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Characterization of Joined Graphs

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Abstract. *The **join** of simple graphs G_1 and G_2*, written by $G_1 \vee G_2$, is the graph obtained from the disjoint union between G_1 and G_2 by adding the edges $\{xy : x \in V(G_1), y \in V(G_2)\}$. We call a simple graph G as a ***joined graph*** if there are G_1 and G_2 that $G = G_1 \vee G_2$. In this paper, we give conditions to determine that which graphs are joined graphs and use its properties to investigate the chromatic number of joined graphs.

1. Introduction and Preliminaries

In this paper, graphs must be simple graphs which can be trivial graphs but not empty graphs. We follow West [2] for terminologies and notations not defined here. Let G_1 and G_2 be any two graphs. The **join** of graphs G_1 and G_2, written by $G_1 \vee G_2$, is the graph obtained from the disjoint union between G_1 and G_2 by adding the edges $\{xy : x \in V(G_1), y \in V(G_2)\}$.

We call a simple graph G as a ***joined graph*** if there are G_1 and G_2 that $G = G_1 \vee G_2$. Clearly that G_1 and G_2 are subgraphs of $G_1 \vee G_2$. If a graph G is a joined graph of G_1 and G_2, $G = G_1 \vee G_2$, we refer G_1 and G_2 as factors of G.

In generally, we may define $G_1 \vee G_2 \vee G_3$ as $G_1 \vee (G_2 \vee G_3)$. We note here that $G_1 \vee (G_2 \vee G_3) = G_1 \vee G_2 \vee G_3 = (G_1 \vee G_2) \vee G_3$ where G_1, G_2 and G_3 are graphs.

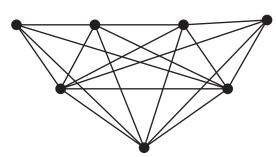

Figure 1. $K_4 \vee K_3 = K_2 \vee K_2 \vee K_3$

Theorem 1.1. *Let G_1 and G_2 be graphs. If H_1 and H_2 are subgraphs of G_1 and G_2, respectively, then $H_1 \vee H_2 \subseteq G_1 \vee G_2$.*

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Proof. Let G_1 and G_2 be graphs. Assume that H_1 and H_2 are subgraphs of G_1 and G_2, respectively. Clearly that $V(H_1 \vee H_2) \subseteq V(G_1 \vee G_2)$. Next, let e be an edge in $H_1 \vee H_2$ with endpoints u and v. If $u, v \in V(H_1)$, then $e \in E(H_1) \subseteq E(G_1) \subseteq E(G_1 \vee G_2)$. Similarly, If $u, v \in V(H_2)$, then $e \in E(H_2) \subseteq E(G_2) \subseteq E(G_1 \vee G_2)$. Suppose that $u \in V(H_1)$ and $v \in V(H_2)$. So $e \in \{uv : u \in V(G_1), v \in V(G_2)\} \subseteq E(G_1 \vee G_2)$. Therefore $H_1 \vee H_2 \subseteq G_1 \vee G_2$. □

In [3], There are Theorems about property of joined graphs as follow

Theorem 1.2. *Any joined graphs are always connected.*

Theorem 1.3. *Any joined graphs are bipartite graphs or contain K_3.*

By applying Theorem 1.2 and Theorem 1.3, we have necessary conditions to be a joined graph as Theorem 1.4.

Theorem 1.4. *Let G be a graph. If G has properties that*

(i) *G is not connected or*
(ii) *G is not a bipartite graph and have no K_3 or*
(iii) *girth of G are not ∞,* 3 *or* 4*,*

then G is not a joined graph.

Because $\overline{K_n}$ where $n \in \mathbb{N}$ is not connected, so $\overline{K_n}$ is not a joined graph. Since girth of C_{2n} where $n \in \mathbb{N}$ and $n > 2$ is $2n$, we have that C_{2n} is not a joined graph. We can conclude that C_4 is the only one bipartite graph that is a joined graph where $C_4 = \overline{K_2} \vee \overline{K_2}$.

We end this section by giving the Theorem about complement of graphs to use in the next section.

Theorem 1.5. *Let G be a graph and let H be a spanning subgraph of G. We have $\overline{G} \subseteq \overline{H}$.*

Proof. Let G be a graph and let H be a spanning subgraph of G. Clearly that $n(\overline{G} = n(\overline{H}$. Let e be an edge in $\overline{G}$ with endpoints u and v. Then $u, v \in V(G) = V(H)$ and u is not adjacent to v in G. Since H is a subgraph of G, we have u is not adjacent to v in H. So $e \in E(H)$. Hence $\overline{G} \subseteq \overline{H}$. □

2. Necessary and Sufficient Conditions

We begin this section by giving the definition of operator + and a relation between + and $\vee$. Let G_1 and G_2 be distinct two graphs. ***The sum of*** G_1 ***and*** G_2, denoted by $G_1 + G_2$, is the graph that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2)$. Clearly that $G_1, G_2 \subseteq G_1 + G_2 \subseteq G_1 \vee G_2$.

Theorem 2.1. *For any graphs G_1 and G_2, $\overline{G_1 \vee G_2} = \overline{G_1} + \overline{G_2}$.*

Proof. Let G_1 and G_2 be graphs. By the definition of the sum of graph, we have $\overline{G_1} + \overline{G_2} \subseteq \overline{G_1 \vee G_2}$. Next, let $e \in E(\overline{G_1 \vee G_2})$ with endpoints u and v. So u and v are not adjacent in $G_1 \vee G_2$. Hence $u, v \in V(G_1)$ or $u, v \in V(G_2)$. Without of loss generality, we may assume that $u, v \in V(G_1)$. Since $G_1 \subseteq G_1 \vee G_2$, we have u and v are not adjacent in G_1. Thus $e \in \overline{G_1} \subseteq \overline{G_1} + \overline{G_2}$. Therefore $\overline{G_1 \vee G_2} = \overline{G_1} + \overline{G_2}$. □

In the previous section, we have only necessary conditions to be a joined graph. We next show the sufficient conditions.

Theorem 2.2. *For any graph G, the following are equivalent(and characterize the joined graph).*

(i) *G is a joined graph.*
(ii) *G have a spanning complete bipartite as a subgraph.*
(iii) *$\overline{G}$ is a disconnected graph.*

Proof. Let G be a graph.

(i)→(ii) Assume that G is a joined graph. Let G_1 and G_2 be graphs that $G = G_1 \vee G_2$. So $n(G_1) + n(G_2) = n(G)$. Let G_i' be a graph obtained by deleting all edges in G_i for all $i = 1, 2$. Then $G_1' \subseteq G_1$ and $G_2' \subseteq G_2$. By Theorem 1.1, we have $G_1' \vee G_2' \subseteq G_1 \vee G_2 = G$ and $n(G_1') + n(G_2') = n(G_1) + n(G_2) = n(G)$. Therefore G have a spanning complete bipartite, $G_1' \vee G_2'$, as a subgraph.

(ii)→(iii) Assume that G have a spanning complete bipartite as a subgraphs, called $H \cong K_{m,n}$ where $m + n = n(G)$. By Theorem 1.5, we have $\overline{G} \subseteq \overline{H} \cong \overline{K_{m,n}}$. Clearly that $\overline{H} \cong \overline{K_{m,n}}$ is disconnected. Therefore $\overline{G}$ is a disconnected graph.

(iii)→(i) We assume that $\overline{G}$ is a disconnected graph. Let H be a connected induce subgraph of $\overline{G}$. So $\overline{G} = H + \overline{G}\backslash H$. By Theorem 2.1, we have that $\overline{\overline{G}} = \overline{H + \overline{G}\backslash H} = \overline{H} \vee \overline{\overline{G}\backslash H}$. Hence $G = \overline{H} \vee \overline{\overline{G}\backslash H}$. Therefore G is a joined graph. □

Corollary 2.3. *Let G be a graph. If $n(G) + e(G) > \frac{n(n-1)}{2} + 1$, then G is a joined graph.*

Proof. Let G be a graph. We assume that $n(G) + e(G) > \frac{n(n-1)}{2} + 1$. We know that $e(\overline{G}) = \frac{n(n-1)}{2} - e(G)$. So $e(\overline{G}) = \frac{n(n-1)}{2} - e(G) < n(G) - 1 = n(\overline{G}) - 1$. Hence $\overline{G}$ is not a connected graph. Therefore G is a joined graph by Theorem 2.2. □

The converse of Corollary 2.3 is not true. For example, K_2 is a joined graph but $n(K_2) + e(K_2) = 3 = \frac{2(1)}{2} + 1$.

Because the complement of the Petersen graph is a connected graph, so we can conclude that the Petersen graph is not a joined graph (see Figure 2).

3. Joined Graphs and It's Chromatic Number

To find the chromatic number of a graph, we use clique number to be a lover bound and find a proper coloring to get a upper bound. Sometime, it's not easy to

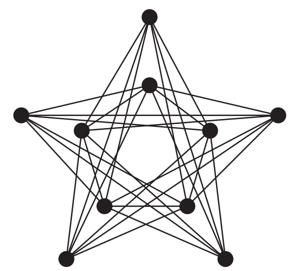

Figure 2. The complement of the Petersen graph

find a clique number for a graph with many edges as Example 3.2, but if we know that a graph is a joined graph, we can find the chromatic number of that graph easier by the next Theorem.

Theorem 3.1. *Let G_1 and G_2 be graphs. Then $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.*

Proof. Let G_1 and G_2 be graphs. Let f and g be proper colorings of G_1 and G_2, respectively. Define $\alpha : V(G_1) \cup V(G_2) \to \{1, 2 \ldots, \chi(G_1) + \chi(G_2)\}$ by for all $v \in G_1 \cup V(G_2)$

$$\alpha(v) = \begin{cases} f(v) & \text{if } v \in V(G_1), \\ \chi(G_1) + g(v) & \text{if } v \in V(G_2). \end{cases}$$

It is easy to see that α is proper. So $\chi(G_1 \vee G_2) \leq \chi(G_1) + \chi(G_2)$. Suppose that $\chi(G_1 \vee G_2) < \chi(G_1) + \chi(G_2)$. There exist $u \in V(G_1)$ and $v \in V(G_2)$ such that $\alpha(u) = \alpha(v)$. So u and v are not adjacent in $G_1 \vee G_2$. This contradicts to the definition of the join graphs. Hence $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$. □

We know that **Wheel** with n vertices, denote by W_n, is a joined graph where $W_n = C_{n-1} \vee K_1$. Since $\chi(C_{n-1}) = 2$ or 3, then by Theorem 3.1 we have that

$$\chi(W_n) = \begin{cases} 3, & \text{if } n \text{ is an even integer,} \\ 4, & \text{if } n \text{ is an odd integer.} \end{cases}$$

Example 3.2. Let G be a graph as Figure 3. We can see that $\overline{G}$ is disconnected. By Theorem 2.2, we have G is a joined graph.

Next, we find factors of G. By following the proof of Theorem 2.2, we get that factors of G are the complement of it's component. So we have factors of G as Figure 4. So $G = H_1 \vee H_2 \vee H_3$ where H_1, H_2 and H_3 are factors of G. Hence $\chi(G) = \chi(H_1) + \chi(H_2) + \chi(H_3) = 2 + 3 + 2 = 7$.

We conclude the results here that a jointed graph is a graph that its complement is disconnected graph and chromatic number of jointed graph is equal to the sum of chromatic number of their factors.

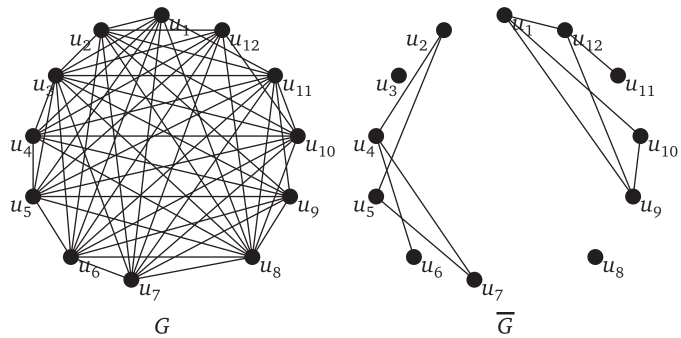

Figure 3. A graph G and it's complement

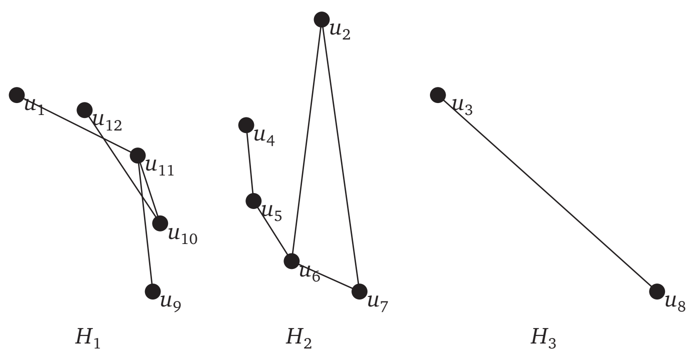

Figure 4. Factors of G

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