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Periodic Boundary Value Problems for the Second Order Impulsive Differential Equations

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Abstract. This paper is concerned with solutions of periodic boundary value problems for the second order impulsive differential equation. We prove the existence of extreme solutions and present the method of lower and upper solutions coupled with monotone iterative technique. Some comparison results are also established.

1. Introduction

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses. Impulsive differential equations have been developed for modelling impulsive problems in physics, medicine, population dynamics, biotechnology, control theory, etc. (see also ref. [1, 5]). The monotone iterative technique combined with lower and upper solutions has been applied to obtain existence results for first order [4, 7] and second order [2, 3, 6] impulsive differential equation with initial or boundary value conditions.

In this paper, we consider the second-order impulsive functional differential equation with boundary conditions,

$$\begin{cases} -x''(t) = f(t, x(t), x(\alpha(t))) - K(t)x'(t) \equiv Fx(t), \\ \qquad\qquad\qquad\qquad t \neq t_k,\ t \in J = [0, T], \\ \Delta x(t_k) = P_k(x(t_k), x'(t_k)), \qquad k = 1, \dots, m, \\ \Delta x'(t_k) = Q_k(x(t_k), x'(t_k)), \qquad k = 1, \dots, m, \\ x(0) = x(T), \\ x'(0) = x'(T), \end{cases} \tag{1.1}$$

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where $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $0 \le \alpha(t) \le t$, $P_k, Q_k \in C(\mathbb{R}^2, \mathbb{R})$, $K \in C(J, \mathbb{R}^+)$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of x at t_k, respectively. Similarly, $x'(t_k^+)$ and $x'(t_k^-)$ denote the right and left limits of x' at t_k, respectively.

This paper is organized as follows. In Section 2, we introduce the concepts of lower and upper solutions and formulate some lemmas which are necessary in our discussion. In section 3, by using the method of lower and upper solutions and the monotone iterative technique we prove the existence of extreme solutions for PBVP (1.1).

2. **Preliminaries**

$J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R},\ x|_{J_k} \in C(J_k, \mathbb{R}),$ $k = 0, 1, \dots, m$ and there exist $x(t_k^+)$ for $k = 1, 2, \dots, m\}$, $PC^1(J, \mathbb{R}) = \{x : J \to \mathbb{R},$ $x|_{J_k} \in C^1(J_k, \mathbb{R})$, $k = 0, 1, \dots, m$ and there exist $x'(t_k^+)$ for $k = 1, \dots, m\}$. Let $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ be Banach spaces with the respective norms,

$$\|x\|_{PC} = \sup_{t \in J} |x(t)|, \quad \|x\|_{PC^1} = \max_{t \in J} \{\|x\|_{PC}, \|x'\|_{PC}\}.$$

A function $x \in PC^1(J, \mathbb{R}) \cap C^2(J, \mathbb{R})$ is called a solution of PBVP (1.1) if it satisfies (1.1).

Definition 2.1. We say that the function $y_0 \in PC^1(J, \mathbb{R}) \cap C^2(J, \mathbb{R})$ is called a lower solution of the PBVP (1.1) if

$$\begin{cases} -y_0''(t) \le f(t, y_0(t), y_0(\alpha(t))) - K(t) y_0'(t) \equiv F y_0(t), & t \in J', \\ \Delta y_0(t_k) = P_k(y_0(t_k), y_0'(t_k)), & k = 1, \dots, m, \\ \Delta y_0'(t_k) \ge Q_k(y_0(t_k), y_0'(t_k)), & k = 1, \dots, m, \\ y_0(0) = y_0(T), \\ y_0'(0) \ge y_0'(T), \end{cases} \tag{2.1}$$

and it is called an upper solution of the PBVP (1.1) if the above inequalities are reversed.

Lemma 2.2. *Assume that* $p \in PC^1(J, \mathbb{R}) \cap C^2(J, \mathbb{R})$,

$$\begin{cases} p''(t) \ge K(t) p'(t) + M(t) p(t) + N(t) p(\alpha(t)), & t \in J', \\ \Delta p(t_k) = L_k p'(t_k), & k = 1, \dots, m, \\ \Delta p'(t_k) \ge L_k^* p'(t_k), & k = 1, \dots, m, \\ p(0) = p(T), \ \ p'(0) \ge p'(T), \end{cases} \tag{2.2}$$

where M, $N \in C(J, \mathbb{R}^+)$. *Also assume that* L_k, $L_k^* \ge 0$ *and*

$$\left(\sum_{i=1}^{m} L_i + T \right) \left(\frac{1}{e^{\int_0^T K(s) ds}} + 1 \right) \\ \times \left(\sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} H^*(s) ds \left[\prod_{j=i}^{m} (1 + L_j^*) \right] + \int_0^T H^*(s) ds \right) \le 1, \tag{2.3}$$

where $H^*(t) \equiv e^{\int_t^T K(s)ds}[M(t)+N(t)]$ *and* $\inf\{H^*(t); t \in J\} > 0$ *hold, then* $p(t) \le 0$ *on* J.

Proof. Firstly, we show that $\inf\{p(t); t \in J\} \le 0$. If $p'(t) > 0$ for all t, it follows that $p(t)$ is increasing and $\Delta p(t_k) = L_k p'(t_k) \ge 0$, then $p(0) < p(T)$, a contradiction. Hence, there exists a point $\check{t} \in J_h$, $h \in \{0, \dots, m\}$, such that $p'(\check{t}) \le 0$. Let $\inf\{p(t); t \in J\} = b$, where b is a constant, and there exists $t_* \in J_r$, $r \in \{0, \dots, m\}$ such that $p(t_*) = b$ or $p(t_r^+) = b$. We only consider $p(t_*) = b$, for the case $p(t_r^+) = b$ the proof is similar. Let $u(t) = (e^{\int_t^T K(s)ds} p'(t))$, we have

$$\begin{aligned} u'(t) &= (e^{\int_t^T K(s)ds} p'(t))' = e^{\int_t^T K(s)ds}[p''(t) - K(t)p'(t)] \\ &\ge e^{\int_t^T K(s)ds}[M(t)p(t) + N(t)p(\alpha(t))]ds \equiv \eta(t), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \Delta u(t_k) &= \Delta(e^{\int_{t_k}^T K(s)ds} p'(t_k)) \ge L_k^* e^{\int_{t_k}^T K(s)ds} p'(t_k), \\ u(t_k^+) &\ge (1 + L_k^*) u(t_k). \end{aligned} \tag{2.5}$$

Then, we obtain

$$u(\check{t}) \ge u(0)\left[\prod_{i=1}^h (1+L_i^*)\right] + \sum_{i=1}^h \int_{t_{i-1}}^{t_i} \eta(s)ds \left[\prod_{j=i}^h (1+L_j^*)\right] + \int_{t_h}^{\check{t}} \eta(s)ds,$$

$$0 \ge u(0)\left[\prod_{i=1}^h (1+L_i^*)\right] + b\sum_{i=1}^h \int_{t_{i-1}}^{t_i} H^*(s)ds \left[\prod_{j=i}^h (1+L_j^*)\right] + b\int_{t_h}^{\check{t}} H^*(s)ds.$$

Thus

$$u(0) \le \frac{-b}{\prod_{i=1}^h (1+L_i^*)} \left(\sum_{i=1}^h \int_{t_{i-1}}^{t_i} H^*(s)ds \left[\prod_{j=i}^h (1+L_j^*)\right] + \int_{t_h}^{\check{t}} H^*(s)ds \right). \tag{2.6}$$

Similarly, we can prove that

$$\begin{aligned} u(T) &\ge u(t)\left[\prod_{i=l+1}^m (1+L_i^*)\right] + \sum_{i=l+1}^m \int_{\bar{t}_{i-1}}^{\bar{t}_i} \eta(s)ds \left[\prod_{j=i}^m (1+L_j^*)\right] + \int_{\bar{t}_m}^T \eta(s)ds \\ &\ge u(t)\left[\prod_{i=l+1}^m (1+L_i^*)\right] + b\sum_{i=l+1}^m \int_{\bar{t}_{i-1}}^{\bar{t}_i} H^*(s)ds \left[\prod_{j=i}^m (1+L_j^*)\right] + b\int_{\bar{t}_m}^T H^*(s)ds, \end{aligned} \tag{2.7}$$

for $t \in J_l = (t_l, t_{l+1}]$, where $\bar{t}_l = t$, $\bar{t}_k = t_k$, $\bar{J}_k = (\bar{t}_k, \bar{t}_{k+1}]$, $k = l+1, l+2, \dots, m$. By using (2.2) and inequality (2.6), (2.7), we get that

$$u(t) \le \frac{-b}{\prod_{i=l+1}^m (1+L_i^*)} \left[\frac{1}{e^{\int_0^T K(s)ds} \prod_{i=1}^h (1+L_i^*)} \right.$$

$$\times\left(\sum_{i=1}^{h}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{h}(1+L_j^*)\right]+\int_{t_h}^{\check{t}}H^*(s)ds\right)$$
$$+\sum_{i=l+1}^{m}\int_{\bar{t}_{i-1}}^{\bar{t}_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_{\bar{t}_m}^{T}H^*(s)ds\Bigg]. \tag{2.8}$$

Substituting $u(t)=(e^{\int_t^T K(s)ds}p'(t))$ into (2.8), we have

$$p'(t) \le \frac{-b}{e^{\int_t^T K(s)ds}\prod_{i=l+1}^{m}(1+L_i^*)}\Bigg[\frac{1}{e^{\int_0^T K(s)ds}\prod_{i=1}^{h}(1+L_i^*)}$$
$$\times\left(\sum_{i=1}^{h}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{h}(1+L_j^*)\right]+\int_{t_h}^{\check{t}}H^*(s)ds\right)$$
$$+\sum_{i=l+1}^{m}\int_{\bar{t}_{i-1}}^{\bar{t}_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_{\bar{t}_m}^{T}H^*(s)ds\Bigg]. \tag{2.9}$$

By (2.9) if $b>0$, which implies $p'(t)<0$ for all t, $\Delta p(t_k)=L_kp'(t_k)\le 0$, $k=1,\dots,m$ and $p(0)>p(T)$ a contradicts. Then, we have $\inf\{p(t);t\in J\}\le 0$.

Next, we will prove that $p(t)\le 0$ for all $t\in J$. Suppose, to the contrary, that $p(t^*)>0$ for some $t^*\in J_v$, $v\in\{0,\dots,m\}$. By letting $b=-d$ where $d\ge 0$ from (2.9), we have

$$p'(t) \le \frac{d}{e^{\int_t^T K(s)ds}\prod_{i=l+1}^{m}(1+L_i^*)}\Bigg[\frac{1}{e^{\int_0^T K(s)ds}\prod_{i=1}^{h}(1+L_i^*)}$$
$$\times\left(\sum_{i=1}^{h}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{h}(1+L_j^*)\right]+\int_{t_h}^{\check{t}}H^*(s)ds\right)$$
$$+\sum_{i=l+1}^{m}\int_{\bar{t}_{i-1}}^{\bar{t}_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_{\bar{t}_m}^{T}H^*(s)ds\Bigg]$$
$$\le d\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right). \tag{2.10}$$

Assume that $t^*>t_*$ then $v>r$. For the case $t^*<t_*$, the proof is similar and thus we omit it. By mean value theorem, we have

$$\begin{aligned}p(t^*)-p(t_v) &= p(t^*)-p(t_v^+)+L_vp'(t_v)\\ &= p'(s_v)(t^*-t_v^+)+L_vp'(t_v), s_v\in(t_v,t^*)\\ &\le d\big(L_v+(t^*-t_v^+)\big)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)\end{aligned}$$

$$\times\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right)$$

$$p(t_\nu)-p(t_{\nu-1}) \le d\big(L_{\nu-1}+(t_\nu-t_{\nu-1}^+)\big)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)$$
$$\times\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right)$$
$$\vdots$$
$$p(t_{r+1})-p(t_*) \le d\big(L_r+(t_{r+1}-t_*)\big)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)$$
$$\times\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right).$$

Summing up we obtain

$$p(t^*)-p(t_*) \le d\left(\sum_{i=1}^{m}L_i+T\right)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)$$
$$\times\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right).$$

Finally, we have the next two cases.
(i) if $b<0$, then

$$0<p(t^*) \le p(t_*)+d\left(\sum_{i=1}^{m}L_i+T\right)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)$$
$$\times\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right)$$
$$0 < -d+d\left(\sum_{i=1}^{m}L_i+T\right)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)$$
$$\times\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right).$$

Hence

$$\left(\sum_{i=1}^{m}L_i+T\right)\left(\frac{1}{e^{\int_0^T K(s)ds}}+1\right)\left(\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}H^*(s)ds\left[\prod_{j=i}^{m}(1+L_j^*)\right]+\int_0^T H^*(s)ds\right)>1,$$

which contradicts (2.3).
(ii) if $b=0$, then $p'(t)\le 0$. This yields $\Delta p(t_k)=L_kp'(t_k)\le 0$ so $p(t)$ is decreasing and therefore $p(T)\le p(0)\le 0$ which is a contradiction. The proof is complete. □

Consider the PBVP

$$\begin{cases} p''(t) = K(t)p'(t) + M(t)p(t) + N(t)p(\alpha(t)) - \sigma(t), & t \in J', \\ \Delta p(t_k) = L_k p'(t_k) + \gamma_k, & k = 1, \dots, m, \\ \Delta p'(t_k) = L_k^* p'(t_k) + \lambda_k, & k = 1, \dots, m, \\ p(0) = p(T), \ \ p'(0) = p'(T), \end{cases} \tag{2.11}$$

with

(H_0): $K, M, N \in C(J, \mathbb{R}^+)$, $\sigma \in C(J, \mathbb{R})$, $\alpha \in C(J, \mathbb{R})$ and $L, L^* \geq 0$, $\gamma_k, \lambda_k \in \mathbb{R}$, $k = 1, \dots, m$.

Lemma 2.3. *Assume that* (H_0) *holds. Then* $p \in PC^1(J, \mathbb{R}) \cap C^2(J, \mathbb{R})$ *is a solution of* (2.11) *if and only if* $p \in PC^1(J, \mathbb{R})$ *is a solution of the following impulsive integral equation:*

$$\begin{aligned} p(t) \ = \ & \int_0^T G_1(t,s)\sigma_1(s, p(s), p(\alpha(s)), p'(s))ds \\ & + \sum_{k=1}^m [-G_1(t, t_k)(L_k^* p'(t_k) + \lambda_k) + G_2(t, t_k)(L_k p'(t_k) + \gamma_k)], \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} & \sigma_1(t, p(t), p(\alpha(t)), p'(t)) \\ & \quad = \ -K(t)p'(t) + \left[W^2 - M(t)\right]p(t) - N(t)p(\alpha(t)) + \sigma(t), \end{aligned} \tag{2.13}$$

for fixed $W > 0$*, and*

$$\begin{aligned} G_1(t,s) &= \frac{1}{2W\left(e^{WT} - 1\right)} \begin{cases} e^{W(T-t+s)} + e^{W(t-s)}, & 0 \leq s < t \leq T, \\ e^{W(T+t-s)} + e^{W(s-t)}, & 0 \leq t \leq s \leq T, \end{cases} \\ G_2(t,s) &= \frac{1}{2\left(e^{WT} - 1\right)} \begin{cases} e^{W(T-t+s)} - e^{W(t-s)}, & 0 \leq s < t \leq T, \\ -e^{W(T+t-s)} + e^{W(s-t)}, & 0 \leq t \leq s \leq T. \end{cases} \end{aligned} \tag{2.14}$$

Proof. Substituting σ_1 from Eq. (2.13) into the differential equation in (2.11), we obtain

$$-p''(t) + W^2 p(t) = \sigma_1(t, p(t), p(\alpha(t)), p'(t)), \quad t \in J'.$$

Then by using Lemma 2.1 of [2] we have the assertion. □

Lemma 2.4. *Let assumption* (H_0) *hold and* $W > 0$*. Moreover, assume that*

$$\xi \equiv \frac{1+e^{WT}}{2W(e^{WT}-1)}\left(\int_0^T K(s)+|W^2-M(s)|+N(s)ds+\sum_{k=1}^m L_k^*\right)+\frac{1}{2}\sum_{k=1}^m L_k<1, \tag{2.15}$$

$$\psi \equiv \frac{1}{2}\left(\int_0^T K(s)+|W^2-M(s)|+N(s)ds+\sum_{k=1}^m L_k^*\right)+\frac{1+e^{WT}}{2(e^{WT}-1)}W\sum_{k=1}^m L_k<1. \tag{2.16}$$

Then problem (2.11) *has a unique solution* $p \in PC^1(J, \mathbb{R}) \cap C^2(J, \mathbb{R})$.

Proof. Some ideas are taken from Lemma 2.3 of [2]. For any $p \in PC^1(J,\mathbb{R}) \cap C^2(J,\mathbb{R})$, define an operator A by

$$\begin{aligned}(Ap)(t) = & \int_0^T G_1(t,s)\sigma_1(s,p(s),p(\alpha(s)),p'(s))ds \\ & + \sum_{k=1}^{m}[-G_1(t,t_k)(L_k^* p'(t_k)+\lambda_k)+G_2(t,t_k)(L_k p'(t_k)+\gamma_k)], \quad t \in J,\end{aligned}$$

where G_1, G_2 are given by Lemma 2.3. We need to find a fixed point of the operator A. By computing directly, we obtain

$$\max_{t\in J}\{G_1(t,s)\} = \frac{1+e^{WT}}{2W(e^{WT}-1)}, \qquad \max_{t\in J}\{G_2(t,s)\} = \frac{1}{2}.$$

For any x, $y \in PC^1(J,\mathbb{R})$, we have

$$\begin{aligned}&\|Ax-Ay\|_{PC} \\ &\quad = \sup_{t\in J}|Ax-Ay| \\ &\quad \le \sup_{t\in J}\left|\int_0^T G_1(t,s)[\sigma_1(s,x(s),x(\alpha(s)),x'(s))-\sigma_1(s,y(s),y(\alpha(s)),y'(s))]ds\right| \\ &\qquad + \sup_{t\in J}\left|\sum_{k=1}^{m}\big[-G_1(t,t_k)L_k^*(x'(t_k)-y'(t_k))+G_2(t,t_k)L_k(x'(t_k)-y'(t_k))\big]\right| \\ &\quad \le \xi\|x-y\|_{PC^1}. \end{aligned} \tag{2.17}$$

Similarly,

$$\begin{aligned}&\|Ax'-Ay'\|_{PC} \\ &\quad = \sup_{t\in J}|Ax'-Ay'| \\ &\quad \le \sup_{t\in J}\left|\int_0^T -G_2(t,s)[\sigma_1(s,x(s),x(\alpha(s)),x'(s))-\sigma_1(s,y(s),y(\alpha(s)),y'(s))]ds\right| \\ &\qquad + \sup_{t\in J}\left|\sum_{k=1}^{m}[G_2(t,t_k)L_k^*(x'(t_k)-y'(t_k))-W^2G_1(t,t_k)L_k(x'(t_k)-y'(t_k))]\right| \\ &\quad \le \psi\|x-y\|_{PC^1}. \end{aligned} \tag{2.18}$$

Hence

$$\|Ax-Ay\|_{PC^1} \le \max\{\xi,\psi\}\|x-y\|_{PC^1}. \tag{2.19}$$

By the Banach fixed point theorem, the operator A has a unique fixed point. This completes the proof. □

3. Main Results

We are now in a position to prove that the problem (1.1) has extremal solutions.

Theorem 3.1. *Let the following assumptions hold:*

(H_1): $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $0 \le \alpha(t) \le t$, P_k, $Q_k \in C(\mathbb{R}^2, \mathbb{R})$ *for* $k = 1, \dots, m$ *and if there exists a point* $\check{t} \in J$ *such that* $\alpha(\check{t}) \in \{t_1, t_2, \dots, t_m\}$, *then* $\check{t} \in \{t_1, t_2, \dots, t_m\}$,

(H_2): y_0, $z_0 \in PC^1(J, \mathbb{R}) \cap C^2(J, \mathbb{R})$ *are lower and upper solutions of problem (1.1), respectively, and* $y_0 \le z_0$ *on* J.

(H_3): *there exist* K, M, $N \in C(J, \mathbb{R}^+)$ *such that*

$$[f(t,u,v) - K(t)w] - [f(t,\bar{u},\bar{v}) - K(t)\bar{w}] \le K(t)(\bar{w} - w) + M(t)(\bar{u} - u) + N(t)(\bar{v} - v),$$

for $y_0(\alpha(t)) \le v \le \bar{v} \le z_0(\alpha(t))$, $y_0(t) \le u \le \bar{u} \le z_0(t)$, $t \in J$,

(H_4): *there exist constants* L_k, L_k^*; $k = 1, \dots, m$, *such that*

$$P_k(r(t_k), r'(t_k)) - P_k(\bar{r}(t_k), \bar{r}'(t_k)) = L_k[r'(t_k) - \bar{r}'(t_k)],$$

$$Q_k(r(t_k), r'(t_k)) - Q_k(\bar{r}(t_k), \bar{r}'(t_k)) \ge L_k^*[r'(t_k) - \bar{r}'(t_k)],$$

for $y_0(t_k) \le r(t_k) \le \bar{r}(t_k) \le z_0(t_k)$, $k = 1, \dots, m$,

(H_5): *the functions* K, M, $N \in C(J, \mathbb{R}^+)$ *and constants* $L_k \ge 0$, $L_k^* \ge 0$, $k = 1, \dots, m$, *satisfy (2.3), (2.15) and (2.16).*

Then problem (1.1) *has extremal solutions in* $[y_0, z_0] = \{w \in PC^1(J, \mathbb{R}) : y_0(t) \le w(t) \le z_0(t),\ t \in J\}$.

Proof. Consider the following sequence:

$$\begin{cases} -y_n''(t) = Fy_{n-1}(t) - K(t)[y_n'(t) - y_{n-1}'(t)] - M(t)[y_n(t) - y_{n-1}(t)] \\ \qquad\qquad -N(t)[y_n(\alpha(t)) - y_{n-1}(\alpha(t))], \quad t \in J', \\ \Delta y_n(t_k) = P_k(y_{n-1}(t_k), y_{n-1}'(t_k)) + L_k[y_n'(t_k) - y_{n-1}'(t_k)], \quad k = 1, \dots, m, \\ \Delta y_n'(t_k) = Q_k(y_{n-1}(t_k), y_{n-1}'(t_k)) + L_k^*[y_n'(t_k) - y_{n-1}'(t_k)], \quad k = 1, \dots, m, \\ y_n(0) = y_n(T), \\ y_n'(0) = y_n'(T), \end{cases}$$

$$\begin{cases} -z_n''(t) = Fz_{n-1}(t) - K(t)[z_n'(t) - z_{n-1}'(t)] - M(t)[z_n(t) - z_{n-1}(t)] \\ \qquad\qquad -N(t)[z_n(\alpha(t)) - z_{n-1}(\alpha(t))], \quad t \in J', \\ \Delta z_n(t_k) = P_k(z_{n-1}(t_k), z_{n-1}'(t_k)) + L_k[z_n'(t_k) - z_{n-1}'(t_k)], \quad k = 1, \dots, m, \\ \Delta z_n'(t_k) = Q_k(z_{n-1}(t_k), z_{n-1}'(t_k)) + L_k^*[z_n'(t_k) - z_{n-1}'(t_k)], \quad k = 1, \dots, m, \\ z_n(0) = z_n(T), \\ z_n'(0) = z_n'(T), \end{cases}$$

for $n = 1, 2, \dots$. Moreover, by Lemma 2.4, we have y_1, z_1 are well defined.

We show first that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \quad t \in J. \tag{3.1}$$

Let $v = y_0 - y_1$. By Definition 2.1 of a lower solution of (1.1), we have

$$\begin{aligned} v''(t) &= y_0''(t) - y_1''(t) \\ &\ge Fy_0(t) - Fy_0(t) - K(t)[y_1'(t) - y_0'(t)] - M(t)[y_1(t) - y_0(t)] \\ &\quad -N(t)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &\ge K(t)[y_0'(t) - y_1'(t)] + M(t)[y_0(t) - y_1(t)] + N(t)[y_0(\alpha(t)) - y_1(\alpha(t))], \end{aligned}$$

or

$$v''(t) \ge K(t)v'(t) + M(t)v(t) + N(t)v(\alpha(t)),$$

and

$$\begin{aligned} \Delta v(t_k) &= \Delta y_0(t_k) - \Delta y_1(t_k) \\ &= P_k(y_0(t_k), y_0'(t_k)) - P_k(y_0(t_k), y_0'(t_k)) - L_k[y_1'(t_k) - y_0'(t_k)] \\ &= L_k v'(t_k), \quad k = 1, \ldots, m, \end{aligned}$$

$$\begin{aligned} \Delta v'(t_k) &= \Delta y_0'(t_k) - \Delta y_1'(t_k) \\ &\ge Q_k(y_0(t_k), y_0'(t_k)) - Q_k(y_0(t_k), y_0'(t_k)) - L_k^*[y_1'(t_k) - y_0'(t_k)] \\ &= L_k^* v'(t_k), \quad k = 1, \ldots, m, \end{aligned}$$

$$\left.\begin{aligned} v(0) = y_0(0) - y_1(0) \\ v(T) = y_0(T) - y_1(T) \end{aligned}\right\} \Rightarrow v(0) = v(T),$$

$$\left.\begin{aligned} v'(0) = y_0'(0) - y_1'(0) \\ v'(T) = y_0'(T) - y_1'(T) \end{aligned}\right\} \Rightarrow v'(0) \ge v'(T).$$

Then, by Lemma 2.2, $v \le 0$, which implies $y_0(t) \le y_1(t)$, $t \in J$. In a similar way, we can show that $z_0(t) \ge z_1(t)$, $t \in J$.
Next, we will show that $y_1(t) \le z_1(t)$, $t \in J$. Let $p = y_1 - z_1$ then, we obtain

$$\begin{aligned} p''(t) &= Fz_0(t) - K(t)[z_1'(t) - z_0'(t)] - M(t)[z_1(t) - z_0(t)] \\ &\quad - N(t)[z_1(\alpha(t)) - z_0(\alpha(t))] - Fy_0(t) + K(t)[y_1'(t) - y_0'(t)] \\ &\quad + M(t)[y_1(t) - y_0(t)] + N(t)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &= Fz_0(t) - Fy_0(t) + K(t)[y_1'(t) - y_0'(t) - z_1'(t) + z_0'(t)] \\ &\quad + M(t)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\quad + N(t)[y_1(\alpha(t)) - y_0(\alpha(t)) - z_1(\alpha(t)) + z_0(\alpha(t))], \end{aligned} \tag{3.2}$$

or

$$p''(t) \geq K(t)p'(t) + M(t)p(t) + N(t)p(\alpha(t)),$$

and

$$\begin{aligned}
\Delta p(t_k) &= \Delta y_1(t_k) - \Delta z_1(t_k) \\
&= P_k(y_0(t_k), y_0'(t_k)) + L_k[y_1'(t_k) - y_0'(t_k)] \\
&\quad - P_k(z_0(t_k), z_0'(t_k)) - L_k[z_1'(t_k) - z_0'(t_k)] \\
&= L_k p'(t_k),\ k = 1, \ldots, m,
\end{aligned}$$

$$\begin{aligned}
\Delta p'(t_k) &= \Delta y_1'(t_k) - \Delta z_1'(t_k) \\
&= Q_k(y_0(t_k), y_0'(t_k)) + L_k^*[y_1'(t_k) \\
&\quad - y_0'(t_k)] - Q_k(z_0(t_k), z_0'(t_k)) - L_k^*[z_1'(t_k) - z_0'(t_k)] \\
&\geq \nu L_k^* p'(t_k),\ k = 1, \ldots, m,
\end{aligned}$$

$$\left.\begin{aligned} p(0) = y_1(0) - z_1(0) \\ p(T) = y_1(T) - z_1(T) \end{aligned}\right\} \Rightarrow p(0) = p(T),$$

$$\left.\begin{aligned} p'(0) = y_1'(0) - z_1'(0) \\ p'(T) = y_1'(T) - z_1'(T) \end{aligned}\right\} \Rightarrow p'(0) \geq p'(T).$$

Still by Lemma 2.2, $p \leq 0$, which implies $y_1(t) \leq z_1(t)$, $t \in J$ as required.

Using mathematical induction, we can show that

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t),$$

for $t \in J$ and $n = 1, 2, \ldots$. Employing a standard argument, we have

$$\lim_{n\to\infty} y_n(t) = y(t), \qquad \lim_{n\to\infty} z_n(t) = z(t)$$

uniformly on $t \in J$, and the limit functions y, z satisfy problem (1.1). Moreover, $y, z \in [y_0, z_0]$.

In the next step we will show that y is the minimal solution and z is the maximal solution of (1.1). To prove it we assume that u is any solution of problem (1.1) such that $u \in [y_0, z_0]$. Let $y_{n-1}(t) \leq u(t) \leq z_{n-1}(t)$, $t \in J$, for some positive integer n. Put $v = y_n - u$. Then

$$\begin{aligned}
v''(t) &= -Fy_{n-1}(t) + K(t)[y_n'(t) - y_{n-1}'(t)] + M(t)[y_n(t) - y_{n-1}(t)] \\
&\quad + N(t)[y_n(\alpha(t)) - y_{n-1}(\alpha(t))] + Fu(t)
\end{aligned} \tag{3.3}$$

or

$$v''(t) \geq K(t)v'(t) + M(t)v(t) + N(t)v(\alpha(t)),$$

and

$$\begin{aligned}\Delta v(t_k) &= \Delta y_n(t_k) - \Delta u(t_k)\\ &= P_k(y_{n-1}(t_k), y'_{n-1}(t_k)) + L_k[y'_n(t_k) - y'_{n-1}(t_k)] - P_k(u(t_k), u'(t_k))\\ &= L_k v'(t_k), \quad k = 1, \dots, m,\end{aligned}$$

$$\begin{aligned}\Delta v'(t_k) &= \Delta y'_n(t_k) - \Delta u'(t_k)\\ &= Q_k(y_{n-1}(t_k), y'_{n-1}(t_k)) + L_k^*[y'_n(t_k) - y'_{n-1}(t_k)] - Q_k(u(t_k), u'(t_k))\\ &\geq L_k^* v'(t_k), \quad k = 1, \dots, m,\end{aligned}$$

$$\left.\begin{aligned}v(0) = y_n(0) - u(0)\\ v(T) = y_n(T) - u(T)\end{aligned}\right\} \Rightarrow v(0) = v(T),$$

$$\left.\begin{aligned}v'(0) = y'_n(0) - u'(0)\\ v'(T) = y'_n(T) - u'(T)\end{aligned}\right\} \Rightarrow v'(0) \geq v'(T).$$

Hence, $y_n \leq u$, $t \in J$, by Lemma 2.2. Similarly like the above, we can show that $u(t) \leq z_n(t)$, $t \in J$. This yields $y_n(t) \leq u(t) \leq z_n(t)$, $t \in J$.

Finally, if $n \to \infty$, then

$$y_0(t) \leq y(t) \leq u(t) \leq z(t) \leq z_0(t), \ t \in J.$$

This completes the proof. □

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