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Research Article

Morphisms Between Grassmannian Complex and Higher Order Tangent Complex

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Abstract. In this article we extend the notion of tangent complex to higher order and propose morphisms between Grassmannian subcomplex and the tangent dialogarithmic complex for a general order. Moreover, we connect both these complexes and prove the commutativity of resulting diagram. The interesting point is the reappearance of classical Newton's Identities here in a completely different context to the one he had.

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1. Introduction

Suslin was the first who introduced the Grassmannian complex and Bloch-Suslin complex [13]. Later, Goncharov used geometric configurations in order to define the motivic complexes (see [5–8]). Cathelineau [1,2] and Gangle [4] studied variants (Infinitesimal and Tangential) of these motivic complexes and gave an expected form of tangent groups. Siddiqui in [10, 11] defined the tangent groups $T\mathscr{B}_2(F)$ and $T\mathscr{B}_3(F)$ and its complexes for the first order. He used geometric configurations to construct cross ratio, triple-ratio and Siegel cross-ratio identity and then proposed various morphisms between Grassmannian complex and first order tangent complex. In [9] we extended his work to second order by defining second order tangent groups $T\mathscr{B}_2^2(F)$ and

$T\mathscr{B}_3^2(F)$ and other ingredients like cross ratio, Goncharov's triple-ratio and Siegel's cross ratio identity. Using these groups we formed tangent complexes for weight two and three and then connected these complexes to the Grassmannian complex. We also proved the commutativity of resulting diagrams.

In this work we are going to extend this notion to a general order "n". For this we define n-th order tangent group $T\mathscr{B}_2^n(F)$ of weight 2 along with its functional equations. We define a map ∂_{ε^n} to construct the following tangent complex of general order

$$\partial_{\varepsilon^n} : T\mathscr{B}_2^n(F) \longrightarrow F \otimes F^\times \oplus \bigwedge^2 F\,.$$

Using the results from [12] we determine the coefficients of the cross ratio, Siegel's cross-ratio identity and determinants of order n. After these constructions we move to find morphisms of connection π^2_{0,ε^n} and π^2_{1,ε^n} between Grassmannian complex and Cathelineau's tangent complex. We show both the maps π^2_{0,ε^n} and π^2_{1,ε^n} are well defined. At last we prove that the diagram (5.1d) is commutative.

2. Newton-Girard Identities

Girard was the first who established in 1629 some identities between the coefficients of polynomial and its roots (see [3]). Later, Newton reformed these relations and proposed a general form which was a recursive formula, i.e., suppose $f(y)$ be a polynomial like

$$f(y) = y^n + c_1 y^{n-1} + c_2 y^{n-2} + c_3 y^{n-3} + \ldots + c_{n-1} y + c_n$$

with "n" of its roots $r_1, r_2, \ldots r_n$. We use the notation "δ_k' for the sum of k-th powers of roots as

$$\delta_k = r_1^k + r_2^k + \ldots + r_n^k \quad \text{where } k \in \mathbb{Z}^+ \text{ and } \delta_k = 0 \text{ for } k > n\,.$$

Then for every $k > 0$ we obtain an identity in recursive form which we call Newton's identity

$$\delta_k + c_1\delta_{k-1} + c_2\delta_{k-2} + s_3 c_{k-3} + \ldots + c_{k-1}\delta_1 + k c_k = 0\,.$$

This identity allows us to deduce relations below

$$\left.\begin{aligned} &\delta_1 + c_1 = 0\,,\\ &\delta_2 + c_1\delta_1 + 2c_2 = 0\,,\\ &\delta_3 + c_1\delta_2 + c_2\delta_1 + 3c_3 = 0\,,\\ &\delta_4 + c_1\delta_3 + c_2\delta_2 + c_3\delta_1 + 4c_4 = 0\,. \end{aligned}\right\} \tag{1}$$

By considering most generalized form of the polynomial

$$f(y) = \sum_{i=0}^{n} t_i y^i\,.$$

Now assume that $t_k = 0$ for $k < 0$, we define δ_k for $k \geqslant 0$ as

$$\delta_k = r_1^k + r_2^k + \ldots + r_n^k\,.$$

By interchanging "k" to "$(-k)$" we obtain

$$\delta_{-k} = r_1^{-k} + r_2^{-k} + \ldots + r_n^{-k}\,.$$

Finally, we can conclude the general form of Newton's identity as

$$t_j(n-j)+t_{j+1}\delta_{-1}+t_{j+2}\delta_{-2}+t_{j+3}\delta_{-3}+\ldots+t_n\delta_{j-n}=0; \quad j\leqslant n. \tag{2}$$

Furthermore, we can deduct the following results

$$\left.\begin{aligned} M_1&=\frac{t_1}{s},\\ M_2&=\frac{2t_2}{s}-\frac{t_1^2}{s^2},\\ M_3&=\frac{3t_3}{s}-\frac{3t_1t_2}{s^2}+\frac{t_1^3}{s^3},\\ M_4&=\frac{4t_4}{s}-\frac{4t_1t_3}{s^2}-\frac{2t_2^2}{s^2}+\frac{4t_1^2t_2}{s^3}-\frac{t_1^4}{s^4}. \end{aligned}\right\} \tag{3}$$

In general notation

$$M_n=\frac{nt_n}{s}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{s}M_r \tag{4}$$

here, we used $M_i=-\delta_{-k}\ \forall\ i=0,1,2,\ldots$ and $t_0=s$.

When we consider the case $t_0=1-s$ the above identities will become

$$\left.\begin{aligned} N_1&=\frac{-t_1}{s-1},\\ N_2&=\frac{-2t_2}{s-1}-\frac{t_1^2}{(s-1)^2},\\ N_3&=\frac{-3t_3}{s-1}-\frac{3t_1t_2}{(s-1)^2}-\frac{t_1^3}{(s-1)^3},\\ N_4&=\frac{-4t_4}{s-1}-\frac{4t_1t_3}{(s-1)^2}-\frac{2t_2^2}{(s-1)^2}-\frac{4t_1^2t_2}{(s-1)^3}-\frac{t_1^4}{(s-1)^4} \end{aligned}\right\} \tag{5}$$

the general form will be

$$N_n=\frac{nt_n}{1-s}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{1-s}N_r. \tag{6}$$

3. Tangent Group of Order 3 in Weight 2

Let $\mathbb{F}[\varepsilon]_4$ be a truncated polynomial ring over an arbitrary field $\mathbb{F}$ then we call the $\mathbb{Z}$-module $T\mathscr{B}_2^3(\mathbb{F})$ a tangent group of order 3 if it is generated by $\langle s;s',s'',s'''] \in \mathbb{Z}[\mathbb{F}[\varepsilon]_4]$ and quotient by the expression

$$\langle s;s',s'',s'''] - \langle t;t',t'',t'''] + \left\langle \frac{t}{s};\left(\frac{t}{s}\right)',\left(\frac{t}{s}\right)'',\left(\frac{t}{s}\right)'''\right] - \left\langle \frac{1-t}{1-s};\left(\frac{1-t}{1-s}\right)',\left(\frac{1-t}{1-s}\right)'',\left(\frac{1-t}{1-s}\right)'''\right]$$
$$+\left\langle \frac{s(1-t)}{t(1-s)};\left(\frac{s(1-t)}{t(1-s)}\right)',\left(\frac{s(1-t)}{t(1-s)}\right)'',\left(\frac{s(1-t)}{t(1-s)}\right)'''\right], \quad s,t\neq 0,1, s\neq t, \tag{7}$$

where $\langle s;s',s''s'''] = [s+s'\varepsilon+s''\varepsilon^2+s''\varepsilon^3]-[s]$ and $s,s',s'',s''' \in \mathbb{F}$. The expressions $\left(\frac{t}{s}\right)'$, $\left(\frac{t}{s}\right)''$, $\left(\frac{1-t}{1-s}\right)'$, $\left(\frac{1-t}{1-s}\right)''$, $\left(\frac{s(1-t)}{t(1-s)}\right)'$ and $\left(\frac{s(1-t)}{t(1-s)}\right)''$ are defined in [9] and [11]. Others are given below

$$\left(\frac{t}{s}\right)'''=\frac{t'''}{s}-\frac{s'}{s}\left(\frac{t}{s}\right)''-\frac{s''}{s}\left(\frac{s}{s}\right)'-\frac{s'''}{s}\left(\frac{t}{s}\right),$$

$$\left(\frac{1-t}{1-s}\right)''' = \frac{s'''}{(1-s)}\left(\frac{1-t}{1-s}\right) + \frac{s''}{(1-s)}\left(\frac{1-t}{1-s}\right)' + \frac{s'}{(1-s)}\left(\frac{1-t}{1-s}\right)'' - \frac{s'''}{(1-s)},$$

$$\left(\frac{s(1-t)}{t(1-s)}\right)'' = \frac{B}{s^3(1-t)^3},$$

where

$$B = (t')^2 s^3 - tt''s^3 + 2tt''s^2 - 2(t')^2 s^2 - tt''s + (t')^2 s + tst's' - ts't' + t^3 ss'' - t^3(s')^2 - t^3 s'' - t^2 ss'' + t^2(s')^2 + t^2 s''$$

Consider the diagram

$$\begin{array}{ccccc} \mathbb{C}_5(\mathbb{A}^2_{\mathbb{F}[\varepsilon]_4}) & \xrightarrow{d} & \mathbb{C}_4(\mathbb{A}^2_{\mathbb{F}[\varepsilon]_4}) & \xrightarrow{d} & \mathbb{C}_3(\mathbb{A}^2_{\mathbb{F}[\varepsilon]_4}) \\ & & \downarrow \pi^2_{1,\varepsilon^3} & & \downarrow \pi^2_{0,\varepsilon^3} \\ & & T\mathscr{B}^3_2(\mathbb{F}) & \xrightarrow{\partial_{\varepsilon^3}} & \mathbb{F}\otimes\mathbb{F}^\times \oplus \bigwedge^2 \mathbb{F} \end{array} \tag{F}$$

here ∂_{ε^3} is a map which behaves like

$$\partial_{\varepsilon^3}\left(\langle s;t_1,t_2,t_3]^3_2\right) = \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\}\otimes(1-s) + \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\}\otimes s$$
$$+ \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \wedge \left\{\frac{3t_3}{1-s} - \left(\frac{3t_1t_2}{(1-s)^2} - \frac{t_1^3}{(1-s)^3}\right)\right\}, \tag{8}$$

where

$$\langle s;t_1,t_2,t_3]^3_2 \in T\mathscr{B}^3_2(\mathbb{F}); \quad s,t_1,t_2,t_3 \in \mathbb{F};\ s \neq 0,1.$$

To minimize the complication we express the map π^2_{0,ε^3} as $\pi^2_{0,\varepsilon^3} = \pi^1 + \pi^2$

$$\pi^1(\mathbb{V}^*_{02}) = \sum_{i=0}^{2}(-1)^i\left\{\left(3\left(\frac{(v_i^*v_{i+1}^*)_{\varepsilon^3}}{(v_iv_{i+1})} - \frac{(v_i^*v_{i+1}^*)_{\varepsilon^2}(v_i^*v_{i+1}^*)_\epsilon}{(v_iv_{i+1})^2}\right) + \frac{{(v_i^*v_{i+1}^*)^3}_\epsilon}{(v_iv_{i+1})^3}\right)\otimes\frac{v_iv_{i+2})}{(v_{i+1}v_{i+2})}\right\};\ i \bmod 3, \tag{9}$$

$$\pi^2(\mathbb{V}^*_{02}) = \sum_{i=0}^{2}(-1)^i\left\{3\frac{(v_i^*v_{i+1}^*)_{\varepsilon^3}}{(v_iv_{i+1})} - 3\frac{(v_i^*v_{i+1}^*)_{\varepsilon^2}(v_i^*v_{i+1}^*)_\epsilon}{(v_iv_{i+1})^2} + \frac{{(v_i^*v_{i+1}^*)^3}_\epsilon}{(v_iv_{i+1})^3}\right.$$
$$\left.\wedge\left(3\frac{(v_i^*v_{i+2}^*)_{\varepsilon^3}}{(v_iv_{i+2})} - 3\frac{(v_i^*v_{i+2}^*)_{\varepsilon^2}(v_i^*v_{i+2}^*)_\epsilon}{(v_iv_{i+2})^2} + \frac{{(v_i^*v_{i+2}^*)^3}_\epsilon}{(v_iv_{i+2})^3}\right)\right\};\ i \bmod 3, \tag{10}$$

$$\pi^2_{1,\varepsilon^3}(\mathbb{V}^*_{03}) = \langle r(v_0,\dots,v_3); r_\epsilon(\mathbb{V}^*_{03}), r_{\varepsilon^2}(\mathbb{V}^*_{03}),, r_{\varepsilon^3}(\mathbb{V}^*_{03})]^3_2, \tag{11}$$

where $\mathbb{V}^*_{0m} = (v_0^*,\dots,v_m^*)$.

Proposition 3.1. *Commutation holds for the diagram (F) of complexes.i.e.*

$$\partial^2_{\varepsilon^3}\circ\pi^2_{1,\varepsilon^3} = \pi^2_{0,\varepsilon^3}\circ d.$$

Proof. Since all maps are already defined so we need direct calculations which gives the conclusion below

$$\partial^2_{\varepsilon^3}\circ\pi^2_{1,\varepsilon^3}(\mathbb{V}^*_{03}) = \left\{3\left(\frac{(v_o^*v_3^*)_{\varepsilon^3}}{(v_0v_3)} + \frac{(v_1^*v_2^*)_{\varepsilon^3}}{(v_1v_2)} - \frac{(v_0^*v_2^*)_{\varepsilon^3}}{(v_0v_2)} - \frac{(v_1^*v_3^*)_{\varepsilon^3}}{(v_1v_3)} + \frac{(v_o^*v_2^*)_\epsilon}{(v_0v_2)}\frac{(v_o^*v_2^*)_{\varepsilon^2}}{(v_0v_2)}\right.\right.$$
$$\left. + \frac{(v_1^*v_3^*)_\epsilon}{(v_1v_3)}\frac{(v_1^*v_3^*)_{\varepsilon^2}}{(v_1v_3)} - \frac{(v_o^*v_3^*)_\epsilon}{(v_0v_3)}\frac{(v_o^*v_3^*)_{\varepsilon^2}}{(v_0v_3)} - \frac{(v_1^*v_2^*)_\epsilon}{(v_1v_2)}\frac{(v_1^*v_2^*)_{\varepsilon^2}}{(v_1v_2)}\right)$$

$$+\frac{(v_o^*v_3^*)^3_\varepsilon}{(v_0v_3)^3}+\frac{(v_1^*v_2^*)^3_\varepsilon}{(v_1v_2)^3}-\frac{(v_0^*v_2^*)^3_\varepsilon}{(v_0v_2)^3}-\frac{(v_1^*v_3^*)^3_\varepsilon}{(v_1v_3)^3}\Bigg\}\otimes\frac{(v_0,v_1)(v_2,v_3)}{(v_0,v_2)(v_1,v_3)}$$
$$+\Bigg\{3\Bigg(\frac{(v_0^*v_2^*)_{\varepsilon^3}}{(v_0v_2)}+\frac{(v_1^*v_3^*)_{\varepsilon^3}}{(v_1v_3)}-\frac{(v_0^*v_1^*)_{\varepsilon^3}}{(v_0v_1)}-\frac{(v_2^*v_3^*)_{\varepsilon^3}}{(v_2v_3)}-\frac{(v_0^*v_2^*)_{\varepsilon^2}}{(v_0v_2)}\frac{(v_0^*v_2^*)_{\varepsilon}}{(v_0v_2)}$$
$$-\frac{(v_1^*v_3^*)_{\varepsilon^2}}{(v_1v_3)}\frac{(v_1^*v_3^*)_{\varepsilon}}{(v_1v_3)}+\frac{(v_0^*v_1^*)_{\varepsilon^2}}{(v_0v_1)}\frac{(v_0^*v_1^*)_{\varepsilon}}{(v_0v_1)}+\frac{(v_2^*v_3^*)_{\varepsilon^2}}{(v_2v_3)}\frac{(v_2^*v_3^*)_{\varepsilon}}{(v_2v_3)}\Bigg)$$
$$+\frac{(v_0^*v_2^*)^3_\varepsilon}{(v_0v_2)^3}+\frac{(v_1^*v_3^*)^3_\varepsilon}{(v_1v_3)^3}-\frac{(v_0^*v_1^*)^3_\varepsilon}{v_0v_1)^3}-\frac{(v_2^*v_3^*)^3_\varepsilon}{(v_2v_3)^3}\Bigg\}\otimes\frac{(v_0,v_3)(v_1,v_2)}{(v_0,v_2)(v_1,v_3)}. \tag{12}$$

Next, we move to evaluate the other side $\pi^2_{0,\varepsilon^3}\circ d(\mathbb{V}^*_{03})$. Since, we have

$$\pi^2_{0,\varepsilon^3}\circ d(\mathbb{V}^*_{03})=\pi^1\circ d(\mathbb{V}^*_{03})+\pi^2\circ d(\mathbb{V}^*_{03}). \tag{13}$$

Applying, the definitions ofπ^2, π^2 and d

$$\pi^1\circ d(\mathbb{V}^*_{03})=\widetilde{\text{Alt}}_{(0123)}\Bigg\{\sum_{i=0}^{2}(-1)^i\Bigg\{\Bigg(3\frac{(v_i^*,v_{i+1}^*)_{\varepsilon^3}}{(v_i,v_{i+1})}-3\frac{(v_i^*,v*{i+1})_{\varepsilon^2}(v_i^*,v_{i+1}^*)_\varepsilon}{(v_i,v_{i+1})^2}+\frac{(v_i^*,v_{i+1}^*)^3{}_\varepsilon}{(v_i,v_{i+1})^3}\Bigg)$$
$$\otimes\frac{(v_i,v_{i+2})}{(v_{i+1},v_{i+2})}\Bigg\}\Bigg\};\quad i \bmod 3. \tag{14}$$

Furthermore, we use the facts $p\otimes\frac{q}{r}=p\otimes q-p\otimes r$ in the expansion of inner sum. This gives us total 18 terms which can further be classified into the terms like $\frac{(u)_{\varepsilon^3}}{u}\otimes v$, $\frac{(u)_{\varepsilon^2}(u)_\varepsilon}{u^2}\otimes v$ and $\frac{(u)^3_\varepsilon}{u^3}\otimes v$. Now, if we expand through the sum of alternation then total number of terms will be raised up to ninety. After cancellations and simplifications we acquire an expression identical with (12). □

3.1 Generalized Tangential Dialogarithmic Group

Let $F[\varepsilon]_4$ be a truncated polynomial ring over an arbitrary field F then the $\mathbb{Z}$-module $T\mathscr{B}_2^n(F)$ is called tangent group of order "n" if it is generated by $\langle s;s_\varepsilon,s_{\varepsilon^2},\dots,s_{\varepsilon^n}]_2^n\in\mathbb{Z}[F[\varepsilon]_{n+1}]$ and quotient by the expression

$$\langle s;s_\varepsilon,s_{\varepsilon^2},\dots,s_{\varepsilon^n}]-\langle t;t_\varepsilon,t_{\varepsilon^2},\dots,t_{\varepsilon^n}]+\Bigg\langle\frac{t}{s};\Big(\frac{t}{s}\Big)_\varepsilon,\Big(\frac{t}{s}\Big)_{\varepsilon^2},\dots,\Big(\frac{t}{s}\Big)_{\varepsilon^n}\Bigg]$$
$$-\Bigg\langle\frac{t-1}{s-1};\Big(\frac{t-1}{s-1}\Big)_\varepsilon,\Big(\frac{t-1}{s-1}\Big)_{\varepsilon^2},\dots,\Big(\frac{t-1}{s-1}\Big)_{\varepsilon^n}\Bigg]$$
$$+\Bigg\langle\frac{s(t-1)}{t(s-1)};\Big(\frac{s(t-1)}{t(s-1)}\Big)_\varepsilon,\Big(\frac{s(t-1)}{t(s-1)}\Big)_{\varepsilon^2},\dots,\Big(\frac{s(t-1)}{t(s-1)}\Big)_{\varepsilon^n}\Bigg],\quad s,t\neq0,1,s\neq t, \tag{15}$$

where

$$\langle s;s_\varepsilon,s_{\varepsilon^2},\dots,s_{\varepsilon^n}]=[s+s_\varepsilon\varepsilon+s_{\varepsilon^2}\varepsilon^2+\ldots+s_{\varepsilon^n}\varepsilon^n]-[s];\quad(s,s_\varepsilon,\dots,s_{\varepsilon^n}\in F),$$
$$\Big(\frac{t}{s}\Big)_{\varepsilon^n}=\frac{t_{\varepsilon^n}}{s}-\frac{s_\varepsilon}{s}\Big(\frac{t}{s}\Big)_{\varepsilon^{(n-1)}}-\frac{s_{\varepsilon^2}}{s}\Big(\frac{t}{s}\Big)_{\varepsilon^{(n-2)}}-\ldots-\frac{s_{\varepsilon^{(n-1)}}}{s}\Big(\frac{t}{s}\Big)_\varepsilon-\frac{s_{\varepsilon^n}}{s}\Big(\frac{t}{s}\Big),$$
$$\Big(\frac{t-1}{s-1}\Big)_{\varepsilon^n}=\frac{s_{\varepsilon^n}}{1-s}\Big(\frac{t-1}{s-1}\Big)+\frac{s_{\varepsilon^{(n-1)}}}{1-s}\Big(\frac{t-1}{s-1}\Big)_\varepsilon+\ldots+\frac{s_\varepsilon}{1-s}\Big(\frac{t-1}{s-1}\Big)_{\varepsilon^{(n-1)}}-\frac{t_{\varepsilon^n}}{1-s},$$
$$\Big(\frac{s(t-1)}{t(s-1)}\Big)_{\varepsilon^n}=\frac{s_{\varepsilon^n}}{t(1-s)}\Big(\frac{s(t-1)}{t(s-1)}\Big)+\frac{s_{\varepsilon^{n-1}}}{t(1-s)}\Big(\frac{s(t-1)}{t(s-1)}\Big)_\varepsilon+\ldots+\frac{s_\varepsilon}{t(1-s)}\Big(\frac{s(1-t)}{t(1-s)}\Big)_{\varepsilon^{(n-1)}}-\frac{t_{\varepsilon^n}}{t(1-s)}.$$

For $n = 1,2$, we obtain the groups $T\mathscr{B}_2(F)$ and $T\mathscr{B}_2^2(F)$ which are discussed in [9, 12], where using these two groups, the tangent or Cathelineau Complexes are formed and are connected with Grassmannian complex.

4. Tangential Analogue of n-th order Complex

We have described the group $T\mathscr{B}_2^n(F)$ earlier and now we propose a map ∂_{ε^n} in order to establish dialogarithmic tangent complex of order "n" i.e.

$$\partial_{\varepsilon^n} : T\mathscr{B}_2^n(F) \longrightarrow F \otimes F^\times \oplus \bigwedge^2 F.$$

The map ∂_{ε^n} has already been defined for $n = 1,2$ as

$$\partial_\varepsilon : \langle s;t]_2 \mapsto \frac{t}{s} \otimes (1-s) + \frac{t}{1-s} \otimes s + \frac{t}{s} \wedge \frac{t}{1-s}; \quad s,t \in F,$$

$$\partial_{\varepsilon^2} : \langle s;t_1,t_2]_2^2 \mapsto \left(\frac{2t_2}{s} - \frac{t_1^2}{s^2}\right) \otimes (1-s) + \left(\frac{2t_2}{(1-s)} + \frac{t_1^2}{(1-s)^2}\right) \otimes s + \left(\frac{2t_2}{s} - \frac{t_1^2}{s^2}\right) \wedge \left(\frac{2t_2}{(1-s)} + \frac{t_1^2}{(1-s)^2}\right); \quad s,t_1,t_2 \in F,$$

(see [11] and [9] for details). From (8), we have

$$\partial_{\varepsilon^3}\left(\langle s;t_1,t_2,t_3]_2^3\right) = \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \otimes (1-s) + \left\{\frac{-3t_3}{s-1} - \left(\frac{3t_1t_2}{(s-1)^2} + \frac{t_1^3}{(s-1)^3}\right)\right\} \otimes s) + \left\{\frac{3t_3}{s} - \left(\frac{3t_1t_2}{s^2} - \frac{t_1^3}{s^3}\right)\right\} \wedge \left\{\frac{-3t_3}{s-1} - \left(\frac{3t_1t_2}{(s-1)^2} + \frac{t_1^3}{(s-1)^3}\right)\right\}. \tag{16}$$

Here, we use the results of (2), to rebuild these maps. So, we write

$$\partial_\varepsilon : \langle s;t_1]_2 \mapsto M_1 \otimes (1-s) + N_1 \otimes s + M_1 \wedge N_1,$$
$$\partial_{\varepsilon^2} : \langle s;t_1,t_2]_2^2 \mapsto M_2 \otimes (1-s) + N_2 \otimes s + M_2 \wedge N_2,$$
$$\partial_{\varepsilon^3} : \langle s;t_1,t_2,t_3]_2^3 \mapsto M_3 \otimes (1-s) + N_3 \otimes s + M_3 \wedge N_3,$$

the pattern above allows us to propose a similar morphism for a general order n.

$$\partial_{\varepsilon^n} : \langle s;t_1,t_2,\dots t_n]_2^n \mapsto M_n \otimes (1-s) + N_n \otimes s + M_n \wedge N_n \tag{17}$$

where M_n and N_n are defined in (4) and (6), respectively. Our aim is to connect this analogue of Grassmannian sub-complex and the n-th order complex in tangential settings. The result of connection of both complexes gives the diagram below.

$$\begin{array}{ccccc} C_5(\mathbb{A}^2_{F[\varepsilon]_{n+1}}) & \xrightarrow{d} & C_4(\mathbb{A}^2_{F[\varepsilon]_{n+1}}) & \xrightarrow{d} & C_3(\mathbb{A}^2_{F[\varepsilon]_{n+1}}) \\ & & \downarrow \pi^2_{1,\varepsilon^n} & & \downarrow \pi^2_{0,\varepsilon^n} \\ & & T\mathscr{B}_2^n(F) & \xrightarrow{\partial_{\varepsilon^n}} & F \otimes F^\times \oplus \bigwedge^2 F \end{array} \tag{5.1d}$$

For $n = 1,2$, the map π^2_{0,ε^n} is given in [9] but here we describe maps $\pi^2_{0,\varepsilon}$ and π^2_{0,ε^2} in a different fashion that is in terms of Newton's relations. This enables us to write this map for the higher values of n.

So, we use the notations $\mathbb{V}^*_{0n} = (v^*_0, \dots, v^*_n)$; $a_{ij} = (v^*_i, v^*_{i+1})_{\varepsilon^j}$, $b_{ij} = (v^*_i, v^*_{i+2})$ and $z_i = (v^*_{i+1}, v^*_{i+2})$ $\forall\ 1 \le i, j \le n$. Consider the polynomials

$$g(t) = \sum_{l=0}^{n} a_{il} t^l \ ; \ \ h(t) = \sum_{l=0}^{n} b_{il} t^l ,$$

$$g(t) = a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3 + \ldots + a_{in}t^n ,$$

$$h(t) = b_{i0} + b_{i1}t + b_{i2}t^2 + b_{i3}t^3 + \ldots + b_{in}t^n ,$$

then the Newton's identities for these polynomial will be

$$M_{i1} = \frac{a_{i1}}{a_{i0}},$$

$$M_{i2} = \frac{2a_{i2}}{a_{i0}} - \frac{a^2_{i1}}{a^2_{i0}},$$

$$M_{i3} = \frac{3a_{i3}}{a_{i0}} - \frac{3a_{i1}a_{i2}}{a^2_{i0}} + \frac{a^3_{i1}}{a^3_{i0}},$$

$$M_{i4} = \frac{4a_{i4}}{a_{i0}} - \frac{4a_{i1}a_{i3}}{a^2_{i0}} - \frac{2a^2_{i2}}{a^2_{i0}} + \frac{4a^2_{i1}a_{i2}}{a^3_{i0}} - \frac{a^4_{i1}}{a^4_{i0}},$$

these relations can be generalized recursively as under

$$M_{in} = \frac{na_{in}}{a_{io}} - \sum_{p=1}^{n-1} \frac{a_{i(n-r)}}{a_{i0}} M_{ip} . \tag{18}$$

Similarly, for the polynomial $h(t)$

$$N_{i1} = \frac{b_{i1}}{b_{i0}},$$

$$N_{i2} = \frac{2b_{i2}}{b_{i0}} - \frac{b^2_{i1}}{b^2_{i0}},$$

$$N_{i3} = \frac{3b_{i3}}{b_{i0}} - \frac{3b_{i1}b_{i2}}{b^2_{i0}} + \frac{b^3_{i1}}{b^3_{i0}},$$

$$N_{i4} = \frac{4b_{i4}}{b_{i0}} - \frac{4b_{i1}b_{i3}}{b^2_{i0}} - \frac{2b^2_{i2}}{b^2_{i0}} + \frac{4b^2_{i1}b_{i2}}{b^3_{i0}} - \frac{b^4_{i1}}{b^4_{i0}}$$

with general term

$$N_{in} = \frac{nb_{in}}{b_{io}} - \sum_{r=1}^{n-1} \frac{b_{i(n-r)}}{b_{i0}} N_{ir} . \tag{19}$$

Now, using above relations we have

$$\pi^2_{0,\varepsilon}(\mathbb{V}^*_{02}) = \sum_{i=0}^{2} (-1)^i \left(M_{i1} \otimes \frac{b_{i0}}{z_{i0}} + M_{i1} \wedge N_{i1} \right), \ \ i \bmod 3,$$

$$\pi^2_{0,\varepsilon^2}(\mathbb{V}^*_{02}) = \sum_{i=0}^{2} (-1)^i \left(M_{i2} \otimes \frac{b_{i0}}{z_{i0}} + M_{i2} \wedge N_{i2} \right), \ \ i \bmod 3 .$$

Hence for order "n" one can write as

$$\pi^2_{0,\varepsilon^n}(\mathbb{V}^*_{02}) = \sum_{i=0}^{2}(-1)^i\left(M_{in}\otimes\frac{b_{i0}}{z_{i0}} + M_{in}\wedge N_{in}\right),\quad i \bmod 3. \tag{20}$$

Now, we come to define π^2_{1,ε^n}. For $n=1,2$ it is already defined in [9] and see (11) for $n=3$ then we can propose such map for a general number.

$$\pi^2_{1,\varepsilon^n}(\mathbb{V}^*_{03}) = \langle r(\mathbb{V}_{03}); r_\epsilon(\mathbb{V}^*_{03}), r_{\epsilon^2}(\mathbb{V}^*_{03}),\dots,r_{\epsilon^n}(\mathbb{V}^*_{03})]^n_2. \tag{21}$$

Lemma 4.1. *The map π^2_{0,ε^n} is free of the choice of volume form Ω.*

Proof. From (20), we have

$$\pi^2_{0,\varepsilon^n}(v^*_0,v^*_1,v^*_2) = \sum_{i=0}^{2}(-1)^i\left(M_{in}\otimes\frac{b_{i0}}{z_{i0}} + M_{in}\wedge N_{in}\right),\quad i \bmod 3, \tag{22}$$

where $M_{in} = \frac{na_{in}}{x_{io}} - \sum_{r=1}^{n-1}\frac{a_{i(n-r)}}{a_{i0}}M_{ir}$ and $N_{in} = \frac{na_{in}}{b_{io}} - \sum_{r=1}^{n-1}\frac{b_{i(n-r)}}{b_{i0}}N_{ir}$. Clearly, the map τ^2_{0,ε^n} consists of two different expressions of the form $M_{in}\otimes\frac{b_{i0}}{z_{i0}}$ and $M_{in}\wedge N_{in}$. Since value of the expression $M_{in}\otimes\frac{b_{i0}}{z_{i0}}$ will remain unchanged if we interchange Ω with $\lambda\Omega$ (see [7, Proposition 3.7]) and the same result holds for $M_{in}\wedge N_{in}$. Hence π^2_{0,ε^n} has no dependence on Ω. □

Lemma 4.2. *$\pi^2_{0,\varepsilon^n}\circ d(\mathbb{V}^*_{03})$ is free of the size of vectors $v_i \in V_2$.*

Proof. Using definitions of τ^2_{0,ε^n} and d when we evaluate the composition $\tau^2_{0,\varepsilon^n}\circ d(\mathbb{V}^*_{03})$ we obtain the combination of homogeneous ratios of determinants like $\frac{\Delta(v^*_i,v^*_j)_{\varepsilon^n}}{\Delta(v_i,v_j)}$. As we have the property

$$\frac{\{\lambda\Delta(v^*_i,v^*_j)\}_{\varepsilon^n}}{\{\lambda\Delta(v_i,v_j)\}} = \frac{\lambda\Delta(v^*_i,v^*_j)_{\varepsilon^n}}{\lambda\Delta(v_i,v_j)} = \frac{\Delta(v^*_i,v^*_j)_{\varepsilon^n}}{\Delta(v_i,v_j)}$$

which ensures the validity of required result. □

Proposition 4.3. *Commutativity holds for the diagram* (5.1d).

$$\partial_{\varepsilon^n}\circ\pi^2_{1,\varepsilon^n} = \pi^2_{0,\varepsilon^n}\circ d.$$

Proof. Chose a tuple $(\mathbb{V}^*_{03}) \in C_4\left(\mathbb{A}^2_{F[\varepsilon]_{n+1}}\right)$, then definition (21) gives us

$$\pi^2_{1,\varepsilon^n}(\mathbb{V}^*_{03}) = \langle r(v_0,\dots,v_3); r_\epsilon(\mathbb{V}^*_{03}), r_{\epsilon^2}(\mathbb{V}^*_{03}),\dots,r_{\epsilon^n}(\mathbb{V}^*_{03})]^n_2$$

employing the description of ∂_{ε^n} given in (17)

$$\begin{aligned}\partial_{\varepsilon^n}\circ\pi^2_{1,\varepsilon^n}(\mathbb{V}^*_{03}) &= \left(\frac{nt_n}{s} - \sum_{r=1}^{n-1}\frac{t_{n-r}}{s}M_r\right)\otimes(1-s) + \left(\frac{nt_n}{1-s} - \sum_{r=1}^{n-1}\frac{t_{n-r}}{1-s}N_r\right)\otimes s\\ &\quad + \left(\frac{nt_n}{s} - \sum_{r=1}^{n-1}\frac{t_{n-r}}{s}M_r\right)\wedge\left(\frac{nt_n}{1-s} - \sum_{r=1}^{n-1}\frac{t_{n-r}}{1-s}N_r\right),\end{aligned} \tag{23}$$

where $s = r(\mathbb{V}_{03})$; $t_1 = r_\epsilon(\mathbb{V}^*_{03})$; $t_2 = r^2_\epsilon(\mathbb{V}^*_{03})$ up to $t_n = r_{\epsilon^n}(\mathbb{V}^*_{03})$.

The phrases $\frac{nt_n}{s} - \sum_{r=1}^{n-1}\frac{t_{n-r}}{s}M_r$ and $\frac{nt_n}{1-s} - \sum_{r=1}^{n-1}\frac{t_{n-r}}{1-s}N_r$ contains expressions like $(\alpha)_{\varepsilon^n}$, $(\alpha)_{\varepsilon^i}(\alpha)_{\varepsilon^j}$; $i+j=n$, $(\alpha)_{\varepsilon^i}(\alpha)_{\varepsilon^j}(\alpha)_{\varepsilon^k}$, $i+j+k=n$, … .

After tensoring with $s = \frac{(v_0,v_3)(v_1,v_2)}{(v_0,v_2)(v_1,v_3)}$, we get terms of the form $(a)_{\varepsilon^n} \otimes (k,l)$, $(a)_{\varepsilon^i}(a)_{\varepsilon^j} \otimes (k,l)$; $i+j=n$, $(a)_{\varepsilon^i}(a)_{\varepsilon^j}(a)_{\varepsilon^k} \otimes (k,l)$, $i+j+k=n$ and so on. The coefficients of these terms will occur according to the pattern of Newton identities.

To find value of right hand side $\pi^2_{0,\varepsilon^n} \circ d$ we use the formula (20)

$$\pi^2_{0,\varepsilon^n} \circ d(\mathbb{V}^*_{03}) = \widetilde{\mathrm{Alt}}_{(0123)} \left(\sum_{i=0}^{2} (-1)^i \left(M_{in} \otimes \frac{y_{i0}}{z_{i0}} + M_{in} \wedge N_{in} \right) \right), \quad i \bmod 3, \tag{24}$$

where M_{in} and N_{in} are defined earlier. First we expand the inner sum which gives us terms of the type $M_{ij} \otimes a$ and $M_i j \wedge M_{kl}$, where M_{ij} and M_{kl} are Newton's identities which are defined in (18) and (19). By substituting these values in (24) and applying the alternation sum we only need a little simplification to achieve the required result. □

5. Conclusion

In this work we have shown that the higher order tangent group $T\mathscr{B}^n_2(F)$ and its defining relations are valid for higher orders. The above results motivate us to compute higher order tangent groups for weight 3. One can find the group $T\mathscr{B}^n_3(F)$ and use it to construct the higher order tangent to Goncharov's complex for weight 3 or even higher.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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