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Research Article

The Domination Number of a Graph $P_k((k_1,k_2),(k_3,k_4))$

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Abstract. For each $k,k_1,k_2,k_3,k_4 \in \mathbb{N}$, we will denote by $P_k\big((k_1,k_2),(k_3,k_4)\big)$ a tree of $k+k_1+k_2+k_3+k_4+1$ vertices with the degree sequence $(1,1,1,1,2,2,2,\dots,2,3,3)$. Let $\alpha_{k_1},\beta_{k_2},\sigma_{k_3}$, and δ_{k_4} be all four endpoints of the graph. Let the distance between both vertices of degree 3 be equal to k. A subset S of vertices of a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ is a dominating set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$ if every vertex in $V\big(P_k\big((k_1,k_2),(k_3,k_4)\big)\big)-S$ is adjacent to some vertex in S. We investigate the dominating set of minimum cardinality of a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ to obtain the domination number of this graph. Finally, we determine an upper bound on the domination number of a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$.

Keywords. Domination number; Tree; A dominating set of a graph; The domination number of a graph; The domination number of a tree

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1. Introduction

There are a variety of topics in graph theory. Due to its wide applications in many computational branches, different types of graphs are investigated for specific reasons. For instances, using Eulerian graphs to solve the transportation problems, applying complete graphs in the proofs about planar graphs, or concluding the famous Hall's marriage theorem by using bipartite graphs as a helpful tool. We see that many parameters on graph theory have been examined.

Perhaps, the fast growth of such many areas as computer science, operations research, electrical engineering, even mathematics, shows that this field of study is being more useful.

Another interesting graph parameter we shall study in this paper is the domination number. In 1998, Haynes, Hedetniemi and Slater [6] described the concept of a dominating set in a graph thoroughly. Several works determined the exact value and bounds of various kinds of graphs. In 2008, Alikhani et al. [1] defined some certain graphs and applied a relevant inequality of a dominating set to find the domination number. In 2009, Kostochka and Stodolsky [9] obtained an upper bound for the domination number of n-vertex connected cubic graphs. Later, in the same year, Kostochka and Stocker [8] improved this bound. In 2010, Huang and Xu [7] presented the domination number for the fundamental graphs such as paths and cycles. In the next year, Murugesan and Nair [10] gave a result on the domination number of a cubic bipartite graph. The number is less than or equal to $\frac{1}{3}$ of the number of vertices. In 2012, Chelvam and Kalaimurugan [4] verified that the domination number of a k-regular graph of n vertices is greater than or equal to $\frac{n}{k+1}$. Furthermore, Bacolod and Baldado Jr. [2] created another new binary operation on graphs in 2014, called acquaint vertex gluing and focused on its domination number. As same as the work of Nupo and Panma [11], they investigated the domination number of Cayley digraphs of rectangular groups.

Since the concept of domination problem was dramatically considered a few decades ago, some results about the domination parameters have been challenging to mathematical researchers. Hereafter, we study the domination number of a certain type of graph which will be introduced in the sequel. Our main result can be applied to obtain the domination number of graphs with more complexity of structures.

2. Preliminaries

Throughout this section, we provide some terminology together with helpful examples for this paper. For more details we refer the reader to [3] and [12].

A graph G consists of a non-empty finite set $V(G)$ of elements called *vertices*, and a finite family $E(G)$ of unordered pairs of these elements of $V(G)$ called *edges*. We call $V(G)$ the *vertex set*, and $E(G)$ the *edge family* of G. An edge $\{v,w\}$ is said to *join* the vertices v and w, and is usually abbreviated to vw. We use the word family to mean a collection of elements, some of which may occur several times.

The *degree* of a vertex v of G is the number of edges incident with v, and is written $\deg(v)$. A vertex of degree 1 is an *endpoint*. The *degree sequence* of a graph consists of the degrees of all vertices of the graph written in an increasing order, with repeats where necessary.

Let S be any subset of vertices of G. Then the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge family consists of all edges in G that have both endpoints in S. The induced subgraph $G[S]$ may also be called the *subgraph induced in G by S*.

A *walk* in G is a finite sequence of vertices in G. A *path* in G is a walk in which no vertex is repeated. The *distance* from a vertex v to a vertex w is the number of edges in the shortest path from v to w.

A *forest* is a simple graph that contains no cycles. A connected forest is a *tree*. A tree with n vertices has $n-1$ edges.

For a vertex v of a graph G, a *neighbor* of v is a vertex adjacent to v in G. The *neighborhood* (or *open neighborhood*) $N(v)$ of v is the set of neighbors of v. The *closed neighborhood* $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$.

A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. Therefore, v dominates $1+\deg(v)$ vertices of G.

For a set S of vertices of G, the *closed neighborhood* $N[S]$ is defined as $N[S] = \bigcup\limits_{v\in S}(N(v)\cup\{v\})$. A set S of vertices of G is said to *dominate* the vertices in $N[S]$.

Definition 1. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex in S.

We shall introduce another definition which is equivalent to Definition 1.

Definition 2. A set S of vertices of G is a *dominating set* of G if every vertex in $V(G)-S$ is adjacent to some vertex in S.

It is obvious that Definition 1 is equivalent to Definition 2.

Example 1. Consider the graph G of Figure 1, the sets $S_1 = \{v_1,v_2,v_3\}$ and $S_2 = \{v_1,v_4\}$ are both dominating sets in G, indicated by solid vertices.

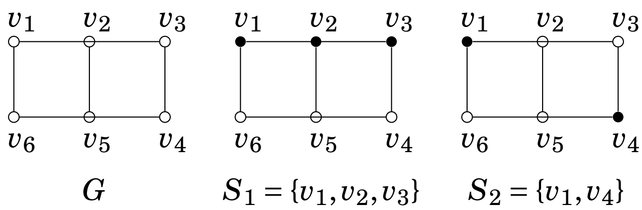

Figure 1. Two dominating sets in a graph G

Definition 3. A *minimum dominating set* in a graph G is a dominating set of minimum cardinality.

Definition 4. The cardinality of a minimum dominating set is called the *domination number* of G, and is denoted by $\gamma(G)$.

Example 2. Since the vertex set of a graph is always a dominating set, the domination number is defined for every graph. If $|V(G)| = n$, then $1 \le \gamma(G) \le n$. A graph G with n vertices has domination number 1 if and only if G contains a vertex v of degree $n-1$, in which case $\{v\}$ is a minimum dominating set; while $\gamma(G)=n$ if and only if $G \cong \overline{K_n}$, in which case $V(G)$ is the unique minimum dominating set.

Example 3. From a graph G in Example 1, we see that the set $S_2=\{v_1,v_4\}$ is a dominating set for G. Therefore, $\gamma(G)\le 2$. To show that the domination number of G is actually 2, it is required to show that there is no dominating set with a vertex. Note that $|V(G)|=6$ and the degree of every vertex of G is at most 3. This means that no vertex can dominate more than 4 vertices. That is, $\gamma(G)>1$ and so $\gamma(G)=2$. Notice that the set S_2 is a dominating set of minimum cardinality.

Definition 5. A dominating set of a graph G with minimum cardinality is called a *γ-set* of G.

Example 4. The set S_2 in Example 1 is a dominating set of minimum cardinality, so we call S_2 a γ-set of G.

The domination number of a cycle or a path is easy to compute.

Theorem 1 ([5]). *For $n\ge 3, \gamma(P_n)=\gamma(C_n)=\left\lceil \frac{n}{3}\right\rceil$.*

3. The Domination Number of a Graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$

Firstly, we introduce a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ which will be stated along the whole paper. As a result, the domination number of the graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ are investigated.

For each $k,k_1,k_2,k_3,k_4\in\mathbb{N}$, let $P_k\big((k_1,k_2),(k_3,k_4)\big)$ be a graph where

$$\begin{aligned}
V\Big(P_k\big((k_1,k_2),(k_3,k_4)\big)\Big) = &\{0,1,\dots,k\}\cup\{\alpha_1,\alpha_2,\dots,\alpha_{k_1}\}\cup\{\beta_1,\beta_2,\dots,\beta_{k_2}\}\\
&\cup\{\sigma_1,\sigma_2,\dots,\sigma_{k_3}\}\cup\{\delta_1,\delta_2,\dots,\delta_{k_4}\}, \text{ and}\\
E\Big(P_k\big((k_1,k_2),(k_3,k_4)\big)\Big) = &\{01,12,\dots,(k-1)k\}\cup\{0\alpha_1,\alpha_1\alpha_2,\alpha_2\alpha_3,\dots,\alpha_{k_1-1}\alpha_{k_1}\}\\
&\cup\{0\beta_1,\beta_1\beta_2,\beta_2\beta_3,\dots,\beta_{k_2-1}\beta_{k_2}\}\cup\{k\sigma_1,\sigma_1\sigma_2,\sigma_2\sigma_3,\dots,\sigma_{k_3-1}\sigma_{k_3}\}\\
&\cup\{k\delta_1,\delta_1\delta_2,\delta_2\delta_3,\dots,\delta_{k_4-1}\delta_{k_4}\}.
\end{aligned}$$

Such a graph is shown in Figure 2.

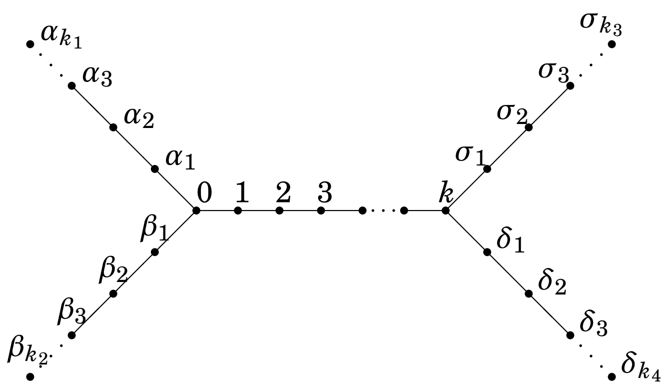

Figure 2. $P_k\big((k_1,k_2),(k_3,k_4)\big)$

We are now prepared to give results on finding the domination number of a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$. We first solve a few useful lemmas, and then determine our main result.

Lemma 1. *Let $k,k_1,k_2,k_3,k_4 \in \mathbb{N}$ and $k_1 = k_2 = k_3 = k_4 = 1$. Then the following statements hold.*

1. *If $k \equiv 0 \pmod 3$, then $S = \{0,3,6,\ldots,(k-3),k\}$ is a dominating set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$.*
2. *If $k \equiv 1 \pmod 3$, then $S = \{0,3,6,\ldots,(k-1),k\}$ is a dominating set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$.*
3. *If $k \equiv 2 \pmod 3$, then $S = \{0,3,6,\ldots,(k-2),k\}$ is a dominating set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$.*

Proof. The reader can easily check that S is a dominating set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$. □

Lemma 2. *Let $k,k_1,k_2,k_3,k_4 \in \mathbb{N}$. Let S be a γ-set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$. Then the following statements hold.*

1. *If $k_1 = k_2 = 1$ then $\alpha_1,\beta_1 \notin S$ and $0 \in S$.*
2. *If $k_3 = k_4 = 1$ then $\sigma_1,\delta_1 \notin S$ and $k \in S$.*

Proof. Let S be a γ-set of $P_k\big((k_1,k_2),(k_3,k_4)\big)$.

1. Assume that $k_1 = k_2 = 1$. We first prove that $\alpha_1 \notin S$. Suppose that $\alpha_1 \in S$. In order to dominate β_1, then β_1 or 0 must be in S. We consider two cases.

Case 1: Suppose that $\beta_{k_2} = \beta_1 \in S$.

Since $P_k\big((k_1,k_2),(k_3,k_4)\big)\big[\{1,2,\ldots,k,\sigma_1\}\big]$ is an induced subgraph of

$$P_k\big((k_1,k_2),(k_3,k_4)\big) \quad \text{and} \quad P_k\big((k_1,k_2),(k_3,k_4)\big)\big[\{1,2,\ldots,k,\sigma_1\}\big] \cong P_{k+1},$$

we have

$$\gamma\big(P_k\big((k_1,k_2),(k_3,k_4)\big)\big[\{1,2,\ldots,k,\sigma_1\}\big]\big) = \gamma(P_{k+1}) = \left\lceil \frac{k+1}{3} \right\rceil.$$

In order to dominate vertices $1,2,\ldots,k,\sigma_1$, then S contains at least $\lceil \frac{k+1}{3} \rceil$ vertices in $\{0,1,2,\ldots,k,\sigma_1,\delta_1\}$. Thus $|S| \ge \lceil \frac{k+1}{3} \rceil + 2 = \lceil \frac{k+7}{3} \rceil$. By Lemma 1, we know that there exists a dominating set of order $\lceil \frac{k+3}{3} \rceil$, a contradiction.

Case 2: Suppose that $0 \in S$.

Since $P_k\big((k_1,k_2),(k_3,k_4)\big)\big[\{2,\ldots,k,\sigma_1\}\big]$ is an induced subgraph of

$$P_k\big((k_1,k_2),(k_3,k_4)\big) \quad \text{and} \quad P_k\big((k_1,k_2),(k_3,k_4)\big)\big[\{2,\ldots,k,\sigma_1\}\big] \cong P_k,$$

we have

$$\gamma\big(P_k\big((k_1,k_2),(k_3,k_4)\big)\big[\{2,\ldots,k,\sigma_1\}\big]\big) = \gamma(P_k) = \left\lceil \frac{k}{3} \right\rceil.$$

In order to dominate vertices $2,3,\ldots,k,\sigma_1$, then S contains at least $\lceil \frac{k}{3} \rceil$ vertices in $\{1,2,\ldots,k,\sigma_1,\delta_1\}$. Thus $|S| \ge \lceil \frac{k}{3} \rceil + 2 = \lceil \frac{k+6}{3} \rceil$. By Lemma 1, we know that there exists a dominating set of order $\lceil \frac{k+3}{3} \rceil$, a contradiction.

Therefore,

$$\text{if } k_1 = k_2 = 1 \text{ then } \alpha_1 \notin S. \tag{3.1}$$

Similarly,

$$\text{if } k_1 = k_2 = 1 \text{ then } \beta_1 \notin S. \tag{3.2}$$

We now prove that $0 \in S$.

Suppose that $0 \notin S$.

Then $\alpha_1, \beta_1 \in S$, contrary to (3.1) and (3.2).

2. Assume that $k_3 = k_4 = 1$. Similarly, the proof follows by the same arguments. □

Lemma 3. *If $k, k_1, k_2, k_3, k_4 \in \mathbb{N}$ and $1 \le k_1, k_2, k_3, k_4 \le 3$ then*

$$\gamma\Big(P_k\big((k_1,k_2),(k_3,k_4)\big)\Big) - \left\lceil \frac{k + \min\{k_1,k_2\} + \min\{k_3,k_4\} + 1}{3} \right\rceil$$
$$= \begin{cases} 0 & \text{if } \max\{k_1,k_2\} = \max\{k_3,k_4\} = 1, \\ 1 & \text{if either } \max\{k_1,k_2\} \ge 2 \text{ or } \max\{k_3,k_4\} \ge 2, \\ 2 & \text{if } \max\{k_1,k_2\} \ge 2 \text{ and } \max\{k_3,k_4\} \ge 2. \end{cases}$$

Proof. We shall consider three cases as follows:
Case 1: if $\max\{k_1,k_2\} = \max\{k_3,k_4\} = 1$;
Case 2: if either $\max\{k_1,k_2\} \ge 2$ or $\max\{k_3,k_4\} \ge 2$;
Case 3: if $\max\{k_1,k_2\} \ge 2$ and $\max\{k_3,k_4\} \ge 2$.

Case 1: Assume that $\max\{k_1,k_2\} = \max\{k_3,k_4\} = 1$.

Then $k_1 = k_2 = k_3 = k_4 = 1$ and we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,1),(1,1))$ (see Figure 3).

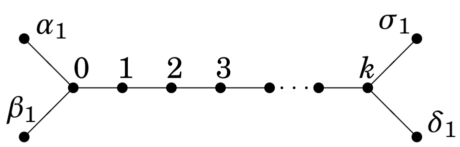

Figure 3. $P_k((1,1),(1,1))$

It is easy to check that if $k \equiv 0 \pmod 3$, then $S = \{0,3,6,\dots,(k-3),k\}$ is a dominating set of $P_k((1,1),(1,1))$. Similarly, if $k \equiv 1 \pmod 3$, then $S = \{0,3,6,\dots,(k-1),k\}$ is a dominating set of $P_k((1,1),(1,1))$. Also, if $k \equiv 2 \pmod 3$, then $S = \{0,3,6,\dots,(k-2),k\}$ is a dominating set of $P_k((1,1),(1,1))$. We thus get $\gamma\big(P_k((1,1),(1,1))\big) \le \lceil \frac{k+3}{3} \rceil$. We next prove that $\gamma\big(P_k((1,1),(1,1))\big) \ge \lceil \frac{k+3}{3} \rceil$. Let S be a γ-set of $P_k((1,1),(1,1))$. By Lemma 2, we have $0, k \in S$. Thus $N[\{0,k\}] = \{0,1,\alpha_1,\beta_1,k,k-1,\sigma_1,\delta_1\}$. Consider $P_k((1,1),(1,1)) - N[\{0,k\}] \cong P_k((1,1),(1,1))[\{2,3,\dots,(k-2)\}] \cong P_{k-3}$. In order to dominate $2,3,\dots,(k-2)$, then $S - \{0,k\}$ contains $\lceil \frac{k-3}{3} \rceil$ vertices. This gives $|S| \ge \lceil \frac{k-3}{3} \rceil + 2 = \lceil \frac{k-3+6}{3} \rceil = \lceil \frac{k+3}{3} \rceil$. We thus get $\gamma\big(P_k((1,1),(1,1))\big) \ge \lceil \frac{k+3}{3} \rceil$. Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k + \min\{k_1,k_2\} + \min\{k_3,k_4\} + 1}{3} \right\rceil.$$

Case 2: Assume that either $\max\{k_1,k_2\} \geq 2$ or $\max\{k_3,k_4\} \geq 2$.

We have divided the proof into two cases:

Case 2.1: $\max\{k_1,k_2\} = 1$ and $\max\{k_3,k_4\} \geq 2$, and

Case 2.2: $\max\{k_1,k_2\} \geq 2$ and $\max\{k_3,k_4\} = 1$.

Case 2.1: We obtain $k_1 = k_2 = 1$. The proof will be considered two cases as follows:

Case 2.1.1: $\min\{k_3,k_4\} = 1$ and $\max\{k_3,k_4\} = 2$ and

Case 2.1.2: $\min\{k_3,k_4\} = 1$ and $\max\{k_3,k_4\} = 3$.

Case 2.1.1: Assume that $\min\{k_3,k_4\} = 1$ and $\max\{k_3,k_4\} = 2$.

Suppose that $\min\{k_3,k_4\} = 1 = k_3$, and $\max\{k_3,k_4\} = 2 = k_4$.

Thus we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,1),(1,2))$ (see Figure 4).

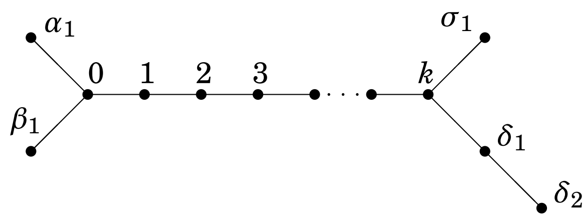

Figure 4. $P_k((1,1),(1,2))$

We can check that if $k \equiv 0 \pmod 3$, then $S = \{0,3,6,\ldots,(k-3),k,\delta_1\}$ is a dominating set of $P_k((1,1),(1,2))$. Similarly, if $k \equiv 1 \pmod 3$, then $S = \{0,3,6,\ldots,(k-1),k,\delta_1\}$ is a dominating set of $P_k((1,1),(1,2))$. Also, if $k \equiv 2 \pmod 3$, then $S = \{0,3,6,\ldots,(k-2),k,\delta_1\}$ is a dominating set of $P_k((1,1),(1,2))$. We thus get $\gamma\big(P_k((1,1),(1,2))\big) \leq \lceil \frac{k+3}{3} \rceil + 1$. We next prove that $\gamma\big(P_k((1,1),(1,2))\big) \geq \lceil \frac{k+3}{3} \rceil + 1$. Let S be a γ-set of $P_k((1,1),(1,2))$. By Lemma 2, we have $0 \in S$. In order to dominate σ_1, then S contains σ_1 or k. Also, in order to dominate δ_2, then S contains δ_1 or δ_2.

Without loss of generality we can assume that $k,\delta_2 \in S$. So we get

$$N[\{0,k,\delta_2\}] = \{\alpha_1,\beta_1,0,1,(k-1),k,\sigma_1,\delta_1,\delta_2\}.$$

Hence

$$P_k((1,1),(1,2)) - N[\{0,k,\delta_2\}] = P_k((1,1),(1,2))[\{2,3,\ldots,(k-2)\}] \cong P_{k-3}.$$

In order to dominate $2,3,\ldots,(k-2)$, then $S-\{0,k,\delta_2\}$ contains $\lceil \frac{k-3}{3} \rceil$ vertices in $\{1,2,3,\ldots,(k-2),(k-1)\}$.

Therefore,

$$|S| \geq \left\lceil \frac{k-3}{3} \right\rceil + 3 = \left\lceil \frac{k-3+6}{3} \right\rceil + 1 = \left\lceil \frac{k+3}{3} \right\rceil + 1.$$

We thus get

$$\gamma\big(P_k((1,1),(1,2))\big) \geq \left\lceil \frac{k+3}{3} \right\rceil + 1.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 1.$$

Case 2.1.2: Assume that $\min\{k_3,k_4\} = 1$ and $\max\{k_3,k_4\} = 3$.

Suppose that $\min\{k_3,k_4\} = 1 = k_3$ and $\max\{k_3,k_4\} = 3 = k_4$. Thus we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,1),(1,3))$ As in the Case 2.1.1, we get

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 1.$$

Case 2.2: The proof is similar to Case 2.1 and we get

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 1.$$

Case 3: Assume that $\max\{k_1,k_2\} \ge 2$ and $\max\{k_3,k_4\} \ge 2$.

Suppose that $\max\{k_1,k_2\} = k_2$ and $\max\{k_3,k_4\} = k_4$.

We divide the proof into four cases as follows:

Case 3.1: $k_2 = 2$ and $k_4 = 2$;

Case 3.2: $k_2 = 3$ and $k_4 = 2$;

Case 3.3: $k_2 = 2$ and $k_4 = 3$;

Case 3.4: $k_2 = 3$ and $k_4 = 3$.

Case 3.1: Assume that $k_2 = 2$ and $k_4 = 2$.

We consider four cases:

Case 3.1.1: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 1$,

Case 3.1.2: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 2$,

Case 3.1.3: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 1$, and

Case 3.1.4: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 2$.

Case 3.1.1: Assume that $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 1$.

We get that $\min\{k_1,k_2\} = k_1 = 1$ and $\min\{k_3,k_4\} = k_3 = 1$.

Thus we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,2),(1,2))$ (see Figure 5).

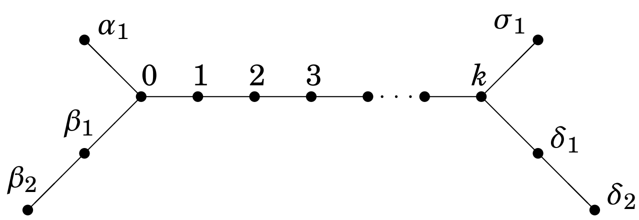

Figure 5. $P_k((1,2),(1,2))$

We can check that if $k \equiv 0 \pmod 3$, then $S = \{0,3,6,\dots,k-3,k,\delta_1,\beta_1\}$ is a dominating set of $P_k((1,2),(1,2))$. Similarly, if $k \equiv 1 \pmod 3$, then $S = \{0,3,6,\dots,k-1,k,\delta_1,\beta_1\}$ is a dominating set of $P_k((1,2),(1,2))$. Also, if $k \equiv 2$

(mod 3), then $S = \{0,3,6,\dots,k-2,k,\delta_1,\beta_1\}$ is a dominating set of $P_k((1,2),(1,2))$. We thus get $\gamma\big(P_k((1,2),(1,2))\big) \le \lceil \frac{k+3}{3} \rceil + 2$. We next prove that $\gamma\big(P_k((1,2),(1,2))\big) \ge \lceil \frac{k+3}{3} \rceil + 2$. Let S be a γ-set of $P_k((1,2),(1,2))$. In order to dominate α_1, then S contains α_1 or 0. Also, in order to dominate σ_1, then S contains σ_1 or k. Without loss of generality we can assume that $0,k \in S$. Similarly, in order to dominate β_2, then S contains β_1 or β_2. Also, in order to dominate δ_2, then S contains δ_1 or δ_2. Without loss of generality we can assume that $\beta_1,\delta_1 \in S$. So we get

$$N[\{0,k,\beta_1,\delta_1\}] = \{\alpha_1,\beta_1,\beta_2,0,1,(k-1),k,\sigma_1,\delta_1,\delta_2\}.$$

Therefore,

$$P_k((1,2),(1,2)) - N[\{0,k,\beta_1,\delta_1\}] = P_k((1,2),(1,2))[\{2,3,\dots,(k-2)\}] \cong P_{k-3}.$$

In order to dominate $2,3,\dots,(k-2)$, then $S-\{0,k,\beta_1,\delta_1\}$ contains $\lceil \frac{k-3}{3} \rceil$ vertices in $\{1,2,3,\dots,(k-2),(k-1)\}$.
Therefore,

$$|S| \ge \left\lceil \frac{k-3}{3} \right\rceil + 4 = \left\lceil \frac{k-3+6}{3} \right\rceil + 2 = \left\lceil \frac{k+3}{3} \right\rceil + 2.$$

We thus get

$$\gamma\big(P_k((1,2),(1,2))\big) \ge \left\lceil \frac{k+3}{3} \right\rceil + 2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.1.2: Assume that $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 2$.
We get that $\min\{k_1,k_2\} = k_1 = 1$ and $\min\{k_3,k_4\} = k_3 = 2$.
Thus we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,2),(2,2))$ (see Figure 6).

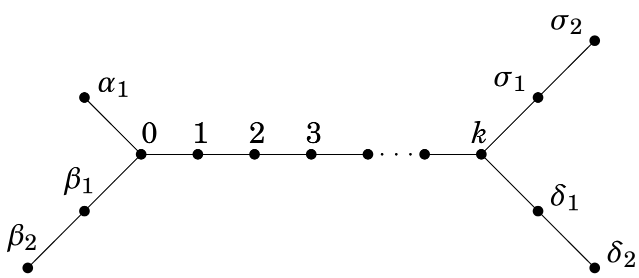

Figure 6. $P_k((1,2),(2,2))$

We can check that if $k \equiv 0 \pmod 3$, then $S = \{0,3,6,\dots,k-3,k,\delta_1,\beta_1,\sigma_1\}$ is a dominating set of $P_k((1,2),(2,2))$. Similarly, if $k \equiv 1 \pmod 3$, then $S = \{0,3,6,\dots,k-1,\sigma_1,\delta_1,\beta_1\}$ is a dominating set of $P_k((1,2),(2,2))$. Also, if $k \equiv 2 \pmod 3$, then $S = \{0,3,6,\dots,k-2,\delta_1,\beta_1,\sigma_1\}$ is a dominating set of $P_k((1,2),(2,2))$. We thus get $\gamma\big(P_k((1,2),(2,2))\big) \le \lfloor \frac{k+3}{3} \rfloor + 3 = \lfloor \frac{k+6}{3} \rfloor + 2 = \lceil \frac{k+4}{3} \rceil + 2$. We next prove that $\gamma\big(P_k((1,2),(2,2))\big) \ge \lceil \frac{k+4}{3} \rceil + 2$. Let S be a γ-set of $P_k((1,2),(2,2))$. In order to dominate α_1, then S contains α_1 or 0. Also, in order to dominate σ_2, then S contains σ_1 or σ_2. Without loss of generality we can assume that $0,\sigma_1 \in S$. Similarly, in order to dominate β_2, then S contains β_1 or β_2. Also, in order to dominate δ_2, then S contains δ_1 or δ_2. Without loss of generality we can assume that $\beta_1,\delta_1 \in S$. So we get

$$N[\{0,\sigma_1,\beta_1,\delta_1\}] = \{\alpha_1,\beta_1,\beta_2,0,1,k,\sigma_1,\sigma_2,\delta_1,\delta_2\}.$$

Therefore,

$$P_k((1,2),(2,2)) - N[\{0,\sigma_1,\beta_1,\delta_1\}] = P_k((1,2),(2,2))[\{2,3,\dots,(k-1)\}] \cong P_{k-2}.$$

In order to dominate $2,3,\dots,(k-1)$, then $S-\{0,\sigma_1,\beta_1,\delta_1\}$ contains $\lceil\frac{k-2}{3}\rceil$ vertices in $\{1,2,3,\dots,(k-2),(k-1),k\}$.

Therefore,

$$|S| \ge \left\lceil\frac{k-2}{3}\right\rceil + 4 = \left\lceil\frac{k-2+6}{3}\right\rceil + 2 = \left\lceil\frac{k+4}{3}\right\rceil + 2.$$

We thus get

$$\gamma\big(P_k((1,2),(2,2))\big) \ge \left\lceil\frac{k+4}{3}\right\rceil + 2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil\frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3}\right\rceil + 2.$$

Case 3.1.3: Assume that $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 1$.

We get that $\min\{k_1,k_2\} = k_1 = 2$ and $\min\{k_3,k_4\} = k_3 = 1$. Thus we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((2,2),(1,2))$. As in the Case 3.1.2, we get

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil\frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3}\right\rceil + 2.$$

Case 3.1.4: Assume that $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 2$.

We get that $\min\{k_1,k_2\} = k_1 = 2$ and $\min\{k_3,k_4\} = k_3 = 2$. Thus we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((2,2),(2,2))$. It is easy to check that if $k \equiv 0 \pmod 3$, then $S = \{\alpha_1,2,5,8,\dots,(k-1),\sigma_1,\delta_1,\beta_1\}$ is a dominating set of $P_k((1,2),(2,2))$. Similarly, if $k \equiv 1 \pmod 3$, then $S = \{\alpha_1,2,5,8,\dots,(k-2),\sigma_1,\delta_1,\beta_1\}$ is a dominating set of $P_k((2,2),(2,2))$. Also, if $k \equiv 2 \pmod 3$, then $S = \{\alpha_1,2,5,8,\dots,(k-3),k,\sigma_1,\delta_1,\beta_1\}$ is a dominating set of $P_k((2,2),(2,2))$. We thus get

$$\gamma\big(P_k((2,2),(2,2))\big) \le \left\lfloor\frac{k+1}{3}\right\rfloor + 4 = \left\lfloor\frac{k+7}{3}\right\rfloor + 2 = \left\lceil\frac{k+5}{3}\right\rceil + 2.$$

We next prove that $\gamma\big(P_k((2,2),(2,2))\big) \ge \lceil\frac{k+5}{3}\rceil + 2$. Let S be a γ-set of $P_k((2,2),(2,2))$. In order to dominate α_2, then S contains α_1 or α_2. Also, in order to dominate σ_2, then S contains σ_1 or σ_2. Without loss of generality we can assume that $\alpha_1,\sigma_1 \in S$. In order to dominate β_2, then S contains β_1 or β_2. Also, in order to dominate δ_2, then S contains δ_1 or δ_2. Without loss of generality we can assume that $\beta_1,\delta_1 \in S$. So we get

$$N[\{\alpha_1,\sigma_1,\beta_1,\delta_1\}] = \{\alpha_1,\alpha_2,\beta_1,\beta_2,0,k,\sigma_1,\sigma_2,\delta_1,\delta_2\}.$$

Therefore,

$$P_k((2,2),(2,2)) - N[\{\alpha_1,\sigma_1,\beta_1,\delta_1\}] = P_k((2,2),(2,2))[\{1,2,3,\dots,(k-1)\}] \cong P_{k-1}.$$

In order to dominate $1,2,3,\dots,(k-1)$, then $S-\{\alpha_1,\sigma_1,\beta_1,\delta_1\}$ contains $\lceil \frac{k-1}{3} \rceil$ vertices in $\{0,1,2,3,\dots,(k-2),(k-1),k\}$.

Therefore,

$$|S| \ge \left\lceil \frac{k-1}{3} \right\rceil + 4 = \left\lceil \frac{k-1+6}{3} \right\rceil + 2 = \left\lceil \frac{k+5}{3} \right\rceil + 2.$$

We thus get

$$\gamma\big(P_k((2,2),(2,2))\big) \ge \left\lceil \frac{k+5}{3} \right\rceil + 2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.2: Assume that $k_2 = 3$ and $k_4 = 2$.

We consider six cases as follows:

Case 3.2.1: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 1$;

Case 3.2.2: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 2$;

Case 3.2.3: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 1$;

Case 3.2.4: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 2$;

Case 3.2.5: if $\min\{k_1,k_2\} = 3$ and $\min\{k_3,k_4\} = 1$;

Case 3.2.6: if $\min\{k_1,k_2\} = 3$ and $\min\{k_3,k_4\} = 2$.

Case 3.2.1: Assume that $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 1$.

We get that $\min\{k_1,k_2\} = k_1 = 1$ and $\min\{k_3,k_4\} = k_3 = 1$.

So in this case, we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,3),(1,2))$ (see Figure 7).

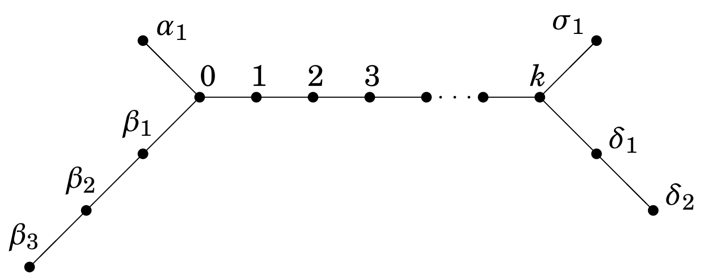

Figure 7. $P_k((1,3),(1,2))$

It is easy to check that if $k \equiv 0 \pmod 3$, then $S = \{0,3,6,\dots,k-3,k,\delta_1,\beta_2\}$ is a dominating set of $P_k((1,3),(1,2))$. Also, if $k \equiv 1 \pmod 3$ and $k \equiv 2 \pmod 3$, then $S = \{0,3,6,\dots,k-1,k,\delta_1,\beta_2\}$ and $S = \{0,3,6,\dots,k-2,k,\delta_1,\beta_2\}$ are both dominating sets of $P_k((1,3),(1,2))$, respectively. We thus get

$$\gamma\big(P_k((1,3),(1,2))\big) \le \left\lceil \frac{k+3}{3} \right\rceil + 2.$$

We next prove that $\gamma\big(P_k((1,3),(1,2))\big) \ge \lceil \frac{k+3}{3} \rceil + 2$. Let S be a γ-set of $P_k((1,3),(1,2))$. In order to dominate α_1, then S must contains α_1 or 0. Also, in order to dominate σ_1, then S must contains σ_1 or k. Without loss of generality we can assume that $0,k \in S$. In order to dominate β_3, then S must contains β_2 or β_3. Also, in order to dominate δ_2, then S must contains δ_1 or δ_2. Without loss of generality we can assume that $\beta_2,\delta_1 \in S$. So we get

$$N[\{0,k,\beta_2,\delta_1\}] = \{\alpha_1,\beta_1,\beta_2,\beta_3,0,1,(k-1),k,\sigma_1,\delta_1,\delta_2\}.$$

Therefore,

$$P_k((1,3),(1,2)) - N[\{0,k,\beta_2,\delta_1\}] = P_k((1,3),(1,2))[\{2,3,\dots,(k-2)\}] \cong P_{k-3}.$$

In order to dominate $2,3,\dots,(k-2)$, then $S-\{0,k,\beta_2,\delta_1\}$ must contains $\lceil \frac{k-3}{3} \rceil$ vertices in $\{1,2,3,\dots,(k-2),(k-1)\}$.

Therefore,

$$|S| \ge \left\lceil \frac{k-3}{3} \right\rceil + 4 = \left\lceil \frac{k-3+6}{3} \right\rceil + 2 = \left\lceil \frac{k+3}{3} \right\rceil + 2.$$

We thus get

$$\gamma\big(P_k((1,3),(1,2))\big) \ge \left\lceil \frac{k+3}{3} \right\rceil + 2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.2.2: Assume that $\min\{k_1,k_2\}=1$ and $\min\{k_3,k_4\}=2$.
We get that $\min\{k_1,k_2\}=k_1=1$ and $\min\{k_3,k_4\}=k_3=2$.
So in this case, we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((1,3),(2,2))$. As in Case 3.1.2, we get

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.2.3: Assume that $\min\{k_1,k_2\}=2$ and $\min\{k_3,k_4\}=1$.
We get that $\min\{k_1,k_2\}=k_1=2$ and $\min\{k_3,k_4\}=k_3=1$.
So in this case, we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((2,3),(1,2))$. As in Case 3.1.3, we get

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.2.4: Assume that $\min\{k_1,k_2\}=2$ and $\min\{k_3,k_4\}=2$.
We get that $\min\{k_1,k_2\}=k_1=2$ and $\min\{k_3,k_4\}=k_3=2$.
So in this case, we get the graph $P_k((k_1,k_2),(k_3,k_4)) = P_k((2,3),(2,2))$. As in

Case 3.1.4, we get

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big)=\left\lceil\frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3}\right\rceil+2.$$

Case 3.2.5: Assume that $\min\{k_1,k_2\}=3$ and $\min\{k_3,k_4\}=1$.
We get that $\min\{k_1,k_2\}=k_1=3$ and $\min\{k_3,k_4\}=k_3=1$.
So in this case, we get the graph $P_k((k_1,k_2),(k_3,k_4))=P_k((3,3),(1,2))$. Next, we check that if $k\equiv 0 \pmod 3$, then $S=\{\alpha_2,1,4,7,\dots,(k-2),k,\delta_2,\beta_2\}$ is a dominating set of $P_k((3,3),(1,2))$. Also, if $k\equiv 1 \pmod 3$ and $k\equiv 2 \pmod 3$, then $S=\{\alpha_2,1,4,7,\dots,(k-3),k,\delta_2,\beta_2\}$ and $S=\{\alpha_2,1,4,7,\dots,(k-1),k,\delta_2,\beta_2\}$ are both dominating sets of $P_k((3,3),(1,2))$, respectively. We thus get

$$\gamma\big(P_k((3,3),(1,2))\big)\le\left\lceil\frac{k+2}{3}\right\rceil+3=\left\lceil\frac{k+5}{3}\right\rceil+2.$$

We next prove that $\gamma\big(P_k((3,3),(1,2))\big)\ge\lceil\frac{k+5}{3}\rceil+2$. Let S be a γ-set of $P_k((3,3),(1,2))$. In order to dominate α_3, then S must contains α_2 or α_3. Also, in order to dominate σ_1, then S must contains k or σ_1. Without loss of generality we can assume that $\alpha_2,k\in S$. In order to dominate β_3, then S must contains β_2 or β_3. Also, in order to dominate δ_2, then S must contains δ_1 or δ_2. Without loss of generality we can assume that $\beta_2,\delta_1\in S$. So we get

$$N[\{\alpha_2,k,\beta_2,\delta_1\}]=\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,(k-1),k,\sigma_1,\delta_1,\delta_2\}.$$

Therefore,

$$P_k((3,3),(1,2))-N[\{\alpha_2,k,\beta_2,\delta_1\}]=P_k((3,3),(1,2))[\{0,1,2,3,\dots,(k-2)\}]\cong P_{k-1}.$$

In order to dominate $0,1,2,3,\dots,(k-2)$, then $S-\{\alpha_2,k,\beta_2,\delta_1\}$ must contains $\lceil\frac{k-1}{3}\rceil$ vertices in $\{\alpha_1,\beta_1,0,1,2,3,\dots,(k-2),(k-1)\}$.
Therefore,

$$|S|\ge\left\lceil\frac{k-1}{3}\right\rceil+4=\left\lceil\frac{k-1+6}{3}\right\rceil+2=\left\lceil\frac{k+5}{3}\right\rceil+2.$$

We thus get

$$\gamma\big(P_k((3,3),(1,2))\big)\ge\left\lceil\frac{k+5}{3}\right\rceil+2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big)=\left\lceil\frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3}\right\rceil+2.$$

Case 3.2.6: Assume that $\min\{k_1,k_2\}=3$ and $\min\{k_3,k_4\}=2$.
We get that $\min\{k_1,k_2\}=k_1=3$ and $\min\{k_3,k_4\}=k_3=2$.
So in this case, we get the graph $P_k((k_1,k_2),(k_3,k_4))=P_k((3,3),(2,2))$. We next check that if $k\equiv 0 \pmod 3$, then $S=\{\alpha_2,1,4,7,\dots,(k-2),\sigma_1,\delta_2,\beta_2\}$ is a dominating set of $P_k((3,3),(2,2))$. Also, if $k\equiv 1 \pmod 3$ and $k\equiv 2 \pmod 3$, then $S=\{\alpha_2,1,4,7,\dots,(k-3),k,\sigma_2,\delta_2,\beta_2\}$ and $S=\{\alpha_2,1,4,7,\dots,(k-1),\sigma_2,\delta_2,\beta_2\}$ are

both dominating sets of $P_k((3,3),(2,2))$, respectively. We thus get

$$\gamma\big(P_k((3,3),(2,2))\big) \le \left\lfloor \frac{k+2}{3} \right\rfloor + 4 = \left\lfloor \frac{k+2+6}{3} \right\rfloor + 2 = \left\lceil \frac{k+6}{3} \right\rceil + 2.$$

We next prove that $\gamma\big(P_k((3,3),(2,2))\big) \ge \lceil \frac{k+6}{3} \rceil + 2$. Let S be a γ-set of $P_k((3,3),(2,2))$. In order to dominate α_3, then S must contains α_2 or α_3. Also, in order to dominate σ_2, then S must contains σ_2 or σ_1. Without loss of generality we can assume that $\alpha_2,\sigma_1 \in S$. In order to dominate β_3, then S must contains β_2 or β_3. Also, in order to dominate δ_2, then S must contains δ_1 or δ_2. Without loss of generality we can assume that $\beta_2,\delta_1 \in S$. So we get

$$N[\{\alpha_2,\sigma_1,\beta_2,\delta_1\}] = \{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,k,\sigma_1,\sigma_2,\delta_1,\delta_2\}.$$

Therefore,

$$P_k((3,3),(2,2)) - N[\{\alpha_2,\sigma_1,\beta_2,\delta_1\}] = P_k((3,3),(2,2))[\{0,1,2,3,\dots,(k-1)\}] \cong P_k.$$

In order to dominate $0,1,2,3,\dots,(k-1)$, then $S-\{\alpha_2,\sigma_1,\beta_2,\delta_1\}$ must contains $\lceil \frac{k}{3} \rceil$ vertices in $\{\alpha_1,\beta_1,0,1,2,3,\dots,(k-2),(k-1),k\}$.
Therefore,

$$|S| \ge \left\lceil \frac{k}{3} \right\rceil + 4 = \left\lceil \frac{k+6}{3} \right\rceil + 2 = \left\lceil \frac{k+6}{3} \right\rceil + 2.$$

We thus get

$$\gamma\big(P_k((3,3),(2,2))\big) \ge \left\lceil \frac{k+6}{3} \right\rceil + 2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.3: Assume that $k_2 = 2$ and $k_4 = 3$.
As in Case 3.2, we obtain

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k+\min\{k_1,k_2\}+\min\{k_3,k_4\}+1}{3} \right\rceil + 2.$$

Case 3.4: Assume that $k_2 = 3$ and $k_4 = 3$.
The proof may be handled in much the same way.
We consider following nine cases. Case 3.4.1: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 1$;
Case 3.4.2: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 2$;
Case 3.4.3: if $\min\{k_1,k_2\} = 1$ and $\min\{k_3,k_4\} = 3$;
Case 3.4.4: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 1$;
Case 3.4.5: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 2$;
Case 3.4.6: if $\min\{k_1,k_2\} = 2$ and $\min\{k_3,k_4\} = 3$;
Case 3.4.7: if $\min\{k_1,k_2\} = 3$ and $\min\{k_3,k_4\} = 1$;
Case 3.4.8: if $\min\{k_1,k_2\} = 3$ and $\min\{k_3,k_4\} = 2$;
Case 3.4.9: if $\min\{k_1,k_2\} = 3$ and $\min\{k_3,k_4\} = 3$.

For Case 3.4.9, we obtain $P_k((k_1,k_2),(k_3,k_4)) = P_k((3,3),(3,3))$ (see Figure 8).

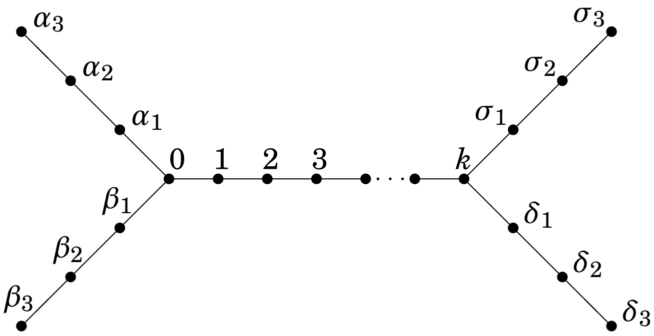

Figure 8. $P_k((3,3),(3,3))$

Similarly, we first check that if $k \equiv 0 \pmod 3$, then $S = \{\alpha_2, 1, 4, 7, \dots, (k-2), k, \sigma_2, \delta_2, \beta_2\}$ is a dominating set of $P_k((3,3),(3,3))$. Also, if $k \equiv 1 \pmod 3$ and $k \equiv 2 \pmod 3$, then $S = \{\alpha_2, 1, 4, 7, \dots, (k-3), k, \sigma_2, \delta_2, \beta_2\}$ and $S = \{\alpha_2, 1, 4, 7, \dots, (k-1), k, \sigma_2, \delta_2, \beta_2\}$ are both dominating sets of $P_k((3,3),(3,3))$, respectively. We thus get $\gamma\big(P_k((3,3),(3,3))\big) \le \lceil \frac{k+1}{3} \rceil + 4 = \lceil \frac{k+7}{3} \rceil + 2$. We next prove that $\gamma\big(P_k((3,3),(3,3))\big) \ge \lceil \frac{k+7}{3} \rceil + 2$. Let S be a γ-set of $P_k((3,3),(3,3))$. In order to dominate α_3, then S must contains α_2 or α_3. Also, in order to dominate σ_3, then S must contains σ_2 or σ_3. Without loss of generality we can assume that $\alpha_2, \sigma_2 \in S$. In order to dominate β_3, then S must contains β_2 or β_3. Also, in order to dominate δ_3, then S must contains δ_2 or δ_3. Without loss of generality we can assume that $\beta_2, \delta_2 \in S$. So we get

$$N[\{\alpha_2, \sigma_2, \beta_2, \delta_2\}] = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3\}.$$

Therefore,

$$P_k((3,3),(3,3)) - N[\{\alpha_2, \sigma_2, \beta_2, \delta_2\}] = P_k((3,3),(3,3))[\{0,1,2,3,\dots,k-1,k\}] \cong P_{k+1}.$$

In order to dominate $0,1,2,3,\dots,k-1,k$, then $S - \{\alpha_2, \sigma_2, \beta_2, \delta_2\}$ must contains $\lceil \frac{k+1}{3} \rceil$ vertices in $\{\alpha_1, \beta_1, 0, 1, 2, 3, \dots, k-1, k, \sigma_1, \delta_1\}$.

Therefore,

$$|S| \ge \left\lceil \frac{k+1}{3} \right\rceil + 4 = \left\lceil \frac{k+7}{3} \right\rceil + 2.$$

We thus get

$$\gamma\big(P_k((3,3),(3,3))\big) \ge \left\lceil \frac{k+7}{3} \right\rceil + 2.$$

Therefore,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k + \min\{k_1,k_2\} + \min\{k_3,k_4\} + 1}{3} \right\rceil + 2.$$

For other subcases of Case 3.4, we leave it to the reader to verify that

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) = \left\lceil \frac{k + \min\{k_1,k_2\} + \min\{k_3,k_4\} + 1}{3} \right\rceil + 2. \qquad \square$$

Finally, we come to the main theorem of this section. The theorem concludes the upper bound of the domination number of a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$.

Theorem 2. *Let $k,k_1,k_2,k_3,k_4 \in \mathbb{N}$ and* $a_i = \begin{cases} 1 & \text{if } k_i \equiv 1 \pmod 3, \\ 2 & \text{if } k_i \equiv 2 \pmod 3, \\ 3 & \text{if } k_i \equiv 0 \pmod 3. \end{cases}$

Then

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) - \sum_{i=1}^{4}\left\lceil \frac{k_i - a_i}{3} \right\rceil - \left\lceil \frac{k + \min\{a_1,a_2\} + \min\{a_3,a_4\} + 1}{3} \right\rceil$$
$$\le \begin{cases} 0 & \text{if } \max\{a_1,a_2\} = \max\{a_3,a_4\} = 1, \\ 1 & \text{if either } \max\{a_1,a_2\} \ge 2 \text{ or } \max\{a_3,a_4\} \ge 2, \\ 2 & \text{if } \max\{a_1,a_2\} \ge 2 \text{ and } \max\{a_3,a_4\} \ge 2. \end{cases}$$

Proof. Let $k,k_1,k_2,k_3,k_4 \in \mathbb{N}$, and

$$a_i = \begin{cases} 1 & \text{if } k_i \equiv 1 \pmod 3, \\ 2 & \text{if } k_i \equiv 2 \pmod 3, \\ 3 & \text{if } k_i \equiv 0 \pmod 3. \end{cases}$$

Then we have a graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ which is as shown in Figure 9.

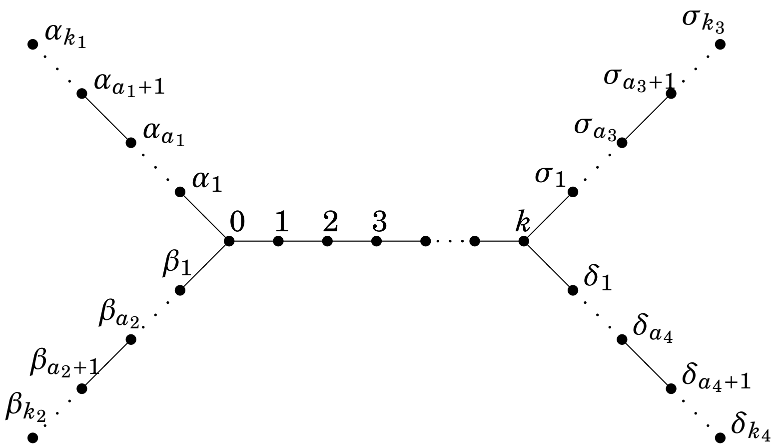

Figure 9. A graph $P_k((k_1,k_2),(k_3,k_4))$ with $\alpha_{a_1},\beta_{a_2},\sigma_{a_3},\delta_{a_4}$

It is evident that

$$P_k((k_1,k_2),(k_3,k_4))\left[\{\alpha_{a_1+1},\alpha_{a_1+2},\dots,\alpha_{k_1}\}\right] \cong P_{k_1-a_1},$$
$$P_k((k_1,k_2),(k_3,k_4))\left[\{\beta_{a_2+1},\beta_{a_2+2},\dots,\beta_{k_2}\}\right] \cong P_{k_2-a_2},$$
$$P_k((k_1,k_2),(k_3,k_4))\left[\{\sigma_{a_3+1},\sigma_{a_3+2},\dots,\sigma_{k_3}\}\right] \cong P_{k_3-a_3}, \quad \text{and}$$
$$P_k((k_1,k_2),(k_3,k_4))\left[\{\delta_{a_4+1},\delta_{a_4+2},\dots,\delta_{k_4}\}\right] \cong P_{k_4-a_4}.$$

Since $a_i = \begin{cases} 1 & \text{if } k_i \equiv 1 \pmod 3 \\ 2 & \text{if } k_i \equiv 2 \pmod 3, \\ 3 & \text{if } k_i \equiv 0 \pmod 3 \end{cases}$ we have $k_i - a_i \equiv 0 \pmod 3$ for all $i = 1,2,3,4$.

Clearly,

$$\gamma\big(P_k((k_1,k_2),(k_3,k_4))\big) \le \gamma\big(P_k((k_1,k_2),(k_3,k_4))\left[\{\alpha_{a_1+1},\alpha_{a_1+2},\dots,\alpha_{k_1}\}\right]\big)$$

$$
\begin{aligned}
&+\gamma\big(P_k((k_1,k_2),(k_3,k_4))\,[\{\beta_{a_2+1},\beta_{a_2+2},\dots,\beta_{k_2}\}]\big)\\
&+\gamma\big(P_k((k_1,k_2),(k_3,k_4))\,[\{\sigma_{a_3+1},\sigma_{a_3+2},\dots,\sigma_{k_3}\}]\big)\\
&+\gamma\big(P_k((k_1,k_2),(k_3,k_4))\,[\{\delta_{a_4+1},\delta_{a_4+2},\dots,\delta_{k_4}\}]\big)\\
&+\gamma\big(P_k((a_1,a_2),(a_3,a_4))\big).
\end{aligned}
$$

By Lemma 3, we obtain

$$
\gamma\big(P_k((k_1,k_2)(k_3,k_4))\big)-\sum_{i=1}^{4}\left\lceil\frac{k_i-a_i}{3}\right\rceil-\left\lceil\frac{k+\min\{a_1,a_2\}+\min\{a_3,a_4\}+1}{3}\right\rceil
$$

$$
\le\begin{cases}0 & \text{if } \max\{a_1,a_2\}=\max\{a_3,a_4\}=1,\\ 1 & \text{if either } \max\{a_1,a_2\}\ge 2 \text{ or } \max\{a_3,a_4\}\ge 2,\\ 2 & \text{if } \max\{a_1,a_2\}\ge 2 \text{ and } \max\{a_3,a_4\}\ge 2.\end{cases} \qquad \square
$$

4. Conclusion

A graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ is a tree of $k+k_1+k_2+k_3+k_4+1$ vertices with the degree sequence $(1,1,1,1,2,2,2,\dots,2,3,3)$ where $k,k_1,k_2,k_3,k_4\in\mathbb{N}$. The distance between both vertices of degree 3 is equal to k. We prove some statements about dominating sets of the graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$ and investigate the minimum cardinality of those dominating sets. Hence we obtain the upper bound for the domination number of the graph $P_k\big((k_1,k_2),(k_3,k_4)\big)$. We can determine the domination number from the result in Theorem 2. The formula is

$$
\gamma\Big(P_k\big((k_1,k_2)(k_3,k_4)\big)\Big)-\sum_{i=1}^{4}\left\lceil\frac{k_i-a_i}{3}\right\rceil-\left\lceil\frac{k+\min\{a_1,a_2\}+\min\{a_3,a_4\}+1}{3}\right\rceil
$$

$$
\le\begin{cases}0 & \text{if } \max\{a_1,a_2\}=\max\{a_3,a_4\}=1,\\ 1 & \text{if either } \max\{a_1,a_2\}\ge 2 \text{ or } \max\{a_3,a_4\}\ge 2,\\ 2 & \text{if } \max\{a_1,a_2\}\ge 2 \text{ and } \max\{a_3,a_4\}\ge 2,\end{cases}
$$

where $a_i=\begin{cases}1 & \text{if } k_i\equiv 1 \pmod 3\\ 2 & \text{if } k_i\equiv 2 \pmod 3\\ 3 & \text{if } k_i\equiv 0 \pmod 3).\end{cases}$

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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