y\in\mathbb{A}$. Moreover, we obtain that $d(f,F)\le\frac{1}{1-L}d(f,Tf)$, which implies that

$$d(f,F)\le\frac{1}{1-L}d(f,Tf)\le\frac{1}{(1-L)}.$$

Therefore, the inequality (2.9) holds. Then we obtain that

$$\begin{aligned}
&\|F(\lambda x+\lambda y,\mu z-\mu w)+F(\lambda x-\lambda y,\mu z+\mu w)-2\lambda\mu F(x,z)+2\lambda\mu F(y,w)\| \\
&=\lim_{n\to\infty}4^n\left\|f\left(\frac{\lambda x+\lambda y}{2^n},\frac{\mu z-\mu w}{2^n}\right)+f\left(\frac{\lambda x-\lambda y}{2^n},\frac{\mu z+\mu w}{2^n}\right)-2\lambda\mu f\left(\frac{x}{2^n},\frac{z}{2^n}\right)+2\lambda\mu f\left(\frac{y}{2^n},\frac{w}{2^n}\right)\right\| \\
&\le\lim_{n\to\infty}4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right) \\
&\le\lim_{n\to\infty}4^n\frac{L}{64}\varphi\left(\frac{x}{2^{n-1}},\frac{y}{2^{n-1}},\frac{z}{2^{n-1}},\frac{w}{2^{n-1}}\right) \\
&\vdots \\
&\le\lim_{n\to\infty}\left(\frac{L}{16}\right)^n\varphi(x,y,z,w)=0
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. This implies that F satisfies (2.7). By Lemma 1.7, the mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ is $\mathbb{C}$-bilinear. It follows that from (2.2) and (2.8) that

$$\begin{aligned}
&\|F([x,y,z],[w,w,w])-[F(x,w),F(y,w),F(z,w)]\| \\
&+\|F([x,x,x],[y,z,w])-[F(x,y),F(x,z),F(x,w)]\| \\
&=\lim_{n\to\infty}64^n\left\|f\left(\left[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right],\left[\frac{w}{2^n},\frac{w}{2^n},\frac{w}{2^n}\right]\right)-\left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right),f\left(\frac{y}{2^n},\frac{w}{2^n}\right),f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\right\| \\
&\quad+\lim_{n\to\infty}64^n\left\|f\left(\left[\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\right],\left[\frac{y}{2^n},\frac{y}{2^n},\frac{w}{2^n}\right]\right)-\left[f\left(\frac{x}{2^n},\frac{y}{2^n}\right),f\left(\frac{x}{2^n},\frac{z}{2^n}\right),f\left(\frac{x}{2^n},\frac{w}{2^n}\right)\right]\right\| \\
&\le\lim_{n\to\infty}64^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right) \\
&\vdots \\
&\le\lim_{n\to\infty}L^n\varphi(x,y,z,w)=0
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. Therefore we get that F is a C^*-ternary bi-homomorphisms. This completes of proof. □

Remark 2.3. Let r and θ be positive real numbers.

- If we take
$$\varphi(x,y,z,w) = \theta\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right)$$
for all $x,y,z,w \in \mathbb{A}$, then we get that
  - (i) If $r < 2$, then Theorem 2.1 recover Theorem 1 in [16] with the Lipschitz $L = 2^{r-2}$.
  - (ii) If $r > 6$, then Theorem 2.2 recover Theorem 2 in [16] with the Lipschitz $L = 2^{6-r}$.
- If we take
$$\varphi(x,y,z,w) = \theta\left(\|x\|^r \cdot \|y\|^r \cdot \|z\|^r \cdot \|w\|^r\right)$$
for all $x,y,z,w \in \mathbb{A}$, then we get that
  - (i) If $r < \frac{1}{2}$, then Theorem 2.1 recover Theorem 3 in [16] with the Lipschitz $L = 2^{4r-2}$.
  - (ii) If $r > \frac{3}{2}$, then Theorem 2.2 recover Theorem 4 in [16] with the Lipschitz $L = 2^{6-4r}$.

3. Stability of C^*-ternary Bi-derivations in C^*-ternary Algebras

We prove the Hyers-Ulam-Rassias stability of C^*-ternary bi-derivations for the functional equation $E_{\lambda,\mu}f(x,y,z,w) = 0$ in C^*-ternary algebras.

Theorem 3.1. *Let $f : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ be a mapping with $f(0,0) = 0$ such that $\varphi : \mathbb{A}^4 \to [0,\infty)$ satisfying*
$$\left\|E_{\lambda,\mu}f(x,y,z,w)\right\| \le \varphi(x,y,z,w) \tag{3.1}$$
and
$$\begin{aligned}&\left\|f([x,y,z],w) - [f(x,w),y,z] - \left[x,f(y,w^*),z\right] - [x,y,f(z,w)]\right\|\\ &+\left\|f(x,[y,z,w]) - [f(x,y),z,w] - \left[y,f(x^*,z),w\right] - [y,z,f(x,w)]\right\|\\ &\le \varphi(x,y,z,w)\end{aligned} \tag{3.2}$$
for all $\lambda,\mu \in \mathbb{S}$ and all $x,y,z,w \in \mathbb{A}$. If there exists a Lipschitz constant $L < 1$ such that
$$\varphi(x,y,z,w) \le 4L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right) \tag{3.3}$$
for all $x,y,z,w \in \mathbb{A}$. Then there exists a unique C^*-ternary bi-homomorphism $\delta : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ such that
$$\|f(x,y) - \delta(x,y)\| \le \frac{L}{1-L}\psi(x,y) \tag{3.4}$$
where $\psi : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ is a function given by
$$\begin{aligned}\psi(x,y) :=& \varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right) + \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right) + \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right)\\ &+\frac{1}{2}\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right) + \frac{1}{2}\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)\end{aligned}$$
for all $x,y \in \mathbb{A}$.

Proof. Let (Ω,d) be the generalized metric space as in the proof of Theorem 2.1. We consider the linear mapping $T : \Omega \to \Omega$ such that
$$Tg(x,y) := \frac{1}{4}g(2x,2y)$$

for all $x,y \in \mathbb{A}$ and for all $g \in \Omega$. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-bilinear mapping $\delta : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is defined by

$$\delta(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x,y \in \mathbb{A}$ and satisfies (3.4). It follows form (3.2) and (3.3) that

$$\begin{aligned}
&\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\| \\
&+ \|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\| \\
&\quad = \lim_{n\to\infty} \frac{1}{16^n} \|f([2^n x, 2^n y, 2^z], 2^n w) - [f(2^n x, 2^n w), 2^n y, 2^n z] \\
&\qquad - [2^n x, f(2^n y, 2^n w^*), 2^n z] - [2^n x, 2^n y, f(2^n z, 2^n w)]\| \\
&\qquad + \lim_{n\to\infty} \frac{1}{16^n} \|f(2^n x, [2^n y, 2^n z, 2^n w]) - [f(2^n x, 2^n y), 2^n z, 2^n w] \\
&\qquad - [2^n y, f(2^n x^*, 2^n z), 2^n w] - [2^n y, 2^n z, f(2^n x, 2^n w)]\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{16^n} \varphi(2^n x, 2^n y, 2^n z, 2^n w) \\
&\quad \vdots \\
&\quad \le \lim_{n\to\infty} \left(\frac{L}{4}\right)^n \varphi(x,y,z,w) = 0
\end{aligned}$$

for all $x,y,z,w \in \mathbb{A}$. Therefore we obtain that

$$\begin{aligned}
\delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)] \\
\delta(x,[y,z,w]) &= [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)]
\end{aligned}$$

for all $x,y,z,w \in \mathbb{A}$, that is, δ is C^*-ternary bi-derivations. □

Theorem 3.2. *Let $f : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ be a mapping with $f(0,0) = 0$ such that $\varphi : \mathbb{A}^4 \to [0,\infty)$ satisfying* (3.1) *and* (3.2)*. If there exists a Lipschitz constant $L < 1$ such that*

$$\varphi(x,y,z,w) \le \frac{L}{16} \varphi(2x,2y,2z,2w) \tag{3.5}$$

for all $x,y,z,w \in \mathbb{A}$. Then there exists a unique C^*-ternary bi-derivation $\delta : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ such that

$$\|f(x,y) - \delta(x,y)\| \le \frac{1}{1-L} \psi(x,y) \tag{3.6}$$

where $\psi : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ is a function given by

$$\begin{aligned}
\psi(x,y) := {}& \varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},-\frac{y}{2}\right) + \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},\frac{y}{2},\frac{y}{2}\right) + \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2},-\frac{y}{2},-\frac{y}{2}\right) \\
&+ \frac{1}{2}\varphi\left(0,0,\frac{y}{2},-\frac{y}{2}\right) + \frac{1}{2}\varphi\left(0,0,-\frac{y}{2},\frac{y}{2}\right)
\end{aligned}$$

for all $x,y \in \mathbb{A}$.

Proof. Let (Ω,d) be the generalized metric space as in the proof of Theorem 2.1. We define a linear mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := 4g\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $g \in \Omega$ and all $x,y \in \mathbb{A}$. By the same reasoning as in the proof of Theorem 2.2, there exists a unique $\mathbb{C}$-bilinear mapping $\delta : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is defined by

$$\delta(x, y) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all $x, y \in \mathbb{A}$ and satisfies the inequality (3.6). It follows from (3.2) and (3.5) that

$$\begin{aligned}
&\left\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\right\| \\
&+ \left\|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\right\| \\
&= \lim_{n\to\infty} 16^n \left(\left\| f\left(\left[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right],\frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right),\frac{y}{2^n},\frac{w}{2^n}\right] - \left[\frac{x}{2^n},f\left(\frac{y}{2^n},\frac{w^*}{2^n}\right),\frac{z}{2^n}\right]\right.\right. \\
&\qquad\qquad \left.\left. - \left[\frac{x}{2^n},\frac{y}{2^n},f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\right\|\right) \\
&\quad + \lim_{n\to\infty} 16^n \left\| f\left(\frac{x}{2^n},\left[\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{z}{2^n},\frac{w}{2^n}\right]\right. \\
&\qquad\qquad \left. - \left[\frac{y}{2^n},f\left(\frac{x^*}{2^n},\frac{z}{2^n}\right),\frac{w}{2^n}\right] - \left[\frac{y}{2^n},\frac{z}{2^n},f\left(\frac{x}{2^n},\frac{w}{2^n}\right)\right]\right\| \\
&\le \lim_{n\to\infty} 16^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right) \\
&\ \ \vdots \\
&\le \lim_{n\to\infty} L^n \varphi(x,y,z,w) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathbb{A}$. Therefore we get that F is C^*-ternary bi-derivations. □

Remark 3.3. Let r and θ be positive real numbers.

- If we take

  $$\varphi(x,y,z,w) = \theta\left(\|x\|^r + \|y\ |^r + \|z\|^r + \|w\|^r\right)$$

  for all $x, y, z, w \in \mathbb{A}$, then we get that

  (i) If $r < 2$, then Theorem 3.1 recover Theorem 5 in [16] with the Lipschitz $L = 2^{r-2}$.
  (ii) If $r > 4$, then Theorem 3.2 recover Theorem 6 in [16] with the Lipschitz $L = 2^{4-r}$.

- If we take

  $$\varphi(x,y,z,w) = \theta\left(\|x\|^r \cdot \|y\|^r \cdot \|z\|^r \cdot \|w\|^r\right)$$

  for all $x, y, z, w \in \mathbb{A}$, then we get that

  (i) If $r < \frac{1}{2}$, then Theorem 3.1 recover Theorem 7 in [16] with the Lipschitz $L = 2^{4r-2}$.
  (ii) If $r > 1$, then Theorem 3.2 recover Theorem 8 in [16] with the Lipschitz $L = 2^{4-4r}$.

4. C^*-ternary Bi-homomorphisms on C^*-ternary Algebras Associated with the Bi-additive Functional Inequalities (1.5) and (1.6)

We prove the Hyers-Ulam-Rassias stability of C^*-ternary bi-homomorphisms on C^*-ternary algebras associated with the bi-additive functional inequalities (1.5) and (1.6).

Theorem 4.1. *Let $f:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ be a mapping with $f(x,0)=f(0,z)=0$ such that $\varphi:\mathbb{A}^2\to[0,\infty)$ satisfying*

$$\begin{aligned}&\left\|f\left(\lambda(x+y),\mu(z-w)\right)+f\left(\lambda(x-y),\mu(z+w)\right)-2\lambda\mu f(x,z)+2\lambda\mu f(y,w)\right\|\\&\quad\le\left\|s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|+\varphi(x,y)\varphi(z,w)\end{aligned}\tag{4.1}$$

for all $\lambda,\mu\in\mathbb{S}$ and for all $x,y,z,w\in\mathbb{A}$. If there exists a Lipschitz constant $L<1$ such that

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right)\le\frac{L}{8}\varphi(x,y)\tag{4.2}$$

for all $x,y\in\mathbb{A}$. Then there exists a unique $\mathbb{C}$-bilinear mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{A}$ such that

$$\|f(x,y)-F(x,y)\|\le\frac{L}{8(1-L)}\varphi(x,x)\varphi(z,0)\tag{4.3}$$

In addition, if the mapping $f:\mathbb{A}^2\to\mathbb{A}$ satisfies

$$\|f([x,y,z],[w,w,w])-[f(x,w),f(y,w),f(z,w)]\|\le\varphi(x,y)\varphi(z,w)\tag{4.4}$$

and

$$\|f([x,x,x],[y,z,w])-[f(x,y),f(x,z),f(x,w)]\|\le\varphi(x,y)\varphi(z,w)\tag{4.5}$$

for all $x,y,z,w\in\mathbb{A}$, then the mapping F is C^*-ternary bi-homomorphisms.

Proof. Let $\lambda=\mu=1$, $w=0$ and $y=x$ in (4.1), we obtain that

$$\begin{aligned}&\|f(2x,z)+f(0,z)-2f(x,z)+2f(y,0)\|\\&\quad\le\|s(2f(x,z)+2f(0,z)-2f(x,z)+2f(y,0))\|+\varphi(x,x)\varphi(z,0),\end{aligned}$$

that is,

$$\|f(2x,z)-2f(x,z)\|\le\varphi(x,x)\varphi(z,0)$$

for all $x,z\in\mathbb{A}$. Substituting x by $\frac{x}{2}$ in the above inequality, we get that

$$\left\|f(x,z)-2f\left(\frac{x}{2},z\right)\right\|\le\varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0)\le\frac{L}{8}\varphi(x,x),\varphi(z,0)\tag{4.6}$$

for all $x,z\in\mathbb{A}$. Consider the set

$$S:=\{g:\mathbb{A}^2\to\mathbb{B}\mid g(x,0)=g(0,y)=0,\ \forall\ x,y\in\mathbb{A}\}$$

and introduce the generalized metric on S as follows:

$$d(g,h)=\inf\left\{M\in[0,\infty)\mid\|g(x,y)-h(x,y)\|\le M\varphi(x,x)\varphi(y,0),\ \forall\ x,y\in\mathbb{A}\right\},$$

Then, (S,d) is a complete generalized metric space (see [15, Theorem 2]). Now we consider the linear mapping $T:S\to S$ such that

$$Tg(x,y):=2g\left(\frac{x}{2},y\right)$$

for all $x,y\in\mathbb{A}$. By the proof Theorem 2 in [15], we have

$$d(Tg,Th)\le Ld(g,h)$$

for all $g,h\in S$. It follows from (4.6) that $d(f,Tf)\le\frac{L}{8}$. By Theorem 1.4, there exists a unique mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ such that the equality

$$F(x,y)=\lim_{n\to\infty}2^nf\left(\frac{x}{2^n},y\right)$$

for all $x,y\in\mathbb{A}$. Moreover, we obtain that $d(f,F)\le\frac{1}{1-L}d(f,Tf)$, which implies that

$$d(f,F)\le\frac{1}{1-L}d(f,Tf)\le\frac{L}{8(1-L)}.$$

Therefore, the inequality (4.3) holds. It follows from (4.1) and (4.2) that we get

$$\begin{aligned}
&\left\|F(\lambda x+\lambda y,\mu z-\mu w)+F(\lambda x-\lambda y,\mu z+\mu w)-2\lambda\mu F(x,z)+2\lambda\mu F(y,w)\right\|\\
&=\lim_{n\to\infty}2^n\left\|f\left(\frac{\lambda x+\lambda y}{2^n},\mu z-\mu w\right)+f\left(\frac{\lambda x-\lambda y}{2^n},\mu z+\mu w\right)-2\lambda\mu f\left(\frac{x}{2^n},z\right)+2\lambda\mu f\left(\frac{y}{2^n},w\right)\right\|\\
&\le\lim_{n\to\infty}2^n\left(\left\|s\left(2f\left(\frac{\frac{x+y}{2}}{2^n},z-w\right)+2f\left(\frac{\frac{x-y}{2}}{2^n},z+w\right)-2f\left(\frac{x}{2^n},z\right)+2f\left(\frac{y}{2^n},w\right)\right)\right\|+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi(z,w)\right)\\
&\le\left\|s\left(2\lim_{n\to\infty}2^nf\left(\frac{\frac{x+y}{2}}{2^n},z-w\right)+2\lim_{n\to\infty}2^nf\left(\frac{\frac{x-y}{2}}{2^n},z+w\right)\right.\right.\\
&\qquad\left.\left.-2\lim_{n\to\infty}2^nf\left(\frac{x}{2^n},z\right)+2\lim_{n\to\infty}2^nf\left(\frac{y}{2^n},w\right)\right)\right\|+\lim_{n\to\infty}2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi(z,w)\\
&\le\left\|s\left(2\lim_{n\to\infty}2^nf\left(\frac{\frac{x+y}{2}}{2^n},z-w\right)+2\lim_{n\to\infty}2^nf\left(\frac{\frac{x-y}{2}}{2^n},z+w\right)\right.\right.\\
&\qquad\left.\left.-2\lim_{n\to\infty}2^nf\left(\frac{x}{2^n},z\right)+2\lim_{n\to\infty}2^nf\left(\frac{y}{2^n},w\right)\right)\right\|+\lim_{n\to\infty}\left(\frac{L}{4}\right)^n\varphi(x,y)\varphi(z,w)
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. Then, we have

$$\begin{aligned}
&\left\|F(\lambda x+\lambda y,\mu z-\mu w)+F(\lambda x-\lambda y,\mu z+\mu w)-2\lambda\mu F(x,z)+2\lambda\mu F(y,w)\right\|\\
&\quad\le\left\|s\left(2F\left(\frac{x+y}{2},z-w\right)+2F\left(\frac{x-y}{2},z+w\right)-2F(x,z)+2F(y,w)\right)\right\|
\end{aligned}\tag{4.7}$$

for all $x,y,z,w\in\mathbb{A}$. Letting $w=0$ and $y=x$ in (4.7), we obtain that

$$F(\lambda 2x,\mu z)=2\lambda\mu F(x,z)\tag{4.8}$$

for all $x,z\in\mathbb{A}$. Replacing x in (4.8) by $\frac{x}{2}$, we get that

$$F(\lambda x,\mu z)=2\lambda\mu F\left(\frac{x}{2},z\right)$$

for all $x,z\in\mathbb{A}$. Let $\lambda=\mu=1$ in the above equation, we have $F(x,z)=2F\left(\frac{x}{2},z\right)$ for all $x,z\in\mathbb{A}$. Substituting $\lambda=\mu=1$ in (4.7), we have

$$\begin{aligned}
&\|F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w)\|\\
&\quad\le\left\|s\left(2F\left(\frac{x+y}{2},z-w\right)+2F\left(\frac{x-y}{2},z+w\right)-2F(x,z)+2F(y,w)\right)\right\|\\
&\quad\le\|s(F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w))\|\\
&\quad\le|s|\,\|F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w)\|
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. Since $|s|<1$, we have

$$F(x+y,z-w)+F(x-y,z+w)=2F(x,z)-2F(y,w)\tag{4.9}$$

for all $x,y,z,w\in\mathbb{A}$. By Theorem 2.1 in [19], we have F is bi-additive. Next, we will show that $F(\lambda x,\mu y)=\lambda\mu F(x,y)$ for all $x,y\in\mathbb{A}$. Letting $w=0$ and $y=x$ in (4.1), we have

$$\left\|f(\lambda 2x,\mu z)-2\lambda\mu f(x,z)\right\|\le\varphi(x,x)\varphi(y,0)$$

for all $x,y,z \in \mathbb{A}$. Replacing x by $\frac{x}{2}$ in the above equation, we obtain that
$$\left\| f(\lambda x,\mu z) - 2\lambda\mu f\left(\frac{x}{2},z\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(y,0) \tag{4.10}$$
for all $x,y,z \in \mathbb{A}$. Putting x by $\frac{x}{2^n}$ in (4.10), we have
$$\left\| f\left(\lambda\frac{x}{2^n},\mu z\right) - 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right)\right\| \le \varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\varphi(y,0) \tag{4.11}$$
for all $x,y,z \in \mathbb{A}$. Letting $\lambda = \mu = 1$ in (4.11), we have
$$\left\| f\left(\frac{x}{2^n},z\right) - 2f\left(\frac{x}{2^{n+1}},z\right)\right\| \le \varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\varphi(y,0) \tag{4.12}$$
for all $x,y,z \in \mathbb{A}$. It follows from (4.11) and (4.12) that we have
$$\begin{aligned}\left\| f\left(\lambda\frac{x}{2^n},\mu z\right) - \lambda\mu f\left(\frac{x}{2^n},z\right)\right\| &\le \left\| f\left(\lambda\frac{x}{2^n},\mu z\right) + 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right) - 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right) - \lambda\mu f\left(\frac{x}{2^n},z\right)\right\| \\
&\le \left\| f\left(\lambda\frac{x}{2^n},\mu z\right) + 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right)\right\| + \left\| 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right) - \lambda\mu f\left(\frac{x}{2^n},z\right)\right\| \\
&\le \left\| f\left(\lambda\frac{x}{2^n},\mu z\right) + 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right)\right\| + |\lambda||\mu| \left\| 2f\left(\frac{x}{2^{n+1}},z\right) - f\left(\frac{x}{2^n},z\right)\right\| \\
&\le \left\| f\left(\lambda\frac{x}{2^n},\mu z\right) + 2\lambda\mu f\left(\frac{x}{2^{n+1}},z\right)\right\| + \left\| 2f\left(\frac{x}{2^{n+1}},z\right) - f\left(\frac{x}{2^n},z\right)\right\| \\
&\le 2\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\varphi(y,0)\end{aligned}$$
for all $x,y,z \in \mathbb{A}$. Then, we have
$$\begin{aligned}\left\| 2^n f\left(\lambda\frac{x}{2^n},\mu z\right) - \lambda\mu 2^n f\left(\frac{x}{2^n},z\right)\right\| &\le 2\cdot 2^n\varphi\left(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\varphi(y,0) \\
&\le \left(\frac{L}{4}\right)^{n+1}\varphi(x,x)\varphi(y,0)\end{aligned}$$
for all $x,y,z \in \mathbb{A}$. Taking limit $n \to \infty$ in the above inequality, we have
$$\begin{aligned}\|F(\lambda x,\mu z) - \lambda\mu F(x,z)\| &= \left\| \lim_{n\to\infty} 2^n f\left(\lambda\frac{x}{2^n},\mu z\right) - \lambda\mu \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},z\right)\right\| \\
&\le \lim_{n\to\infty}\left(\frac{L}{4}\right)^{n+1}\varphi(x,x)\varphi(y,0) = 0,\end{aligned}$$
that is, $F(\lambda x,\mu z) = \lambda\mu F(x,z)$ for all $x,z \in \mathbb{A}$. By Lemma 1.7, we obtain that the mapping F is $\mathbb{C}$-bilinear. It follows from (4.4) that
$$\begin{aligned}&\|F([x,y,z],[w,w,w]) - [F(x,w),F(y,w),F(z,w)]\| \\
&\quad\le \lim_{n\to\infty} 64^n \left\| f\left(\left[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right],\left[\frac{w}{2^n},\frac{w}{2^n},\frac{w}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right),f\left(\frac{y}{2^n},\frac{w}{2^n}\right),f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\right\| \\
&\quad\le \lim_{n\to\infty} 64^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi\left(\frac{z}{2^n},\frac{w}{2^n}\right) \\
&\quad\ \vdots \\
&\quad\le \lim_{n\to\infty} L^{2n}\varphi(x,y)\varphi(z,w) = 0\end{aligned}$$
for all $x,y,z,w \in \mathbb{A}$. Thus
$$F([x,y,z],[w,w,w]) = [F(x,w),F(y,w),F(z,w)]$$
for all $x,y,z,w \in \mathbb{A}$. By using (4.5), we also have
$$F([x,x,x],[y,z,w]) = [F(x,y),F(x,z),F(x,w)]$$
for all $x,y,z,w \in \mathbb{A}$. Therefore the mapping $F : \mathbb{A}\times\mathbb{A} \to \mathbb{B}$ is a C^*-ternary bi-homomorphisms. □

Theorem 4.2. *Let $f : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ be a mapping with $f(x,0) = f(0,z) = 0$ for all $x,z \in \mathbb{A}$ such that $\varphi : \mathbb{A}^2 \to [0,\infty)$ satisfying* (4.1)*. If there exists a Lipschitz constant $L < 1$ such that*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{4.13}$$

for all $x,y \in \mathbb{A}$. Then there exists a unique $\mathbb{C}$-bilinear mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that

$$\|f(x,y) - F(x,y)\| \le \frac{1}{2(1-L)}\varphi(x,x)\varphi(z,0) \tag{4.14}$$

In addition, if the mapping $f : \mathbb{A}^2 \to \mathbb{A}$ satisfies (4.4) *and* (4.5) *for all $x,y,z,w \in \mathbb{A}$, then the mapping F is C^*-ternary bi-homomorphisms.*

Proof. Let (S,d) be the generalized metric space defined in proof of Theorem 4.1.

Now, we consider the linear mapping $T : S \to S$ such that

$$Tg(x,y) = \frac{1}{2}g(2x,y)$$

for all $x,y \in \mathbb{A}$ and $g \in S$. Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x,y) - h(x,y)\| \le \varepsilon\varphi(x,x)\varphi(y,0)$$

for all $x,y \in \mathbb{A}$. Hence we have

$$\begin{aligned}
\|Tg(x,y) - Th(x,y)\| &= \left\|\frac{1}{2}g(2x,y) - \frac{1}{2}h(x,y)\right\| \\
&\le \frac{1}{2}\varepsilon\varphi(2x,2x)\varphi(y,0) \\
&\le \frac{1}{2}\varepsilon 2L\varphi(x,x)\varphi(y,0) \\
&= L\varepsilon(x,x)\varphi(y,0)
\end{aligned}$$

for all $x,y \in \mathbb{A}$. This implies that $d(Tg,Th) \le L\varepsilon$. This mean that

$$d(Tg,Th) \le Ld(g,h)$$

for all $g,h \in S$. By the same argument of Theorem 4.1, we have

$$\left\|\frac{1}{2}f(2x,y) - f(x,y)\right\| \le \frac{1}{2}\varphi(x,x)\varphi(y,0)$$

for all $x,y \in \mathbb{A}$. So $d(f,Tf) \le \frac{1}{2}$. By Theorem 1.4, there exists a unique mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ such that the equality

$$F(x,y) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x, y\right)$$

for all $x,y \in \mathbb{A}$. Moreover, we obtain that $d(f,F) \le \frac{1}{1-L}d(f,Tf)$, which implies that

$$d(f,F) \le \frac{1}{1-L}d(f,Tf) \le \frac{1}{2(1-L)}.$$

Therefore, the inequality (4.14) holds.

It follows from (4.1) and (4.13) that we get

$$\begin{aligned}
&\left\|F(\lambda x + \lambda y, \mu z - \mu w) + F(\lambda x - \lambda y, \mu z + \mu w) - 2\lambda\mu F(x,z) + 2\lambda\mu F(y,w)\right\| \\
&= \lim_{n\to\infty} \frac{1}{2^n} \left\|f\left(2^n(\lambda x + \lambda y), \mu z - \mu w\right) + f\left(2^n(\lambda x - \lambda y), \mu z + \mu w\right) - 2\lambda\mu f\left(2^n x, z\right) + 2\lambda\mu f\left(2^n y, w\right)\right\|
\end{aligned}$$

$$\begin{aligned}
&\le \lim_{n\to\infty}\frac{1}{2^n}\left(\left\|s\left(2f\left(2^n\left(\frac{x+y}{2}\right),z-w\right)+2f\left(2^n\left(\frac{x-y}{2}\right),z+w\right)\right.\right.\right.\\
&\qquad \left.\left.\left.-2f\left(2^n x,z\right)+2f\left(2^n y,w\right)\right)\right\|+\varphi\left(2^n x,2^n y\right)\varphi(z,w)\right)\\
&=\left\|s\left(2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n\left(\frac{x+y}{2}\right),z-w\right)+2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n\left(\frac{x-y}{2}\right),z+w\right)\right.\right.\\
&\qquad \left.\left.-2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x,z\right)+2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n y,w\right)\right)\right\|+\lim_{n\to\infty}\frac{1}{2^n}\varphi\left(2^n x,2^n y\right)\varphi(z,w)\\
&\le\left\|s\left(2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n\left(\frac{x+y}{2}\right),z-w\right)+2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n\left(\frac{x-y}{2}\right),z+w\right)\right.\right.\\
&\qquad \left.\left.-2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x,z\right)+2\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n y,w\right)\right)\right\|+\lim_{n\to\infty}L^n\varphi(x,y)\varphi(z,w)
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. Then we have

$$\begin{aligned}
&\|F(\lambda x+\lambda y,\mu z-\mu w)+F(\lambda x-\lambda y,\mu z+\mu w)-2\lambda\mu F(x,z)+2\lambda\mu F(y,w)\|\\
&\quad\le\left\|s\left(2F\left(\frac{x+y}{2},z-w\right)+2F\left(\frac{x-y}{2},z+w\right)-2F(x,z)+2F(y,w)\right)\right\|
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. By the same argument of Theorem 4.1, we obtain that F is bi-additive. Next, we will show that $F(\lambda x,\mu z)=\lambda\mu F(x,z)$ for all $x,z\in\mathbb{A}$. Letting $w=0$ and $y=x$ in (4.1), we have

$$\|f(\lambda 2x,\mu z)-2\lambda\mu f(x,z)\|\le\varphi(x,x)\varphi(y,0)$$

for all $x,z\in\mathbb{A}$. Replacing x in $2^{n-1}x$ in the above equation, we obtain that

$$\|f(\lambda 2^n x,\mu z)-2\lambda\mu f(2^{n-1}x,z)\|\le\varphi(2^{n-1},2^{n-1}x)\varphi(y,0) \tag{4.15}$$

for all $x,z\in\mathbb{A}$. Letting $\lambda=\mu=1$ in (4.15), we have

$$\|f(2^n x,z)-2f(2^{n-1}x,z)\|\le\varphi(2^{n-1},2^{n-1}x)\varphi(y,0) \tag{4.16}$$

for all $x,z\in\mathbb{A}$. It follows from (4.15) and (4.16) that we have

$$\begin{aligned}
&\left\|f\left(\lambda 2^n x,\mu z\right)-\lambda\mu f\left(2^n x,z\right)\right\|\\
&\quad\le\left\|f\left(\lambda 2^n x,\mu z\right)+2\lambda\mu f(2^{n-1}x,z)-2\lambda\mu f(2^{n-1}x,z)-\lambda\mu f\left(2^n x,z\right)\right\|\\
&\quad\le\left\|f\left(\lambda 2^n x,\mu z\right)+2\lambda\mu f(2^{n-1}x,z)\right\|+\left\|2\lambda\mu f(2^{n-1}x,z)-\lambda\mu f\left(2^n x,z\right)\right\|\\
&\quad=\left\|f\left(\lambda 2^n x,\mu z\right)+2\lambda\mu f(2^{n-1}x,z)\right\|+|\lambda||\mu|\left\|2f(2^{n-1}x,z)-f\left(2^n x,z\right)\right\|\\
&\quad\le\left\|f\left(\lambda 2^n x,\mu z\right)+2\lambda\mu f(2^{n-1}x,z)\right\|+\left\|2f(2^{n-1}x,z)-f\left(2^n x,z\right)\right\|\\
&\quad\le 2\varphi(2^{n-1},2^{n-1}x)\varphi(y,0)\\
&\quad\ \vdots\\
&\quad\le 2^n L^n\varphi(x,x)\varphi(y,0)
\end{aligned}$$

for all $x,z\in\mathbb{A}$. Then we have

$$\left\|\frac{1}{2^n}f\left(\lambda 2^n x,\mu z\right)-\lambda\mu\frac{1}{2^n}f\left(2^n x,z\right)\right\|\le L^n\varphi(x,x)\varphi(y,0)$$

for all $x,z\in\mathbb{A}$. Taking limit $n\to\infty$, we have

$$\begin{aligned}
\|F(\lambda x,\mu z)-\lambda\mu F(x,z)\|&=\left\|\lim_{n\to\infty}\frac{1}{2^n}f\left(\lambda 2^n x,\mu z\right)-\lambda\mu\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x,z\right)\right\|\\
&\le\lim_{n\to\infty}L^n\varphi(x,x)\varphi(y,0)=0,
\end{aligned}$$

that is, $F(\lambda x,\mu z)=\lambda\mu F(x,z)$ for all $x,z\in\mathbb{A}$. By Lemma 1.7, we obtain that the mapping F is

$\mathbb{C}$-bilinear. It follows from (4.4) that

$$\begin{aligned}
&\|F([x,y,z],[w,w,w])-[F(x,w),F(y,w),F(z,w)]\| \\
&\le \lim_{n\to\infty}\frac{1}{64^n}\left\|f\left([2^nx,2^ny,2^nz],[2^nw,2^nw,2^nw]\right)-\left[f\left(2^nx,2^nw\right),f\left(2^ny,2^nw\right),f\left(2^nz,2^nw\right)\right]\right\| \\
&\le \lim_{n\to\infty}\frac{1}{64^n}\varphi\left(2^nx,2^ny\right)\varphi\left(2^nz,2^nw\right) \\
&\vdots \\
&\le \lim_{n\to\infty}\left(\frac{L}{16}\right)^n\varphi(x,y)\varphi(z,w)=0
\end{aligned}$$

for all $x,y,z,w\in\mathbb{A}$. Thus

$$F([x,y,z],[w,w,w])=[F(x,w),F(y,w),F(z,w)]$$

for all $x,y,z,w\in\mathbb{A}$. Similarly, we also have

$$F([x,x,x],[y,z,w])=[F(x,y),F(x,z),F(x,w)]$$

for all $x,y,z,w\in\mathbb{A}$. Therefore the mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ is a C^*-ternary bi-homomorphisms. □

Theorem 4.3. *Let $f:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ be a mapping with $f(x,0)=f(0,z)=0$ for all $x,z\in\mathbb{A}$ such that $\varphi:\mathbb{A}^2\to[0,\infty)$ satisfying* (4.2) *and*

$$\begin{aligned}
&\left\|2f\left(\lambda\frac{x+y}{2},\mu(z-w)\right)+2f\left(\lambda\frac{x-y}{2},\mu(z+w)\right)-2\lambda\mu f(x,z)+2\lambda\mu f(y,w)\right\| \\
&\le \|s(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w))\|+\varphi(x,y)\varphi(z,w) \qquad (4.17)
\end{aligned}$$

for all $\lambda,\mu\in\mathbb{S}$ and for all $x,y,z,w\in\mathbb{A}$. Then there exists a unique $\mathbb{C}$-bilinear mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{A}$ such that

$$\|f(x,y)-F(x,y)\|\le\frac{1}{2(1-L)}\varphi(x,0)\varphi(z,0) \qquad (4.18)$$

In addition, if the mapping $f:\mathbb{A}^2\to\mathbb{A}$ satisfies (4.4) *and* (4.5) *for all $x,y,z,w\in\mathbb{A}$, then the mapping F is C^*-ternary bi-homomorphisms.*

Proof. Letting $\lambda=\mu=1$ and $y=w=0$ in (4.17), we get

$$\left\|4f\left(\frac{x}{2},z\right)-2f(x,z)\right\|\le\varphi(x,0)\varphi(z,0), \qquad (4.19)$$

for all $x,z\in\mathbb{A}$. Consider the set

$$S:=\{g:\mathbb{A}^2\to\mathbb{B}\mid g(x,0)=g(0,y)=0,\ \forall\ x,y\in\mathbb{A}\}$$

and introduce the generalized metric on S as follows:

$$d(g,h)=\inf\left\{M\in[0,\infty)\mid \|g(x,y)-h(x,y)\|\le M\varphi(x,0)\varphi(y,0),\ \forall\ x,y\in\mathbb{A}\right\},$$

Then, (S,d) is a complete generalized metric space (see [15, Theorem 4]). Now we consider the linear mapping $T:S\to S$ such that

$$Tg(x,y):=2g\left(\frac{x}{2},y\right)$$

for all $x,y\in\mathbb{A}$. By the proof of Theorem 4 in [15], we get that

$$d(Tg,Th)\le Ld(g,h)$$

for all $g,h \in S$. It follows from (4.19) that

$$\left\| f(x,z) - 2f\left(\frac{x}{2},z\right)\right\| \le \frac{1}{2}\varphi(x,0)\varphi(z,0),$$

for all $x,z \in \mathbb{A}$. So we obtain that $d(f,Tf) \le \frac{1}{2}$. By Theorem 1.4, there exists a unique mapping $F : \mathbb{A}\times\mathbb{A} \to \mathbb{B}$ such that the equality

$$F(x,y) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},y\right)$$

for all $x,y \in \mathbb{A}$. Moreover, we obtain that $d(f,F) \le \frac{1}{1-L}d(f,Tf)$, which implies that

$$d(f,F) \le \frac{1}{1-L}d(f,Tf) \le \frac{1}{2(1-L)}.$$

Therefore, the inequality (4.18) holds. It follows from (4.17) and (4.2) that we get

$$\begin{aligned}
&\left\|2F\left(\lambda\frac{x+y}{2},\mu(z-w)\right)+2F\left(\lambda\frac{x-y}{2},\mu(z+w)\right)-2\lambda\mu F(x,z)+2\lambda\mu F(y,w)\right\| \\
&= \lim_{n\to\infty} 2^n \left\|2f\left(\lambda\frac{\frac{x}{2^n}+\frac{y}{2^n}}{2},\mu z-\mu w\right)+2f\left(\lambda\frac{\frac{x}{2^n}-\frac{y}{2^n}}{2},\mu z+\mu w\right)-2\lambda\mu f\left(\frac{x}{2^n},z\right)+2\lambda\mu f\left(\frac{y}{2^n},w\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \left(\left\|s\left(f\left(\frac{x}{2^n}+\frac{y}{2^n},z-w\right)+f\left(\frac{x}{2^n}-\frac{y}{2^n},z+w\right)-2f\left(\frac{x}{2^n},z\right)+2f\left(\frac{y}{2^n},w\right)\right)\right\|\right. \\
&\quad \left. +\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi(z,w)\right) \\
&\le \left\|s\left(\lim_{n\to\infty} 2^n f\left(\frac{x+y}{2^n},z-w\right)+\lim_{n\to\infty} 2^n f\left(\frac{x-y}{2^n},z+w\right)\right.\right. \\
&\quad \left.\left. -2\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},z\right)+2\lim_{n\to\infty} 2^n f\left(\frac{y}{2^n},w\right)\right)\right\| + \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi(z,w) \\
&\le \left\|s\left(\lim_{n\to\infty} 2^n f\left(\frac{x+y}{2^n},z-w\right)+\lim_{n\to\infty} 2^n f\left(\frac{x-y}{2^n},z+w\right)\right.\right. \\
&\quad \left.\left. -2\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},z\right)+2\lim_{n\to\infty} 2^n f\left(\frac{y}{2^n},w\right)\right)\right\| + \lim_{n\to\infty}\left(\frac{L}{4}\right)^n \varphi(x,y)\varphi(z,w)
\end{aligned}$$

for all $x,y,z,w \in \mathbb{A}$. Then we have

$$\begin{aligned}
&\left\|2F\left(\lambda\frac{x+y}{2},\mu(z-w)\right)+2F\left(\lambda\frac{x-y}{2},\mu(z+w)\right)-2\lambda\mu F(x,z)+2\lambda\mu F(y,w)\right\| \\
&\quad \le \|s(F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w))\|
\end{aligned} \tag{4.20}$$

for all $x,y,z,w \in \mathbb{A}$. Letting $y = w = 0$ and $\lambda = \mu = 1$ in (4.20), we obtain that

$$2F\left(\frac{x}{2},z\right) = F(x,z) \tag{4.21}$$

for all $x,z \in \mathbb{A}$. For $\lambda = \mu = 1$ and using (4.21), we have

$$\begin{aligned}
&\|F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w)\| \\
&\quad \le \left\|2F\left(\frac{x+y}{2},z-w\right)+2F\left(\frac{x-y}{2},z+w\right)-2F(x,z)+2F(y,w)\right\| \\
&\quad \le \|s(F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w))\| \\
&\quad \le |s|\,\|F(x+y,z-w)+F(x-y,z+w)-2F(x,z)+2F(y,w)\|.
\end{aligned}$$

for all $x,y,z,w \in \mathbb{A}$. Since $|s| < 1$, we have

$$F(x+y,z-w)+F(x-y,z+w) = 2F(x,z)-2F(y,w)$$

for all $x,y,z,w \in \mathbb{A}$. By Theorem 2.1 in [19], we have F is bi-additive.

Next, we will show that $F(\lambda x,\mu z)=\lambda\mu F(x,z)$ for all $x,z\in\mathbb{A}$. Replacing x by $2x$ in (4.21), we have

$$2F(x,z)=F(2x,z)$$

for all $x,z\in\mathbb{A}$. Letting $y=x$ and $w=0$ in (4.20) and using the fact of the above equation, we obtain that

$$\left\|2F(\lambda x,\mu z)-2\lambda\mu F(x,z)\right\|\le\|s(F(2x,z)-2F(x,2))\|=0$$

for all $x,z\in\mathbb{A}$. Then we have $F(\lambda x,\mu z)=\lambda\mu F(x,z)$ for all $x,z\in\mathbb{A}$. By Lemma 1.7, we obtain that the mapping F is $\mathbb{C}$-bilinear. The rest of the proof is similar to the proof of the Theorem 4.1. □

Theorem 4.4. *Let $f:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ be a mapping with $f(x,0)=f(0,z)=0$ for all $x,z\in\mathbb{A}$ such that $\varphi:\mathbb{A}^2\to[0,\infty)$ satisfying* (4.2) *and* (4.17)*. Then there exists a unique $\mathbb{C}$-bilinear mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{A}$ such that*

$$\|f(x,y)-F(x,y)\|\le\frac{L}{4(1-L)}\varphi(x,0)\varphi(z,0) \tag{4.22}$$

In addition, if the mapping $f:\mathbb{A}^2\to\mathbb{A}$ satisfies (4.4) *and* (4.5) *for all $x,y,z,w\in\mathbb{A}$, then the mapping F is C^*-ternary bi-homomorphisms.*

Proof. Let (S,d) be the generalized metric space defined in proof of Theorem 4.3.

Now we consider the linear mapping $T:S\to S$ such that

$$Tg(x,y)=\frac{1}{2}g(2x,y)$$

for all $x,y\in\mathbb{A}$ and $g\in S$. By the same argument of Theorem 4.2, we also have T is strictly contractive mapping with Lipschitz constant L. It follows from (4.19) and (4.17) that we have

$$\left\|f(x,z)-\frac{1}{2}f(2x,z)\right\|\le\frac{1}{4}\varphi(2x,0)\varphi(z,0)\le\frac{L}{2}\varphi(x,0)\varphi(z,0)$$

for all $x,z\in\mathbb{A}$. So $d(f,Tf)\le\frac{L}{2}$. By Theorem 1.4, there exists a unique mapping $F:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ such that the equality

$$F(x,y)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx,y\right)$$

for all $x,y\in\mathbb{A}$. Moreover, we obtain that $d(f,F)\le\frac{1}{1-L}d(f,Tf)$, which implies that

$$d(f,F)\le\frac{1}{1-L}d(f,Tf)\le\frac{L}{2(1-L)}.$$

Therefore, the inequality (4.22) holds. The rest of the proof is similar to the proofs of Theorem 4.2 and 4.3. □

5. C^*-ternary Bi-derivations on C^*-ternary Algebras Associated with the Bi-additive Functional Inequalities (1.5) and (1.6)

Theorem 5.1. *Let $f:\mathbb{A}\times\mathbb{A}\to\mathbb{B}$ be a mapping with $f(x,0)=f(0,z)=0$ for all $x,z\in\mathbb{A}$ such that $\varphi:\mathbb{A}^2\to[0,\infty)$ satisfying* (4.1)*. If there exists a Lipschitz constant $L<1$ such that*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right)\le\frac{L}{4}\varphi(x,y) \tag{5.1}$$

for all $x, y \in \mathbb{A}$. Then there exists a unique $\mathbb{C}$-bilinear mapping $F : \mathbb{A}\times\mathbb{A} \to \mathbb{A}$ such that

$$\|f(x,y) - F(x,y)\| \le \frac{L}{4(1-L)}\varphi(x,x)\varphi(z,0) \tag{5.2}$$

In addition, if the mapping $f : \mathbb{A}^2 \to \mathbb{A}$ satisfies and

$$\left\|f([x,y,z],w) - [f(x,w),y,z] - \left[x, f(y,w^*),z\right] - [x,y,f(z,w)]\right\| \le \varphi(x,y)\varphi(z,w) \tag{5.3}$$

and

$$\left\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\right\| \le \varphi(x,y)\varphi(z,w) \tag{5.4}$$

for all $x, y, z, w \in \mathbb{A}$, then the mapping F is a C^*-ternary bi-derivation.

Proof. By the same reasoning as in the proof of Theorem 4.1 we get that

$$\left\|f(x,z) - 2f\left(\frac{x}{2},z\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0) \le \frac{L}{4}\varphi(x,x),\varphi(z,0) \tag{5.5}$$

for all $x, z \in \mathbb{A}$. Let (S,d) be the generalized metric space defined in proof of Theorem 4.1. Now we consider the linear mapping $T : S \to S$ such that

$$Tg(x,y) := 2g\left(\frac{x}{2},y\right)$$

for all $x, y \in \mathbb{A}$. By the same argument of Theorem 4.1, we also have T is strictly contractive mapping with Lipschitz constant L. It follows from (5.5) that we have $d(f,Tf) \le \frac{L}{4}$. By Theorem 1.4, there exists a unique $\mathbb{C}$-bilinear mapping $F : \mathbb{A}\times\mathbb{A} \to \mathbb{B}$ such that

$$F(x,y) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},y\right)$$

for all $x, y \in \mathbb{A}$. Moreover, we obtain that $d(f,F) \le \frac{1}{1-L}d(f,Tf)$, which implies that

$$d(f,F) \le \frac{1}{1-L}d(f,Tf) \le \frac{L}{4(1-L)}.$$

Therefore, the inequality (5.2) holds. It follows from (5.3) that

$$\begin{aligned}
&\left\|F([x,y,z],w) - [F(x,w),y,z] - \left[x,F(y,w^*),z\right] - [x,y,F(z,w)]\right\| \\
&\quad \le \lim_{n\to\infty} 16^n \left(\left\| f\left(\left[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right],\frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n},\frac{w}{2^n}\right),\frac{y}{2^n},\frac{z}{2^n}\right]\right.\right. \\
&\qquad \left.\left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n},\frac{w^*}{2^n}\right),\frac{z}{2^n}\right] - \left[\frac{x}{2^n},\frac{y}{2^n}, f\left(\frac{z}{2^n},\frac{w}{2^n}\right)\right]\right\|\right) \\
&\quad \le \lim_{n\to\infty} 16^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi\left(\frac{z}{2^n},\frac{w}{2^n}\right) \\
&\quad \vdots \\
&\quad \le \lim_{n\to\infty} L^n \varphi(x,y)\varphi(z,w) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathbb{A}$. Thus, we have

$$F([x,y,z],w) = [F(x,w),y,z] + \left[x,F(y,w^*),z\right] + [x,y,F(z,w)]$$

for all $x, y, z, w \in \mathbb{A}$. Similarity, one can show that

$$F(x,[y,z,w]) = [F(x,y),z,w] + [y,F(x^*,z),w] + [y,z,F(x,w)]$$

for all $x, y, z, w \in \mathbb{A}$. Hence the mapping $F : \mathbb{A}^2 \to \mathbb{B}$ is a C^*-ternary bi-derivation. □

Theorem 5.2. *Let $f : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ be a mapping with $f(x,0) = f(0,z) = 0$ for all $x,z \in \mathbb{A}$ such that $\varphi : \mathbb{A}^2 \to [0,\infty)$ satisfying* (4.1) *and* (4.13)*. Then there exists a unique $\mathbb{C}$-bilinear mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ satisfying* (4.14)*. In addition, if the mapping $f : \mathbb{A}^2 \to \mathbb{A}$ satisfies* (5.2) *and* (5.3)*, then the mapping F is a C^*-ternary bi-derivation.*

Proof. The proof is similar to the proof of Theorem 4.2 and Theorem 5.1. □

Theorem 5.3. *Let $f : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ be a mapping with $f(x,0) = f(0,z) = 0$ for all $x,z \in \mathbb{A}$ such that $\varphi : \mathbb{A}^2 \to [0,\infty)$ satisfying* (5.1) *and* (4.17)*. Then there exists a unique $\mathbb{C}$-bilinear mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ satisfying* (4.18)*. In addition, if the mapping $f : \mathbb{A}^2 \to \mathbb{A}$ satisfies* (5.2) *and* (5.3)*, then the mapping F is a C^*-ternary bi-derivation.*

Theorem 5.4. *Let $f : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ be a mapping with $f(x,0) = f(0,z) = 0$ for all $x,z \in \mathbb{A}$ such that $\varphi : \mathbb{A}^2 \to [0,\infty)$ satisfying* (4.2) *and* (4.17)*. Then there exists a unique $\mathbb{C}$-bilinear mapping $F : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that*

$$\|f(x,y) - F(x,y)\| \le \frac{L}{4(1-L)}\varphi(x,0)\varphi(z,0)$$

In addition, if the mapping $f : \mathbb{A}^2 \to \mathbb{A}$ satisfies (5.2) *and* (5.3)*, then the mapping F is a C^*-ternary bi-derivation.*

6. Conclusion

The main results of this paper are Hyers-Ulam-Rassias stability of C^*-ternary bi-homomorphisms and C^*-ternary bi-derivations in C^*-ternary algebras associated with the bi-additive functional equation and the bi-additive functional inequalities by using alternative fixed point theorem.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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